A COMPLETE CLASS OF PROGRESSIVITY AND REDISTRIBUTION MEASURES

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1. INTRODUCTION

This paper proposes a generalization of tax progressivity and redistribution measures that defines a complete dominance class. They are illustrated by useful progressivity and redistribution curves, defined for a set of normative parameters, which compare tax schedules under the concentration and Lorenz dominance principles. The well-known partial welfare ordering generated by Lorenz dominance criterium justifies our proposed redistribution measures.

Pfähler (1987), Lambert (1987), Duclos (1993) generalized the progressivity and the redistribution using the ATR and AIR classes of measures. Our proposed progressivity measure preserves the property of tax concentration dominance as a sufficient condition for progressivity. Moreover, it also satisfies the property of tax concentration dominance as a necessary condition for progressivity. This property is important as guarantees that if non-dominance between the concentration and the Lorenz curve occurs, there always exists a normative parameter for which the sign of progressivity changes. This property is not satisfied by classical ATR measures, such as Kakwani (1977) or Suits (1977) measures.

Both necessary and sufficient conditions define the so-called complete dominance class, so that, there is a complete equivalence between the unambiguous sign of all the measures (for all the normative parameters defined) and the tax concentration dominance.

Similarly, we extend the analysis to redistribution by constructing the analogous general redistribution measures. Again, the sign of all our redistribution indices is not only a sufficient but also a necessary condition for Lorenz dominance. This is not the case for classical AIR measures, such as reformulated Reynolds-Smolensky index.

Concentration/Lorenz dominance is an extreme concept: it is either satisfied or not. We introduce the possibility to evaluate intermediate cases when concentration/lorenz curves cross, according to our complete class. The critical aversion parameter for which our progressivity/redistribution index changes from a positive to a negative value can be reinterpreted as the probability (percentage of normative parameter values ensuring progressivity/redistribution) of taxes to be progressive/redistributive. This critical value has also the important empirical property of being revealed by the data.

We observe that this 'probability' of being progressive/redistributive and the progressivity/redistribution itself are different concepts. For example, the probability of being progressive increases, given a constant positive non-zero

Kakwani index, when concentration curves approximate to each other. As we will see, both concepts (progressivity and the probability of being progressive) are well captured by the general progressivity curve. Similarly, redistribution and the probability of being redistributive are well captured by the general redistribution curve.

Finally, we illustrate the study with an example of the general progressivity and redistribution for the interregional direct tax, social security contributions and tranfers in Spain, for the year 1991.

The paper is structured as follows. Next section provides the definition of our progressivity index. In section 3, an extension to redistribution is made. In section 4 we compare both concepts and in section 5, we make an application from regional Spanish data. Finally, concluding remarks are presented in section 6.

2. A COMPLETE MEASURE OF PROGRESSIVITY

Consider N individuals $i=1,...N$, whose income before tax are denoted $x=(x_1,...,x_i,...,x_N)$. These incomes are assumed to be ordered as $0<x_1 \leq ... \leq x_i \leq ... \leq x_N$. Denote $T=(T_1,...,T_N)$, the associated tax burden vector and $t=(t_1,...t_N)$, the associated average tax rate vector, with $t_i=T_i/x_i$. Denote μ_x and μ_T, the mean of the before tax income and of the tax burden respectively.

The Lorenz curve $L_x(i/N)$ for gross income is defined as

$$L_x(i/N) = \sum_{j=1}^{i} x_j / N\, m_x \qquad 1$$

The concentration curve of the tax burden $L_T(i/N)$ is defined as

$$L_T(i/N) = \sum_{j=1}^{i} T_j / N\, m_T \qquad 2$$

Define the equivalent proportional tax schedule as $PT=(\mu_t x_1,...,\mu_t x_N)$, being $\mu_t=\mu_T/\mu_x$ the efective T average tax rate. Note that $L_x(i/N)=L_{PT}(i/N)$.

Given any two tax schedule T^1 and T^2, define T^1 concentration dominates (CD) T^2 iff:

$$\begin{aligned} T^1\ CD\ T^2 \ \equiv\ & L_{T^2}(i/N) \geq L_{T^1}(i/N)\ \ \forall i = 1,\ldots,N \\ & \exists i \mid\ L_{T^2}(i/N) > L_{T^1}(i/N) \end{aligned} \qquad 3$$

Define a progressivity index $P:R^{2N} \rightarrow R$ over a vector $(x_1,...,x_N;T_1,...,T_N)$ which satisfies the following property:

$$P(x,T) = \sum_{i'=1}^{N} w(i'/N)[\,L_x(i/N) - L_T(i/N)\,]_{i'} \qquad 4$$

being the weighting scheme $w:R_+ \rightarrow R_+$ any normalized positive-valued function and being i' the associated rank of the new distribution ordered by $[L_X(i/N)-L_T(i/N)]$.

In particular we propose the following general progressivity class:

$$P(x,T,v) = \sum_{i'=1}^{N} \frac{2}{A} (i' / N)^{f(v)} [L_x(i/N) - L_T(i/N)]_{i'} \qquad 5$$

being A the normalization term:

$$A = \sum_{i'=1}^{N} (i' / N)^{f(v)} \qquad 6$$

and being $\phi: v \rightarrow (-\infty,+\infty)$, $v \in (-1,1)$, $\phi'(v)>0$ and let $\phi(v)$ be a symmetric function with $\phi(0)=0$. Note that it is a generalization of the classical Kakwani (1977) index for v=0. A particular operative case is:

$$f(v) = \frac{v}{(1-v^2)^{1/2}} \qquad 7$$

Note also that the partial derivative of P(x,T,v) with respect to v is non-negative and it is positive when concentration and Lorenz curves do not coincide, so v can be seen as a progressivity sensitivity parameter, by giving higher values to individuals whose distances between the concentration and the

Lorenz curves are higher.

PROPERTY 1: Given any x and T, the following relations are satisfied:

$$T\ CD\ PT \_ [P(x,T,v) > 0 \quad \forall\ v \in (-1,1)] \tag{8}$$

$$PT\ CD\ T \_ [P(x,T,v) < 0 \quad \forall\ v \in (-1,1)] \tag{9}$$

$$[\ L_{PT}(i / N) = L_{T}(i / N) \ \ \forall i] \_ [P(x,T,v) = 0 \quad \forall\ v \in (-1,1)] \tag{10}$$

Notice that concentration dominance is not only a sufficient condition but a necessary condition for the adequate sign for all v-progressivity index. Proof of the sufficient condition is straightforward from equation 4. Necessary condition follows from the fact that assuming that T neither dominates nor is dominated by PT, there always exists a v value such that P(x,T,v)>0 and a another v value such that P(x,T,v)<0.

COROLLARY: Given any x, T^1 and T^2, the following propositions are satisfied:

$$T^1 \ CD \ T^2 \_ [P(x,T^1,v) > P(x,T^2,v) \quad \forall\, v \in (-1,1)] \tag{11}$$

$$[\, L_{T^1}(i/N) = L_{T^2}(i/N) \quad \forall i] \_ [P(x,T^1,v) = P(x,T^2,v) \quad \forall\, v \in (-1,1)] \tag{12}$$

DEFINITION: Given x and T, denote v^* the critical value of v for which the progressivity index is zero, that is

$$v^* = \{v \mid P(x,T,v) = 0\} \tag{13}$$

if $L_{PT}(i/N) \neq L_T(i/N)$ for any i,

and $v^*=0$ when $L_{PT}(i/N)=L_T(i/N)$ $\forall i$

PROPERTY 2.1: Given any x and T then

$$T \_ CD\, PT \wedge PT \_ CD\, T \_ \exists\, v^* \mid P(x,T,v) = 0 \tag{14}$$

Moreover v^* is unique. Proof is based on Property 1 and the fact that partial derivative of P(x,T,v) with respect to v is positive (when concentration and Lorenz curves do not coincide). This is a useful result on the necessary and sufficient conditions required for the existence and uniqueness of the critical value v^*.

PROPERTY 2.2: In the complementary cases v^* does not exist. Given any x and T then

$$T\ CD\ PT\ \_\ \_\ v^* \in (-1,1), v^* \to -1 \qquad 15$$

$$PT\ CD\ T\ \_\ \_\ v^* \in (-1,1), v^* \to 1 \qquad 16$$

DEFINITION: Let define the degree of tax concentration dominance for which the tax can be considered as progressive $d^*=(1-v^*)/2$ according to P(x,T,v). Note that $d^* \in (0,1)$ can be seen as the probability of tax being progressivity, according to the percentage of the complete normative paremeter values indicating positive progressivity. This definition generalizes the concept of classical concentration dominance to the degree of concentration dominance associated to any P(x,T,v) defined in equation (5).

We have now two indices that can be very useful in empirical work as summarize the information about tax systems. Nevertheless both concepts differs: d^* is related to the probability of T being progressive/regressive and P(x,T,v) is the measure of such a progressivity/regressivity.

DEFINITION: Given x, T^1 and T^2, denote v^{*T1T2} the critical value of v for which the progressivity indices of both taxes are the same, that is

$$v^{*T1T2} = \{v \mid P(x,T^1,v) = P(x,T^2,v)\} \qquad 17$$

PROPERTY 3: Given any x, T^1 and T^2 then there always exists a v^{*T1T2} such that:

$$T^1 \_CD\ T^2 \wedge T^2 \_CD\ T^1 \_\exists v^{*T1T2} \in (1,-1) \qquad 18$$

3. MEASURES OF REDISTRIBUTION

Denote $y=(y_1,...y_N)$, the x-associated net income vector, being $y_i=x_i-T_i=x_i(1-t_i)$. The Lorenz curve $L_y(j/N)$ for the net income y is defined as

$$L_y(j/N) = \sum_{k=1}^{j} y_k / N\, m_y \qquad 19$$

being μ_y the mean of the after-tax income and j is the rank of the following ordered distribution $Y=(y_1,...,y_j,...,y_N)$. These incomes are assumed to be ordered as $0<y_1\leq...\leq y_j\leq...\leq y_N$.

Let y-distribution Lorenz dominates (LD) x-distribution iff:

$$y\,LD\,x \equiv \; L_y(j/N) \geq L_x(i/N) \;\; \forall i = j = 1,\ldots,N$$
$$\exists i = j \mid \; L_y(j/N) > L_x(i/N) \qquad \textbf{20}$$

Define a redistribution index RE:$R^{2N}_{+} \longrightarrow R$ over a vector $(x_1,...,x_N;y_1,...,y_N)$ which satisfies the following property:

$$RE(x,y,v) = \sum_{i''=1}^{N} w(i''/N)[\,L_y(j/N) - L_x(i/N)\,]_{i''} \quad j = i \qquad \textbf{21}$$

being w:$R_{+} \longrightarrow R_{+}$ any positive-valued function and being i'' the associated rank of the new distribution ordered by $[L_y(j/N)-L_x(i/N)]$, i=j. In particular we define same w(i''/N) as in (5)-(7). It is a generalization of the Reynolds-Smolensky index (1977) or a generalization of their reformulated version, Lambert (1993).

PROPERTY 4: Given any x and y, the following relations are satisfied:

$$y\;LD\;x \,\_\, [RE(x,y,v) > 0 \quad \forall\; v \in (-1,1)] \qquad \textbf{22}$$

$$[\,L_x(i/N) = L_y(i/N) \;\; \forall i]\,\_\,[RE(x,y,v) = 0 \quad \forall\; v \in (-1,1)] \qquad \textbf{23}$$

Notice that Lorenz dominance is not only a sufficient condition but a necessary condition for the adequate sign for all v-redistribution index. Necessary condition follows from the fact that assuming that y neither dominates nor is dominated by x, there always exists a v value such that RE(x,y,v)>0 and a another v value such that RE(x,y,v)<0.

COROLLARY 1:

If$\mu_x=\mu_y$, then:

$$RE(x,y,v) > 0 \ \forall V \_ W(y) > W(x) \qquad \textbf{24}$$

For all Social Welfare Functions (SWF) W being strictly S-concave[1]. The proof is an application of theorem in Atkinson (1970) and Dasgupta, Sen and Starrett (1973). An extension proposed by Shorrocks (1983) allows to solve some additional cases when $\mu_x \neq \mu_y$, using the generalized Lorenz curves dominance.

DEFINITION: Given x and T, denote v^{**} the critical value of v for which the redistributive index is zero, that is

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[1]The reader will notice that the following implicit inequality index is involved in our redistribution index:

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$$I(x,v) = \sum_{k=1}^{N} w(k / N)[(i / N) - L_x(i / N)]_k$$ ¡Error!Sólo el documen

being k the associated rank of the new distribution ordered by $[(i/N)-L_x(i/N)]$. Notice that I(x,v) is a strictly S-convex index, consistent with any strictly S-concave SWF. As constructed, I(x,v) is a relative index, consistent with any weakly homothetic SWF (see Dutta and Esteban (1992)) although it can be generalized to the absolute and intermediate indices, see Bossert and Pfingsten (1996).
The Lorenz properties of these indices are not satisfied by the classical Atkinson (1970), the general entropy indices (Cowell (1977) and the extended Gini coefficients (Yitzaki (1983)).

$$v^{**} = \{v \mid RE(x,y,v) = 0\} \qquad 25$$

if $L_x(i/N) \neq L_y(j/N)$ for any i=j,

and $v^{**}=0$ when $L_x(i/N)=L_y(j/N)$ $\forall i=j$

PROPERTY 5.1: Given any x and T then

$$y \_LD\, x \wedge x \_LD\, y \_ \exists v^{**} \mid RE(x,y,v) = 0 \qquad 26$$

Moreover v^{**} is unique. Proof is analogous to Property 2.1.

PROPERTY 5.2: In the complementary cases v^{**} does not exist. Given any x and T then

$$y\, LD\, x \_\_ v^{**} \in (-1,1), v^{**} \rightarrow -1 \qquad 27$$

$$x\, LD\, y \_\_ v^{**} \in (-1,1), v^{**} \rightarrow 1 \qquad 28$$

DEFINITION: Let define the degree of tax redistribution dominance for which the tax can be considered as regressive $d^{**}=(1-v^{**})/2$

according to RE(x,y,v). Note that $d^{**} \in (0,1)$ can be seen as the probability of tax being redistributive, according to the percentage of the complete normative paremeter values indicating positive redistribution. This definition generalizes the concept of classical Lorenz dominance to the degree of Lorenz dominance associated to any RE(x,y,v) defined in equation (21).

DEFINITION: Given x, T^1 and T^2, denote v^{**T1T2} the critical value of v for which the redistribution indices of both taxes are the same, that is

$$v^{**T1T2} = \{v \mid RE(x,y^1,v) = RE(x,y^2,v)\} \qquad 29$$

where y^1 and y^2 are the T^1 and T^2 after-tax income distributions respectively.

PROPERTY 6: Given any x, T^1 and T^2 then there always exists a v^{**T1T2} such that:

$$y^1 \ \_LD \ y^2 \ \wedge \ y^2 \ \_LD \ y^1 \ \_\exists\, v^{**T1T2} \in (1,-1) \qquad 30$$

4. PROGRESSIVITY, REDISTRIBUTION AND RERANKING

PROPERTY 7: Given any x and y then redistribution index can be decomposed as follows:

$$RE(x,y,v) = P(x,T,v)\frac{m_t}{1-m_t} - R(x,y,v) \qquad 31$$

being R(x,y,v) a non-negative reranking index:

$$\begin{aligned} R(x,y,v) = {} & \sum_{i'=1}^{N} w(i'/N)[L_y(i/N) - L_x(i/N)]_{i'} \\ & - \sum_{i''=1}^{N} w(i''/N)[L_y(j/N) - L_x(i/N)]_{i''} \ \ j = i \end{aligned} \qquad 32$$

Notice that $R(x,y,v)$ index belongs to the concentration based reranking indices as in Duclos (1993) and Lerman and Yitzaki (1995). It satisfies the important properties that it is positive if (and only if) reranking occurs and it is zero if (and only if) no reranking is produced.

PROPERTY 8: Given any x and T, $v^{*}<v^{**}$ iff $R>0$ and $v^{*}=v^{**}$ iff $R=0$. The proof is a direct application of equation (31) and the fact that R is greater or equal to zero.

5. EMPIRICAL RESULTS

In these section we carry out some illustrative empirical examples applied to the Spanish regional data, elaborated by BBV for the year 1991. The data base gives regional information about income before tax and transfers, about direct taxes, social security contributions and social transfers.

We have computed and represented in TABLES 1, 2 and 3 the progressivity and redistribution curves. It shows the $P(v)$ and $RE(v)$ values for all the normative parameter v values. The crossing point of $P(v)$ with the horizontal axis is the value of the v^{*} parameter. The crossing point of $RE(v)$ with the horizontal axis is the value of the v^{**} parameter. The intersection of $P(v)$ and $RE(v)$ with the vertical axis are the Kakwani (K) and reformulated Reynolds-Smolensky (RS^{*}) indices, respectively.

For the case of interregional Social Security contributions (TABLE 1), P(x,C,0) is equal to -0.00106 in 1991, showing a slight regressivity. The value $d^*=0.3926$ ($v^*=0.21480$), far from being a triviality, is consistent with the real case of a great proximity of the concentration curve to the Lorenz curve of the original income distribution. This is an interesting additional information that reinforces the fact that regressivity is also accepted for a great variety of normative v parameters (60.74%).

The redistribution, measured by the reformulated Reynolds-Smolemsky index, caused by the Social Security Contributions is RE(v=0)=-0.00027. A majority of normative parameters (being $v^{**}=0.23995$) supports that Social Security Contributions cause a very little anti-redistributive effect.

Another revealing example is the case of the Direct Taxes (TABLE 2) and the Social Transfers in Spain in 1991 (TABLE 3), being the respective Kakwani indices 0.08944 and 0.09371, indicating a significant progressivity in both cases. The respective v^* values are -0.86332 ($d^*=0.93166$) and -1 ($d^*=1$), which are also revealing, indicating nearly perfect dominance in the first case (consistent with an only minor crossing detected at the very upper side of the distribution) and perfect dominance in the other case.

The redistribution caused by the Direct Taxes and the Social Transfers, measured by the RE(v=0) is 0.011394 for Direct Taxes and 0.01612 for Social Transfers. The degree of probability associated with these results, measured by v^{**} values, is

-0.86316 (d^{**}=0.93158) and -1 (d^{**}=1), respectively. Note that we observe that in any case, the reranking effect (R) is very small.

Apart from the qualitative different progressivity and redistribution effects of the Direct Taxes and Social Transfers (concentration and Lorenz dominance in the second case, and not in the first case), it is interesting to compare directly the progressivity and redistribution implied by both public policies, in line with Properties (1) and (4). TABLE 4 shows that Social Transfers curves are above the Direct Taxes curves for all normative parameters. So we can conclude that perfect dominance of Social Transfers with respect to Direct Taxes occurs.

6. CONCLUSION

In this paper we have proposed a generalization of tax progressivity and redistribution measures that defines a complete dominance class. Implicitely we have characterized a general inequality index with the same dominance properties, that are not satisfied by any other general inequality index in the literature.

The essencial property of our indices is the equivalence established between Lorenz/concentration dominance and the sign of our indices for all the normative parameters. The well-known partial welfare ordering generated by Lorenz dominance criterium, for a wide class of Social Welfare Functions, justifies the use of our proposed redistribution measures. An example of the Spanish economy illustrates the potential use of the progressivity and redistribution curves.

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TABLE 1

Progressivity and Redistribution Curves

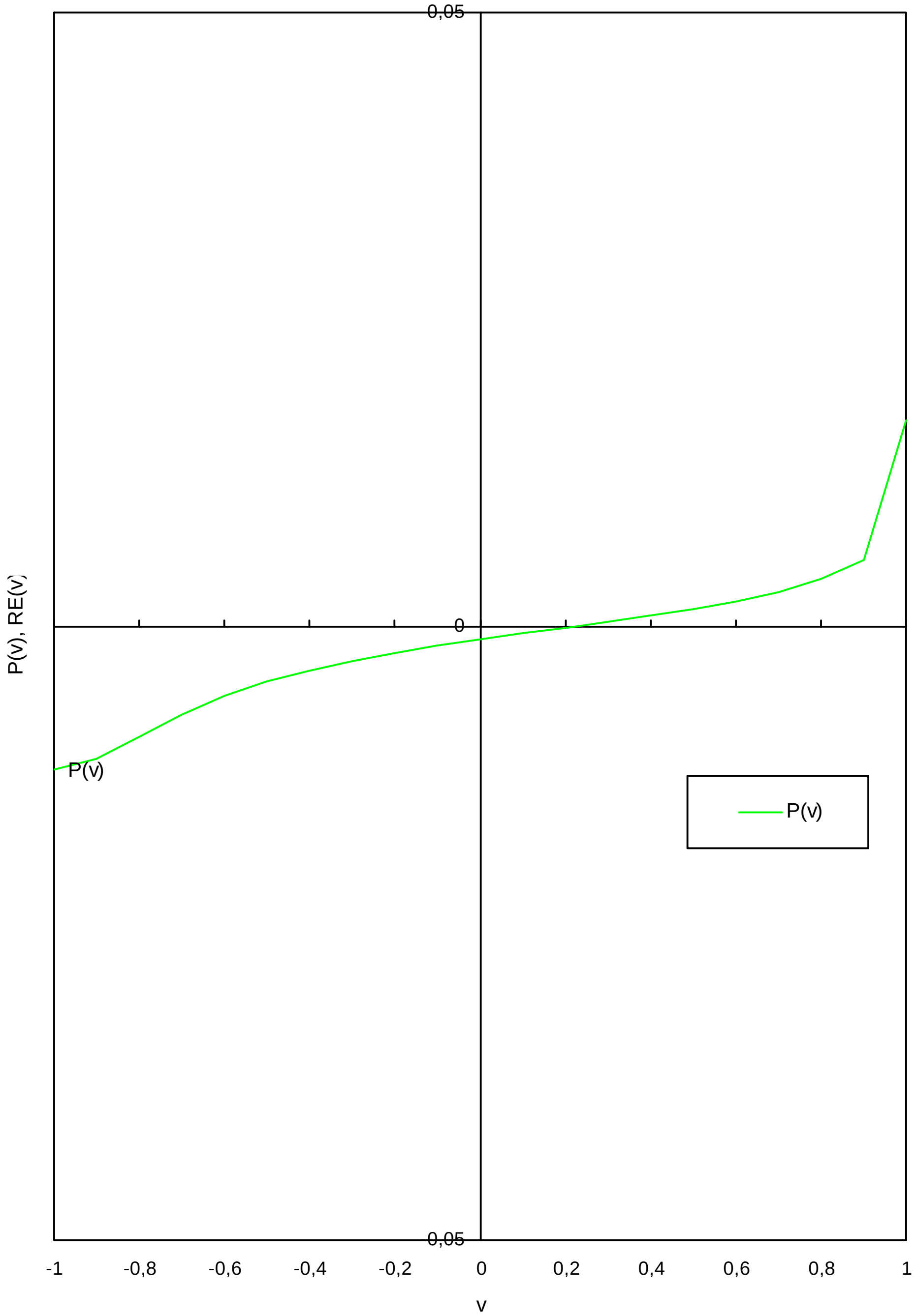

TABLE 2

Progressivity and Redistribution Curves

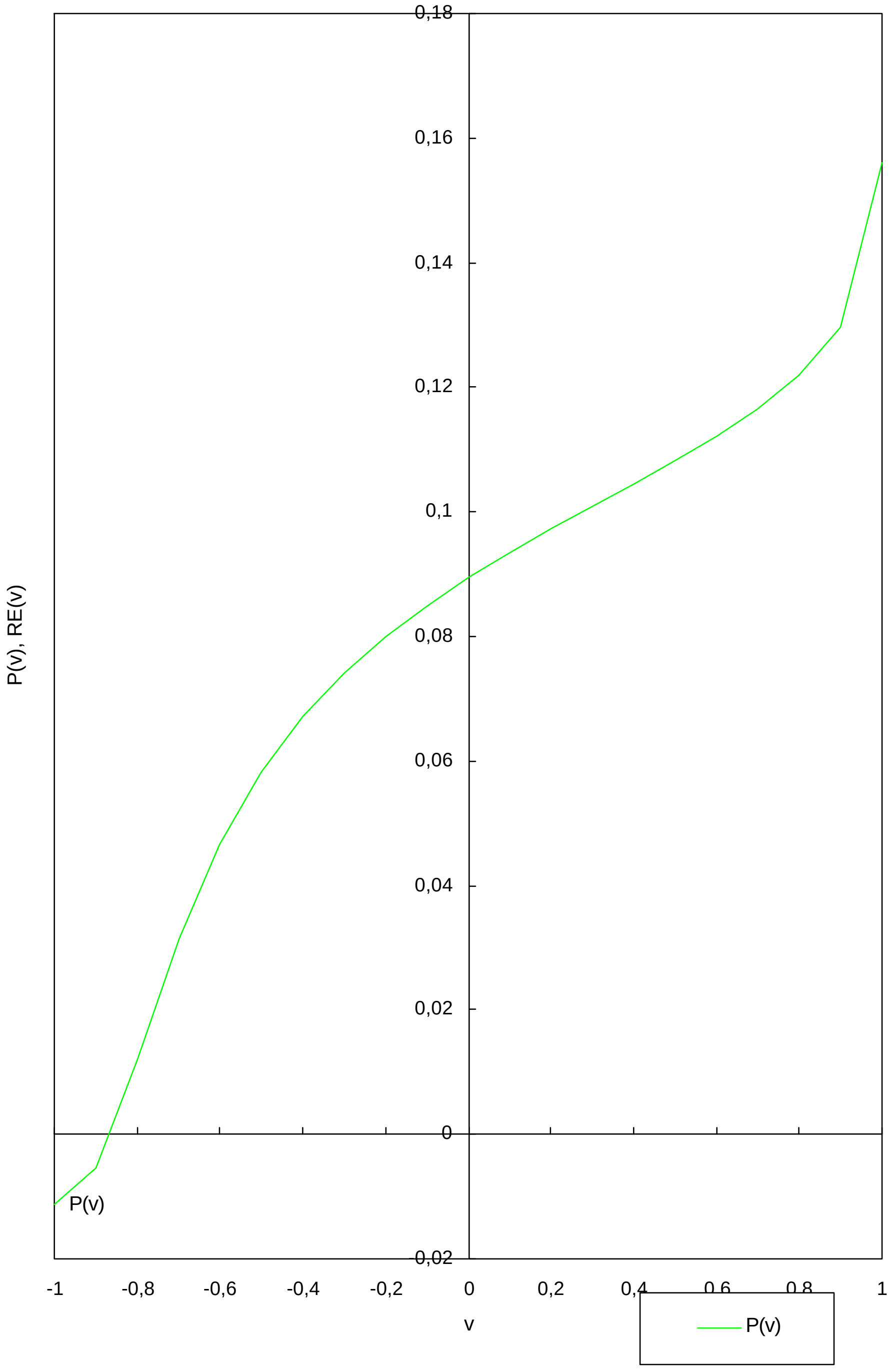

TABLE 3

Progressivity and Redistribution Curves

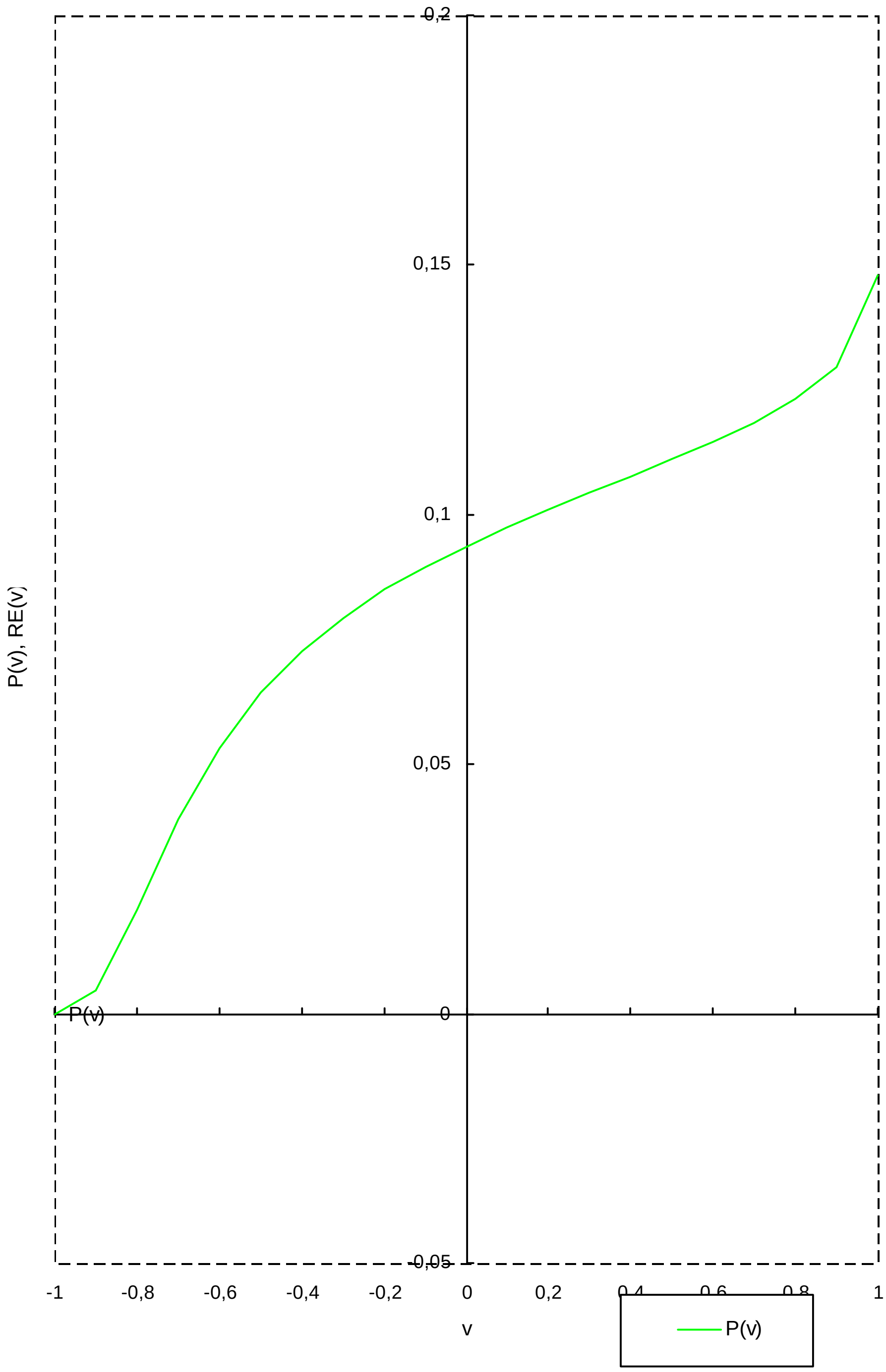

TABLE 4

Direct Taxes and Social Transfers Progressivity and Redistribution Curves