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EXISTENCE RESULTS FOR A SUPERLINEAR SINGULAR EQUATION OF CAFFARELLI-KOHN-NIRENBERG TYPE

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ABSTRACT. In this paper, using the Mountain Pass Lemma and the Linking Argument, we prove the existence of nontrivial weak solutions for the Dirichlet problem for the superlinear equation of Caffarelli-Kohn-Nirenberg type in the case where the parameter $\lambda \in (0, \lambda_2)$, λ_2 being the second positive eigenvalue of the quasilinear elliptic equation of Caffarelli-Kohn-Nirenberg type.

KEY WORDS: singular equation, Caffarelli-Kohn-Nirenberg inequality, Mountain Pass Lemma, Linking Argument.
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1. INTRODUCTION.

In this paper, we investigate the existence of weak solutions for the following Dirichlet problem for the superlinear singular equation of Caffarelli-Kohn-Nirenberg type:

(1.1)
$$\begin{cases} -\mathrm{div}\,(|x|^{-ap}|Du|^{p-2}Du) = \lambda|x|^{-(a+1)p+c}|u|^{p-2}u + |x|^{-bq}f(u), \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < n$, $0 \le a < \frac{n-p}{p}$, $a \le b \le a+1$, $q < p^*(a,b) = \frac{np}{n-dp}$, $d = 1+a-b \in [0,\ 1]$, and $c > 0$.

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For $a=0,\ c=p$, many results of linking-type for critical points have been obtained (e.g. [1, 2, 6] for $p=2$, [12] for $p\neq 2$ and [14] for the case with indefinite weights).
The starting point of the variational approach to these problems with $a\geq 0$ is the following weighted Sobolev-Hardy inequality due to Caffarelli, Kohn and Nirenberg [4], which is called the Caffarelli-Kohn-Nirenberg inequality. Let $1<p<n$. For all $u\in C_0^\infty(\mathbb{R}^n)$, there is a constant $C_{a,b}>0$ such that

$$\Big(\int_{\mathbb{R}^n}|x|^{-bq}|u|^q\,dx\Big)^{p/q}\leq C_{a,b}\int_{\mathbb{R}^n}|x|^{-ap}|Du|^p\,dx, \tag{1.2}$$

where

$$-\infty<a<\frac{n-p}{p},\ a\leq b\leq a+1,\ q=p^*(a,b)=\frac{np}{n-dp},\ d=1+a-b. \tag{1.3}$$

Let $\Omega\subset\mathbb{R}^n$ be an open bounded domain with C^1 boundary and $0\in\Omega$, $\mathcal{D}_a^{1,p}(\Omega)$ be the completion of $C_0^\infty(\mathbb{R}^n)$, with respect to the norm $\|\cdot\|$ defined by

$$\|u\|=\Big(\int_\Omega|x|^{-ap}|Du|^p\,dx\Big)^{1/p}.$$

From the boundedness of Ω and the standard approximation argument, it is easy to see that (1.2) holds for any $u\in\mathcal{D}_a^{1,p}(\Omega)$ in the sense:

$$\Big(\int_\Omega|x|^{-\alpha}|u|^r\,dx\Big)^{p/r}\leq C\int_\Omega|x|^{-ap}|Du|^p\,dx, \tag{1.4}$$

for $1\leq r\leq\frac{np}{n-p},\ \alpha\leq(1+a)r+n(1-\frac{r}{p})$, that is, the embedding $\mathcal{D}_a^{1,p}(\Omega)\hookrightarrow L^r(\Omega,|x|^{-\alpha})$ is continuous, where $L^r(\Omega,|x|^{-\alpha})$ is the weighted L^r space with norm:

$$\|u\|_{r,\alpha}:=\|u\|_{L^r(\Omega,|x|^{-\alpha})}=\Big(\int_\Omega|x|^{-\alpha}|u|^r\,dx\Big)^{1/r}.$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem (cf. [7] for $p=2$ and [16] for the general case). For convenience of the readers, we include the proof here.

Theorem 1 (Compact embedding theorem)**.** *Suppose that $\Omega\subset\mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0\in\Omega$, $1<p<n$, $-\infty<a<\frac{n-p}{p}$. The embedding $\mathcal{D}_a^{1,p}(\Omega)\hookrightarrow L^r(\Omega,|x|^{-\alpha})$ is compact if $1\leq r<\frac{np}{n-p}$, $\alpha<(1+a)r+n(1-\frac{r}{p})$.*

Proof. The continuity of the embedding is a direct consequence of the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4). To prove the compactness, let $\{u_m\}$ be a bounded sequence in $\mathcal{D}_a^{1,p}(\Omega)$. For any $\rho>0$, if $B_\rho(0)\subset\Omega$ is the ball centered at the origin with radius ρ, it holds that $\{u_m\}\subset W^{1,p}(\Omega\setminus B_\rho(0))$.

Then the classical Rellich-Kondrachov compactness theorem guarantees the existence of a convergent subsequence of $\{u_m\}$ in $L^r(\Omega \setminus B_\rho(0))$. By taking a diagonal sequence, we can assume without loss of generality that $\{u_m\}$ converges in $L^r(\Omega \setminus B_\rho(0))$ for any $\rho > 0$.
On the other hand, for any $1 \le r < \frac{np}{n-p}$, there exists a $b \in (a, a+1]$ such that $r < q = p^*(a,b) = \frac{np}{n-dp}$, $d = 1 + a - b \in [0,\ 1)$. From the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4), $\{u_m\}$ is also bounded in $L^q(\Omega, |x|^{-bq})$. By the Hölder inequality, for any $\delta > 0$, it holds that

$$\begin{aligned} \int_{|x|<\delta} |x|^{-\alpha}|u_m - u_j|^r\, dx &\le \Big(\int_{|x|<\delta} |x|^{-(\alpha-br)\frac{q}{q-r}}\, dx\Big)^{1-\frac{r}{q}} \\ &\quad \times\Big(\int_\Omega |x|^{-br}|u_m-u_j|^r\, dx\Big)^{r/q} \\ &\le C\Big(\int_0^\delta r^{n-1-(\alpha-br)\frac{q}{q-r}}\, dr\Big)^{1-\frac{r}{q}} \\ &= C\delta^{n-(\alpha-br)\frac{q}{q-r}}, \end{aligned} \tag{1.5}$$

where $C > 0$ is a constant independent of m. Since $\alpha < (1+a)r + n(1-\frac{r}{p})$, it holds that $n - (\alpha - br)\frac{q}{q-r} > 0$. Therefore, for a given $\varepsilon > 0$, we first fix $\delta > 0$ such that

$$\int_{|x|<\delta} |x|^{-\alpha}|u_m - u_j|^r\, dx \le \frac{\varepsilon}{2},\ \forall\, m, j \in \mathbb{N},$$

then we choose $N \in \mathbb{N}$ such that

$$\int_{\Omega\setminus B_\delta(0)} |x|^{-\alpha}|u_m - u_j|^r\, dx \le C_\alpha \int_{\Omega\setminus B_\delta(0)} |u_m - u_j|^r\, dx \le \frac{\varepsilon}{2},\ \forall\, m, j \ge N,$$

where $C_\alpha = \delta^{-\alpha}$ if $\alpha \ge 0$ and $C_\alpha = (\mathrm{diam}\,(\Omega))^{-\alpha}$ if $\alpha < 0$. Thus

$$\int_\Omega |x|^{-\alpha}|u_m - u_j|^r\, dx \le \varepsilon,\ \forall\, m, j \ge N,$$

that is, $\{u_m\}$ is a Cauchy sequence in $L^q(\Omega, |x|^{-bq})$. □

Our results will rely mainly on the results of the eigenvalue problem corresponding to problem (1.1) in [15]. Let us first recall the main results of [15]. Consider the nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}\,(|x|^{-ap}|Du|^{p-2}Du) = \lambda|x|^{-(a+1)p+c}|u|^{p-2}u, \text{ in } \Omega \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < n,\ 0 \le a < \frac{n-p}{p},\ c > 0$.
Let us introduce the following functionals in $\mathcal{D}_a^{1,p}(\Omega)$:

$$\Phi(u) := \int_\Omega |x|^{-ap}|Du|^p\, dx, \text{ and } J(u) := \int_\Omega |x|^{-(a+1)p+c}|u|^p\, dx.$$

For $c > 0$, J is well-defined. Furthermore, $\Phi, J \in C^1(\mathcal{D}_a^{1,p}(\Omega), \mathbb{R})$, and a real value λ is an eigenvalue of problem (1.6) if and only if there exists $u \in \mathcal{D}_a^{1,p}(\Omega) \setminus \{0\}$ such that $\Phi'(u) = \lambda J'(u)$. At this point, we introduce the set

$$\mathcal{M} := \{u \in \mathcal{D}_a^{1,p}(\Omega) \ : \ J(u) = 1\}.$$

Then $\mathcal{M} \neq \emptyset$ and $\mathcal{M}$ is a C^1 manifold in $\mathcal{D}_a^{1,p}(\Omega)$. It follows from the standard Lagrange multipliers arguments that the eigenvalues of (1.6) correspond to the critical values of $\Phi|_{\mathcal{M}}$. From Theorem 1, Φ satisfies the (PS) condition on $\mathcal{M}$. Thus a sequence of critical values of $\Phi|_{\mathcal{M}}$ comes from the Ljusternik-Schnirelman critical point theory on C^1 manifolds. Let $\gamma(A)$ denote the Krasnoselski's genus on $\mathcal{D}_a^{1,p}(\Omega)$ and for any $k \in \mathbb{N}$, set

$$\Gamma_k := \{A \subset \mathcal{M} \ : \ A \text{ is compact, symmetric and } \gamma(A) \geq k\}.$$

Then the values

$$\lambda_k := \inf_{A \in \Gamma_k} \max_{u \in A} \Phi(u) \tag{1.7}$$

are critical values and hence are eigenvalues of problem (1.6). Moreover, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$.

One can also define another sequence of critical values minimaxing Φ along a smaller family of symmetric subsets of $\mathcal{M}$. Denote by S^k the unit sphere of $\mathbb{R}^{k+1}$ and

$$\mathcal{O}(S^k, \mathcal{M}) := \{h \in C(S^k, \mathcal{M}) \ : \ h \text{ is odd}\}.$$

Then for any $k \in \mathbb{N}$, the value

$$\mu_k := \inf_{h \in \mathcal{O}(S^{k-1}, \mathcal{M})} \max_{t \in S^{k-1}} \Phi(h(t)) \tag{1.8}$$

is an eigenvalue of (1.6). Moreover $\lambda_k \leq \mu_k$. This new sequence of eigenvalues was first introduced in [11] and later used in [10, 9] for $a = 0, c = p$.

In [15], we proved that the first positive eigenvalue $\lambda_1 = \mu_1$ is simple, isolated and it is the unique eigenvalue with positive eigenfunction, and $\underline{\lambda}_2 := \inf\{\lambda \in \mathbb{R} \ : \ \lambda \text{ is eigenvalue and } \lambda > \lambda_1\} = \lambda_2 = \mu_2$.

In this paper, based on the Mountain Pass Lemma and the Linking Argument, we will prove the existence of nontrivial weak solutions to problem (1.1) in the case where the parameter $\lambda \in (0, \lambda_2)$.

2. Linking results

Let $e_k \in \mathcal{M}$ be the eigenfunction associated to λ_k, then $\|e_k\|^p_{\mathcal{D}_a^{1,p}(\Omega)} = \lambda_k$. Denote $G = \{u \in \mathcal{M} : \ \Phi(u) < \lambda_2\}$. Obviously, G is an open set containing e_1 and e_2. Moreover $-G = G$. First we prove the following Lemma.

Lema 2. *e_1 and $-e_1$ do not belong to the same connected component of G.*

Proof. Otherwise, there exists a continuous curve σ connecting e_1 and $-e_1$ in G. Let $A = \sigma \cup \{-\sigma\}$, then from the definition of $\mathcal{M}$, $0 \notin A$, hence $\gamma(A) > 1$, by connectedness of A, so $A \in \Gamma_2$. Hence, as A is a compact set in G, and from the definition of G, we have $\max\{\Phi(u);\ u \in A\} < \lambda_2$ and this contradicts the definition of λ_2. Q.E.D.

Let G_1 be the connected component of G containing e_1, then $-G_1$ is the connected component of G containing $-e_1$. Let

$$K_1 = \{tu :\ u \in G_1, t > 0\}, \quad K = -K_1 \cup K_1.$$

Then

$$\int_\Omega |x|^{-ap}|Du|^p\,dx < \lambda_2 \int_\Omega |x|^{-(a+1)p+c}|u|^p\,dx,\ \forall u \in K, \tag{2.1}$$

and

$$\int_\Omega |x|^{-ap}|Du|^p\,dx = \lambda_2 \int_\Omega |x|^{-(a+1)p+c}|u|^p\,dx,\ \forall u \in \partial K, \tag{2.2}$$

where ∂K is the boundary of K in $X = \mathcal{D}_a^{1,p}(\Omega)$. Let $(\partial K)_\rho = \{u \in \partial K :\ \|u\| = \rho\}$.
Set

$$\mathcal{E}_1 = \operatorname{span}\{e_1\},\ \mathcal{E}_2 = \operatorname{span}\{e_1,\ e_2\},$$

$$\mathcal{Z} = \{u \in X :\ \int_\Omega |Du|^p = \lambda_2 \int_\Omega V(x)|u|^p\}, then$$

(2.2) implies $\partial K \subset \mathcal{Z}$.
In a similar way to Proposition 2.1-2.2 in [12] and Lemma 2.1-2.2 in [14], we obtain the following two linking results.

Theorem 3. *Assume that $v \in \mathcal{E}_1,\ v \neq 0,\ Q = [-v,\ v]$ is the line segment connecting $-v$ and v, $\partial Q = \{-v,\ v\}$. Then $\partial Q \subset Q$ and $\mathcal{Z}$ link in X, that is,*

- (i) *$\partial Q \cap \mathcal{Z} = \emptyset$ and*
- (ii) *For any continuous map $\psi : Q \to X$ with $\psi|_{\partial Q} = id$, it follows that $\psi(Q) \cap \mathcal{Z} \neq \emptyset$.*

Proof. It is obvious that $\partial Q \cap \mathcal{Z} = \emptyset$. Now let $\psi : Q = [-v, v] \to X$ be continuous and $\psi|_{\partial Q} = \mathrm{id}$. From the definition of K and Lemma 2, K has two connected components K_1 and $-K_1$. Assume $v \in K_1,\ -v \in -K_1$, then $\psi(Q)$ is a continuous curve connecting v and $-v$, therefore it holds that $\psi(Q) \cap \partial K \neq \emptyset$ and thus $\psi(Q) \cap \mathcal{Z} \neq \emptyset$. □

Theorem 4. *Assume $0 < \rho < r < \infty$, let $\tilde{e}_1 = e_1/\lambda_1^{1/p},\ \tilde{e}_2 = e_2/\lambda_2^{1/p}$, and*

$$Q = Q_r = \{u = t_1\tilde{e}_1 + t_2\tilde{e}_2 :\ \|u\| \leq r, t_2 \geq 0\},$$

$$\partial Q = \partial Q_r = \{u = t_1\tilde{e}_1 : \ |t_1| \le r\} \cup \{u \in Q_r : \ \|u\| = r\},$$

$$Z_\rho = \{u \in \mathcal{Z} : \ \|u\| = \rho\}.$$

Then $\partial Q_r \subset Q_r$ and Z_ρ link in X.

Proof. $\partial Q_r \cap Z_\rho = \emptyset$ is obvious. Let $\psi : Q_r \to X$ be continuous and $\psi|_{\partial Q_r} = \mathrm{id}$. Denote $d_1 = \mathrm{dist}\,(\tilde{e}_1, \partial K)$ and define the mapping $P : X \to \mathcal{E}_2$ as follows:

$$P(u) = \begin{cases} \big(\min\{\mathrm{dist}\,(u, \partial K), rd_1\}\big)\tilde{e}_1 + (\|u\| - \rho)\tilde{e}_2, & \text{if } u \notin -K_1; \\ -\big(\min\{\mathrm{dist}\,(u, \partial K), rd_1\}\big)\tilde{e}_1 + (\|u\| - \rho)\tilde{e}_2, & \text{if } u \in -K_1. \end{cases}$$

It is easy to see that P is continuous, and that P maps $v = r\tilde{e}_1$ to $v_1 = Pv = rd_1\tilde{e}_1 + (r - \rho)\tilde{e}_2$, the origin 0 to $0_1 = P0 = -\rho\tilde{e}_2$, the line segment $[v, 0]$ onto the line segment $[v_1, 0_1]$ homeomorphically; $-v = -r\tilde{e}_1$ to $v_2 = P(-v) = -rd_1\tilde{e}_1 + (r - \rho)\tilde{e}_2$, the line segment $[0, -v]$ onto a line segment $[0_1, v_2]$ homeomorphically; and the half circle $\{u \in \partial Q : \ \|u\| = r\}$ which is from v to$-v$ in ∂Q onto the line segment $[v_1, v_2]$, where $P(r\tilde{e}_2) = (r - \rho)\tilde{e}_2$.
Let $f = P \circ \psi : Q \to \mathcal{E}_2$. From the discussion above, it holds that $0 \notin f(\partial Q)$, and when u turns a circuit along ∂Q counterclockwise , $f(u)$ also moves a circuit around the original 0 in $\mathcal{E}_2$ counterclockwise . Hence by a degree argument, it holds that $\deg\,(f, Q, 0) = 1$. So there exists some $u \in Q$ such that $f(u) = 0$, i.e., $P(\psi(u)) = 0$, which implies that $\psi(u) \in \partial K$ and $\|\psi(u)\| = \rho$. Thus $\psi(u) \in (\partial K)_\rho$ and $\psi(Q) \cap (\partial K)_\rho \neq \emptyset$. Since $(\partial K)_\rho \subset Z_\rho$, it follows that $\psi(Q) \cap \mathcal{Z} \neq \emptyset$ □

3. Existence results for problem (1.1)

In this section, we will give some conditions on $f(u)$ to guarantee that the functional associated to problem (1.1) satisfies the Palais-Smale condition ((PS) condition) for $\lambda \in (0, \lambda_2)$, the geometric assumptions of the Mountain Pass Lemma (cf. Theorem 6.1 in Chapter 2 of [13]) in the case of $0 < \lambda < \lambda_1$, and those of the linking theorem (cf. Theorem 8.4 in Chapter 2 of [13]) in the case of $\lambda_1 \le \lambda < \lambda_2$.
Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies:

- (f_1) (Subcritical growth) $|f(s)| \le c_1|s|^{q-1} + c_2$, $\forall s \in \mathbb{R}$, where $1 < q < p^*(a, b) = \frac{Np}{N-dp}$.
- (f_2) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $uf(u) \ge 0$, $u \in \mathbb{R}$.
- (f_3) (Asymptotic property at infinity) $\exists\, \theta \in (p,\ p^*(a, b))$ and $M > 0$ such that $0 < \theta F(u) \le uf(u)$ for $|u| \ge M$, where $F(s) = \displaystyle\int_0^s f(t)dt$.
- (f_4) (Asymptotic property at $u = 0$) $\lim_{s \to 0} f(s)/|s|^{p-1} = 0$.

Theorem (1) and (f_1) imply that the functional $I:\ X \to \mathbb{R}$:

$$I(u) = \frac{1}{p}\int_\Omega |x|^{-ap}|Du|^p\,dx - \frac{\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|u|^p\,dx - \int_\Omega |x|^{-bq}F(u)dx$$

is well-defined and $I \in C^1(X;\mathbb{R})$ and that the weak solutions of problem (1.1) is equivalent to the critical points of I. (f_2) implies that 0 is a trivial solution to problem (1.1).

Lema 5. *If f satisfies assumptions (f_1)-(f_3), then I satisfies the (PS) condition for $\lambda \in (0, \lambda_1)$.*

Proof. 1. The boundedness of (PS) sequences of I.
Suppose $\{u_m\}$ is a (PS) sequence of I, that is, there exists $C > 0$ such that $|I(u_m)| \le C$ and $I'(u_m) \to 0$ in X', the dual space of X, as $m \to \infty$. The properties of the first eigenvalue λ_1 imply that for any $u \in X$, one has

$$\lambda_1 \int_\Omega |x|^{-(a+1)p+c}|u|^p\,dx \le \int_\Omega |x|^{-ap}|Du|^p\,dx.$$

Let $c := \sup_m I(u_m)$. Then by the above inequality and (f_3), as $m \to \infty$, it holds that

$$\begin{aligned}
c - \frac{1}{\theta}o(1)\|u_m\| &= (\frac{1}{p} - \frac{1}{\theta})\int_\Omega |x|^{-ap}|Du_m|^p\,dx \\
&\quad -\lambda(\frac{1}{p} - \frac{1}{\theta})\int_\Omega |x|^{-(a+1)p+c}|u_m|^p\,dx + \int_\Omega |x|^{-bq}(\frac{1}{\theta}f(u_m)u_m - F(u_m))\,dx \\
&\ge (\frac{1}{p} - \frac{1}{\theta})(1 - \frac{\lambda}{\lambda_1})\int_\Omega |x|^{-ap}|Du_m|^p\,dx \\
&\quad + \int_{\Omega(u_m \ge M)} |x|^{-bq}(\frac{1}{\theta}f(u_m)u_m - F(u_m))\,dx \\
&\quad + \int_{\Omega(u_m < M)} |x|^{-bq}(\frac{1}{\theta}f(u_m)u_m - F(u_m))\,dx \\
&\ge (\frac{1}{p} - \frac{1}{\theta})(1 - \frac{\lambda}{\lambda_1})\|u_m\|^p - C_1,
\end{aligned}$$

where $C_1 \ge 0$ is a constant independent of u_m. The above estimate implies the boundedness of $\{u_m\}$ for $0 < \lambda < \lambda_1$.
2. By (f_1), f satisfies the subcritical growth condition and by a standard argument there exists a convergent subsequence of $\{u_m\}$ as a consequence of the boundedness of $\{u_m\}$ in X. □

Theorem 6. *If f satisfies assumptions (f_1)-(f_4), then problem (1.1) has a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ provided that $0 < \lambda < \lambda_1$.*

Proof. We will verify the geometric assumptions of the Mountain Pass Lemma (cf. [13] Chapter 2, Theorem 6.1):

(1) $I(0)=0$ is obvious;
(2) $\exists\,\rho>0,\ \exists\,\alpha>0:\ \|u\|=\rho\implies I(u)\geq\alpha$;

In fact, $\forall\,u\in X$, it follows

$$I(u)\geq\frac{1}{p}(1-\frac{\lambda}{\lambda_1})\int_\Omega|x|^{-ap}|Du|^p\,dx-\int_\Omega|x|^{-bq}F(u)\,dx. \tag{3.1}$$

From (f_4), $\forall\,\epsilon>0$, $\exists\,\rho_0=\rho_0(\epsilon)$ such that if $0<\rho=\|u\|<\rho_0$, then $|f(u)|<\epsilon|u|^{p-1}$, thus

$$\int_\Omega|x|^{-bq}F(u)\,dx\leq\int_\Omega|x|^{-bq}\int_0^{u(x)}f(t)\,dt\,dx\leq\frac{\epsilon}{p}\int_\Omega|x|^{-bq}|u|^p dx\leq\frac{c_0\epsilon}{p}\|u\|.$$

Choose $c_0\epsilon_0=(1-\frac{\lambda}{\lambda_1})/2>0,\ \rho=\dfrac{\rho_0(\epsilon_0)}{2}$, from (3.1), one has

$$I(u)\geq\frac{1}{p}(1-\frac{\lambda}{\lambda_1}-c_0\epsilon_0)\int_\Omega|x|^{-ap}|Du|^p\,dx\geq\frac{\lambda_1-\lambda}{2\lambda_1 p}\cdot\rho=:\alpha>0. \tag{3.2}$$

(3) $\exists\,u_1\in X:\ \|u_1\|\geq\rho$ and $I(u_1)<0$.

In fact, from (f_2) and (f_3), one can deduce that there exist constants $c_3,c_4>0$ such that

$$F(s)\geq c_3|s|^\theta-c_4,\ \forall s\in\mathbb{R}. \tag{3.3}$$

Since $\theta>p$, a simple calculation shows that as $t\to\infty$, it holds that

$$\begin{aligned}I(te_1)\ &\leq\frac{t^p}{p}\int_\Omega|x|^{-ap}|De_1|^p\,dx-\frac{\lambda t^p}{p}\int_\Omega|x|^{-(a+1)p+c}|e_1|^p\,dx\\ &\quad -c_3t^\theta\int_\Omega|x|^{-bq}|e_1|^\theta\,dx+c_4\int_\Omega|x|^{-bq}\,dx\\ &\to-\infty,\end{aligned} \tag{3.4}$$

which implies that $I(te_1)<0$ for $t>0$ large enough.
Thus Lemma 5 and the Mountain Pass Lemma imply that value

$$\beta=\inf_{p\in P}\sup_{u\in p}E(u)\geq\alpha>0$$

is critical, where $P=\{p\in C^0([0,1];X):\ p(0)=0,p(1)=u_1\}$. That is, there is a $u\in X$, such that

$$E'(u)=0,\ E(u)=\beta>0.$$

$E(u)=\beta>0$ implies $u\not\equiv0$. □

Lema 7. *Assume that $\lambda_1\leq\lambda<\lambda_2$ and f satisfies assumptions (f_1)-(f_3). Then I satisfies the $(C)_c$ condition introduced by Cerami in* [5]*, that is, any sequence $\{u_m\}\subset X$ such that $I(u_m)\to c$ and $(1+\|u_m\|)\|I'(u_m)\|_{X'}\to0$ possesses a convergent subsequence.*

Proof.
1. The boundedness of the $(\mathrm{C})_c$ sequences in X.
Let $\{u_m\} \subset X$ be such that $I(u_m) \to c$ and $(1+\|u_m\|)\|I'(u_m)\|_{X'} \to 0$. Then from (f_2), (f_3) and (3.3), as $m \to \infty$, one has
(3.5)
$$\begin{aligned} pc + o(1) &= pI(u_m) - < I'(u_m), u_m > \\ &= \int_\Omega |x|^{-bq}(u_m f(u_m) - pF(u_m))\,dx \\ &= \int_\Omega |x|^{-bq}(u_m f(u_m) - \theta F(u_m))\,dx + (\theta - p)\int_\Omega \theta |x|^{-bq} F(u_m)\,dx \\ &\geq -C_1 + (\theta - p)c_3 |u_m|^\theta_{L^\theta(\Omega,|x|^{-bq})} - C_4 \int_\Omega |x|^{-bq}\,dx. \end{aligned}$$

Thus $\theta > p$ implies the boundedness of $\{u_m\}$ in $L^\theta(\Omega, |x|^{-bq})$.
On the other hand, a simple calculation shows that

$$\begin{aligned} \theta c + o(1) &= \theta I(u_m) - < I'(u_m), u_m > \\ &= (\frac{\theta}{p} - 1)\|Du_m\|^p_{L^p(\Omega,|x|^{-ap})} - \lambda(\frac{\theta}{p} - 1)\int_\Omega |x|^{-(a+1)p+c}|u_m|^p\,dx \\ &\quad + \int_\Omega |x|^{-bq}(u_m f(u_m) - \theta F(u_m))\,dx \\ &\geq (\frac{\theta}{p} - 1)\int_\Omega |x|^{-ap}|Du_m|^p\,dx - C \\ &\quad + \int_{\Omega(u_m<M)} |x|^{-bq}(u_m f(u_m) - \theta F(u_m))\,dx \\ &\quad + \int_{\Omega(u_m\geq M)} |x|^{-bq}(u_m f(u_m) - \theta F(u_m))\,dx \\ &\geq (\frac{\theta}{p} - 1)\|Du_m\|^p_{L^p(\Omega,|x|^{-ap})} - C, \end{aligned} \tag{3.6}$$

where $C > 0$ is a universal constant independent of u_m, which may be different from line to line. Thus $\theta > p$ and (3.6) imply the boundedness of $\{u_m\}$ in X.
2. By (f_1), f satisfies the subcritical growth condition, by a standard argument, one can obtain that there exists a convergent subsequence of $\{u_m\}$ based on the boundedness of $\{u_m\}$ in X. □

Theorem 8. *Suppose f satisfies assumptions (f_1)-(f_4), and furthermore, $\theta > ps/(s-1)$ in (f_3). Then problem (1.1) has a nontrivial weak solution $u \in X$ provided that $\lambda_1 \leq \lambda < \lambda_2$.*

Proof. It was shown in [3] that the $(C)_c$ condition actually suffices to get a deformation theorem (Theorem 1.3 in [3], and it was also remarked in [8] that

the proofs of the standard Mountain Pass Lemma and saddle-point theorem go through without change once the deformation theorem (Theorem 1.3 in [3] is obtained with the $(C)_c$ condition. Here we verify the assumptions of standard Linking Argument Theorem(cf. [13] Chapter 2, Theorem 8.4) hold with the $(C)_c$ condition replacing the $(\mathrm{PS})_c$ condition.
Since $\partial Q_r \subset Q_r$ and Z_ρ link in X, it suffices to show that

(1) $\alpha_0 = \sup_{u\in\partial Q_r} I(u) \le 0$, when $r > 0$ is large enough.
(2) $\alpha = \inf_{u\in Z_\rho} I(u) > 0$, when $\rho > 0$ is small enough.

In fact, let $u = te_1 \in \mathcal{E}_1$, from assumption ($\mathrm{f}_2$), $F(x, s) \ge 0$ for all $s \in \mathbb{R}$ and almost every $x \in \Omega$, thus it holds that
(3.7)

$$\begin{aligned} I(u) = I(te_1) &\le \frac{|t|^p}{p}\int_\Omega |x|^{-ap}|De_1|^p\,dx - \frac{|t|^p\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|e_1|^p\,dx \\ &= \frac{|t|^p}{p}(1-\frac{\lambda}{\lambda_1})\|e_1\| \le 0. \end{aligned}$$

Noticing that

$$|u_m|_{L^\theta(\Omega,|x|^{-bq})} = \Big(\int_\Omega |x|^{-bq}|u|^\theta\Big)^{1/\theta},$$

is a norm on $\mathcal{E}_2$, and that the norms of finite dimensional space are equivalent, it follows that there exists a constant $c_5 > 0$ such that

$$\int_\Omega |x|^{-bq}|u|^\theta\,dx \ge c_5\|u\|^\theta,$$

From (3.3), it holds that

$$I(u) \le \frac{1}{p}\|u\|^p - c_3c_5\|u\|^\theta + c_4|\Omega|. \tag{3.8}$$

Since $\theta > p$, it follows

$$I(u) \to -\infty, \text{ as } \|u\| \to \infty,\ u \in \mathcal{E}_2,$$

this implies (1).
From (f_4) and (f_1), it follows that

$$\int_\Omega |x|^{-bq}F(u)\,dx = o(\|u\|^p) \text{ as } u \to 0 \text{ in } X,$$

then for any $u \in Z$, it holds that

$$I(u) = \frac{1}{p}(1-\frac{\lambda}{\lambda_2})\int_\Omega |x|^{-ap}|Du|^p\,dx + o(\|u\|^p). \tag{3.9}$$

Since $\lambda < \lambda_2$, (3.9) implies (2).

Thus the Linking Argument Theorem (cf. [13] Chapter 2, Theorem 8.4) implies that the value

$$\beta = \inf_{h\in\Gamma} \sup_{u\in Q} E(h(u)) \geq \alpha > 0$$

is critical, where $\Gamma = \{h \in C^0(X;X);\ h|_{\partial Q} = \mathrm{id}\}$. That is, there is a $u \in X$, such that

$$E'(u) = 0,\ E(u) = \beta > 0.$$

$E(u) = \beta > 0$ implies $u \not\equiv 0$. □

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