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MULTIPLE STATIONARY SOLUTIONS TO GKP EQUATION IN A BOUNDED DOMAIN

BENJIN XUAN (*)

ABSTRACT. In this paper, we study the existence of multiple stationary solutions of Generalized Kadomtsev-Petviashvili (Abbr. GKP) equation in a bounded domain with smooth boundary and for superlinear nonlinear term $f(u) = \lambda|u|^{p-2}u + |u|^{q-2}u$ where $1 \leq p, q < 2_* = \frac{2(2n-1)}{2n-3}$. Our methods are based on variational methods, and the results are divided into two cases according to the different values of the parameters p, q.

Key words and phrases. GKP equation, Stationary solution, Symmetric Mountain Pass Lemma, Kransnoselskii genus

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1. Introduction.

Kadomtsev-Petviashvili equation and its generalization appear in many advances in Physics (cf. [3], [4], [5], [7], [9], [10] and the references therein). Generally, it reads

$$w_t + w_{xxx} + (f(w))_x = D_x^{-1}\Delta_y w, \tag{1}$$

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Department of Mathematics
Universidad Nacional de Colombia
University of Science and Technology of China
e-mail:bjxuan@matematicas.unal.edu.co.

where $(t,x,y)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^{n-1},\ n\geq 2,\ D_x^{-1}h(x,y)=\int_{-\infty}^x h(s,y)ds,\ \Delta_y := \frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+\cdots+\frac{\partial^2}{\partial y_{n-1}^2}$.

In [4] and [5], using the constrained minimization method, De Bouard and Saut obtained the existence and nonexistence of solitary waves in the case where the power nonlinearities are $f(u)=u^p,\ p=k/l$, with k,l relatively prime and l is odd. In the Chapter 7 of [9], Willem extended the results of [4] to the case where $n=2,\ f(u)$ is a continuous function satisfying some structure conditions. In paper [10], we extended the results of [4], [5] and [9] to higher dimensional spaces for a more general nonlinearity $f(u)$ which satisfies some structure conditions.

In this paper, we shall investigate the existence of multiple stationary solutions to generalized Kadomtsev-Petviashvili equation in a bounded domain in $\mathbb{R}^n$, that is,

$$\begin{cases} u_{xxx}+(f(u))_x = D_x^{-1}\Delta_y u, & \text{in } \Omega, \\ D_x^{-1}u|_{\partial\Omega}=0,\ u|_{\partial\Omega}=0, \end{cases} \tag{2}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $f(u)=\lambda|u|^{p-2}u+|u|^{q-2}u$, $1\leq p,q<2_*=\frac{2(2n-1)}{2n-3}$, $\lambda>0$ is a parameter. First, we rewrite (2) in the following form:

$$\begin{cases} -u_{xx}+D_x^{-2}\Delta_y u = f(u), & \text{in } \Omega, \\ D_x^{-1}u|_{\partial\Omega}=0,\ u|_{\partial\Omega}=0, \end{cases} \tag{3}$$

Our methods are based on the variational methods. To do this, we apply the following functional setting:

Definition 1. For $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, on $Y:=\{g_x\,|\,g\in C_0^\infty(\Omega)\}$, we define the inner product:

$$(u,v):=\int_\Omega \left[u_x v_x + D_x^{-1}\nabla_y u\cdot D_x^{-1}\nabla_y v\right]dV, \tag{4}$$

where $\nabla_y=(\dfrac{\partial}{\partial y_1},\cdots,\dfrac{\partial}{\partial y_{n-1}}),\ dV=dxdy$, and the corresponding norm

$$\|u\|:=\Big(\int_\Omega \left[u_x^2+|D_x^{-1}\nabla_y u|^2\right]dV\Big)^{1/2}. \tag{5}$$

A function $u:\Omega\to R$ belongs to X, if there exists $\{u_m\}_{m=1}^{+\infty}\subset Y$ such that:

(a) $u_m\to u$ a.e. on Ω;

(b) $\|u_j - u_k\| \to 0$ as $j, k \to \infty$.

Note that the space X with inner product (4) and norm (5) is a Hilbert space of infinite dimensions. In [10], using the imbedding theorem for anisotropic Sobolev spaces (cf. Theorem 15.6 in [2]), we have shown that the exponent $2_* = \frac{2(2n-1)}{2n-3} > 2$ is as critical as the critical Sobolev exponent $p^* = \frac{np}{n-p}$, that is, there exists a constant $C > 0$ such that the estimate

$$\|u\|_{L^{2_*}(\mathbb{R}^n)} \leq C\Big(\int_{\mathbb{R}^n} \big[u_x^2 + |D_x^{-1}\nabla_y u|^2\big]\, dV\Big)^{1/2} \tag{6}$$

holds for all functions $u \in C_0^\infty(\mathbb{R}^n)$. Furthermore, there is the following continuous and compact imbedding theorem about X (cf. Lemmas 2.2 and 2.3 in [11]):

Lemma 1 (Continuous and compact imbedding)**.** Imbedding $X \hookrightarrow L^r(\Omega)$ is continuous if $1 \leq r \leq 2_*$, and compact if $1 \leq r < 2_*$.

For the convenience, we include next its proof.

Proof.
1. The continuity of the imbedding is a direct consequence of estimate (6) and the boundedness of Ω.

2. Compactness of the imbedding. By the interpolation theorem, it suffices to prove that the imbedding $X \hookrightarrow\hookrightarrow L^2(\Omega)$ is compact.
Suppose $\{u_m\}_{m=1}^\infty \subset X$ be bounded in norm (9). Then without loss of generality, assume that $u_m \rightharpoonup u = 0$ in X, and that there exists $\{g_m\}_{m=1}^\infty \subset L^2(\Omega)$ such that $u_m = \partial_x g_m$. Let $v_m = (v_{m,1},\ v_{m,2},\ \cdots,\ v_{m,n-1}) = \nabla_y g_m \in (L^2(\Omega))^{n-1}$. Denote by $F[u](r,s)$ the Fourier transform of $u(x,y)$.
Let

$$\begin{aligned}
&Q_{-1} = \{(r,s) \in R^n \mid |r| \leq \rho,\ |s_i| \leq \rho^2, i = 1, 2, \cdots, n-1\},\\
&Q_0 = \{(r,s) \in R^n \mid |r| > \rho\},\\
&Q_1 = \{(r,s) \in R^n \mid |r| < \rho,\ |s_1| > \rho^2\}, \cdots,\\
&Q_i = \{(r,s) \in R^n \mid |r| < \rho,\ |s_1| < \rho^2, \cdots,\ |s_{i-1}| < \rho^2,\ |s_i| > \rho^2\}, \cdots,\\
&Q_{n-1} = \{(r,s) \in R^n \mid |r| < \rho,\ |s_1| < \rho^2, \cdots,\ |s_{n-2}| < \rho^2,\ |s_{n-1}| > \rho^2\}.
\end{aligned}$$

Then $R^n = \bigcup_{i=-1}^{n-1} Q_i$ and $Q_i \cap Q_j = \emptyset,\ i \neq j$.

For $\rho > 0$, we have

$$\int_\Omega |u_m|^2\,dV = \int_{R^n} |F[u_m]|^2\,drds = \sum_{i=-1}^{n-1} \int_{Q_i} \big|F[u_m]\big|^2\,drds. \tag{7}$$

It is clear that

$$\int_{Q_0} \big|F[u_m]\big|^2\,drds = \int_{Q_0} \frac{1}{4\pi^2 r^2}\big|F[\partial_x u_m]\big|^2\,drds \le \frac{1}{4\pi^2\rho^2}|\partial_x u_m|_2^2,$$

and for $i = 1, \cdots, n-1$, we have

$$\int_{Q_i} \big|F[u_m]\big|^2\,dxdy = \int_{Q_i} \frac{r^2}{|s_i|^2}\big|F[v_{m,i}]\big|^2\,drds \le \frac{1}{\rho^2}|v_m|_2^2.$$

For any $\varepsilon > 0$, there exists $\rho > 0$ large enough, such that

$$\sum_{i=0}^{n-1} \int_{Q_i} \big|F[u_m]\big|^2\,drds \le \varepsilon/2.$$

From part 1, the continuity of the imbedding, $\{u_m\}$ is bounded in $L^2(\Omega)$, thus up to a subsequence, still denoted by $\{u_m\}$, such that $u_m \rightharpoonup 0$ in $L^2(\Omega)$, therefore

$$F[u_m](r,s) = \int_\Omega u_m(x,y)e^{-2i\pi(xr+y\cdot s)}\,dV \to 0, \text{ as } m \to \infty,$$

and

$$\big|F[u_m](r,s)\big| \le c_0|u_m|_2 \le c_1.$$

Lebesgue dominated convergence theorem implies that

$$\int_{Q_{-1}} \big|F[u_m]\big|^2\,drds \to 0, \text{ as } m \to \infty.$$

Thus we have proved that $u_m \to 0$ in $L^2(\Omega)$. □

From Lemma 1, for $1 \le p,\, q < 2_*$, on X define a functional as

$$E_\lambda(u) := \frac{1}{2}\int_\Omega [u_x^2 + |D_x^{-1}\nabla_y u|^2]\,dV - \frac{\lambda}{p}\int_\Omega |u|^p\,dV - \frac{1}{q}\int_\Omega |u|^q\,dV.$$

Clearly, $E_\lambda \in C^1(X, \mathbb{R})$ and a critical point of E_λ is a weak solution to problem (3) or (2).

In the case where $2 \le p < q < 2_*$, the existence of infinitely many critical points of E_λ is a consequence of Lemma 1 which ensures that E_λ satisfies $(\mathrm{PS})_c$ condition in X and the Symmetric Mountain Pass Lemma due to Ambrosetti and Rabinowitz [1]; but for the case where $1 \le p < 2 < q < 2_*$, the geometric conditions of Mountain Pass Lemma are difficult to verify due to the sublinear term $\lambda|u|^{p-2}u$, so we apply the Index theory of minimax methods, namely the concept of Kransnoselskii genus, to a truncated functional $J \in C^1(X, \mathbb{R})$ instead of E_λ. The ideas to work with a truncated functional were developed by Garcia and Peral [6] for p-Laplacian equation involving critical Sobolev exponents.

In Section 2, we study the case where $2 \leq p < q < 2_*$; while Section 3 is concerned with the case where $1 \leq p < 2 < q < 2_*$.

2. Case $2 \leq p < q < 2_*$

In this section, we study the existence of infinitely many critical points of E_λ on X in the case where $2 \leq p < q < 2_*$.

Lemma 2 ((PS)$_c$ condition)**.** For $2 \leq p < q < 2_*$, $\lambda > 0$, E_λ satisfies (PS)$_c$ condition in X, i.e., any (PS)$_c$ sequence contains a strongly convergent subsequence in X.

Proof.
1. Boundedness of the (PS)$_c$ sequence.
Suppose that $\{u_m\}_{m=1}^\infty \subset X$ is a (PS)$_c$ sequence of E_λ, i.e., $|E_\lambda(u_m)| \leq M$ and $E'_\lambda(u_m) \to 0$ in X' as $m \to +\infty$. Thus, as $m \to +\infty$, we have

(8)
$$\begin{aligned} E_\lambda(u_m) &= \frac{1}{2}\int_\Omega [u_{mx}^2 + |D_x^{-1}\nabla_y u_m|^2]\, dV - \frac{\lambda}{p}\int_\Omega |u_m|^p\, dV - \frac{1}{q}\int_\Omega |u_m|^q\, dV \\ &\leq M + o(1), \end{aligned}$$

and

$$(DE_\lambda(u_m), u_m) = \|u_m\|_X^2 - \lambda\|u_m\|_p^p - \|u_m\|_q^q = o(1)\|u_m\|, \tag{9}$$

where, and in what follows, we use $o(1)$ to denote any quantity that tends to zero as $m \to \infty$.
If $p > 2$, from (8) and (9), a simple calculation shows that, as $m \to +\infty$, we have

$$\begin{aligned} pM + o(1) + o(1)\|u_m\| &\geq pE_\lambda(u_m) - (DE_\lambda(u_m), u_m) \\ &= (\frac{p}{2} - 1)\|u_m\|_X^2 + (1 - \frac{p}{q})\|u_m\|_q^q \\ &\geq (\frac{p}{2} - 1)\|u_m\|^2, \end{aligned} \tag{10}$$

the last inequality follows from the fact that $p < q$, and (10) implies that $\{u_m\}_{m=1}^{\infty}$ is bounded in X.
If $p = 2$, from (8) and (9), it follows

$$\begin{aligned} 2M + o(1) + o(1)\|u_m\| &\geq 2E_\lambda(u_m) - (DE_\lambda(u_m), u_m) \\ &= (1 - \frac{2}{q})\|u_m\|_q^q. \end{aligned} \tag{11}$$

If on the contrary there is a subsequence, still denoted by $\{u_m\}_{m=1}^{\infty}$, such that $\|u_m\| \to \infty$ as $m \to +\infty$, then from (11) there exists $C > 0$ such that

$$\|u_m\|_q^q \leq C\|u_m\|,$$

for all m. In what follows, C denotes the universal positive constant which is independent of m. The Hölder inequality implies that

$$\|u_m\|_2^2 \leq C\|u_m\|^{2/q}.$$

Therefore, the estimate (8) can not hold as $m \to \infty$, which is a contradiction. Thus $\{u_m\}_{m=1}^{\infty}$ is bounded in X in this case.
2. There exists a convergent subsequence of the $(\mathrm{PS})_c$ sequence.
From the boundedness of $(\mathrm{PS})_c$ sequence $\{u_m\}_{m=1}^{\infty}$, up to a subsequence, still denoted by $\{u_m\}_{m=1}^{\infty}$, there exists an $u_0 \in X$ such that, $u_m \rightharpoonup u_0$ weakly in X and $u_m \to u_0$ strongly in $L^r(\Omega)$ for any $r \in [1, 2_*)$ by Lemma 1. Since X is a Hilbert space, $\lim\limits_{m\to+\infty} \left(\|u_m\|^2 - \|u_m - u_0\|^2\right) = \|u_0\|^2$. It suffices to show that $\lim\limits_{m\to+\infty} \|u_m\| = \|u_0\|$.
From the boundedness of the $(\mathrm{PS})_c$ sequence and $E'_\lambda(u_m) \to 0$ in X' as $m \to +\infty$, there hold as $m \to +\infty$

$$(DE_\lambda(u_m), u_m) = \|u_m\|_X^2 - \lambda\|u_m\|_p^p - \|u_m\|_q^q \to 0 \tag{12}$$

and

$$(DE_\lambda(u_m), u_0) \to \|u_0\|_X^2 - \lambda\|u_0\|_p^p - \|u_0\|_q^q = 0. \tag{13}$$

Thus from (12) and (13), as $m \to +\infty$, there holds

$$\|u_m\|_X^2 \to \lambda\|u_0\|_p^p + \|u_0\|_q^q = \|u_0\|_X^2,$$

which implies the strong convergence $u_m \to u_0$ in X. □

Next, we recall the Symmetric Mountain Pass Lemma due to Ambrosetti and Rabinowitz [1] as our Lemma 3.

Lemma 3 (Symmetric Mountain Pass Lemma)**.** Suppose X is an infinite dimensional Banach space, $E \in C^1(X, \mathbb{R})$ satisfies the $(\mathrm{PS})_c$ condition, $E(u) = E(-u)$ for all u, and $E(0) = 0$. Suppose $X = X^- \oplus X^+$, where X^- is finite dimensional, and assume that the following conditions hold:

(1) $\exists\, \alpha > 0,\ \rho > 0:\ \|u\| = \rho,\ u \in X^+ \Longrightarrow E(u) \geq \alpha$;

(2) For any finite dimensional subspace $W \subset X$, there exists $R = R(W) > 0$ such that $E(u) \leq 0$ for all $u \in W,\ \|u\| \geq R$.

Then E possesses an unbounded sequence of critical values.

Theorem 1. For all $\lambda > 0$, if $2 < p < q < 2_*$, then there are infinitely many critical points of E_λ on X, hence infinitely many solutions to problem (3) or (2).

Proof. For $2 < p < q < 2_*$, we verify the assumptions of the Symmetric Mountain Pass Lemma-Lemma 3 with $X^- = \emptyset, X^+ = X$. From Lemma 2, E_λ satisfies $(\mathrm{PS})_c$ condition in X. It suffices to verify the two geometric conditions.
1. $\exists\, \alpha > 0,\ \rho > 0:\ \|u\| = \rho,\ u \in X^+ \Longrightarrow E_\lambda(u) \geq \alpha$.
In fact, from Lemma 1, there exist $c_1, c_2 > 0$ such that

$$\|u\|_p \leq c_1 \|u\|_X,\ \|u\|_q \leq c_2 \|u\|_X. \tag{14}$$

Thus, for any $u \in X$, we have

$$\begin{aligned} E_\lambda(u) &= \frac{1}{2}\|u\|_X^2 - \frac{\lambda}{p}\|u\|_p^p - \frac{1}{q}\|u\|_q^q \\ &\geq \frac{1}{2}\|u\|_X^2 - \frac{\lambda}{p}c_1^p\|u\|_X^p - \frac{c_2^q}{q}\|u\|_X^q \\ &= \|u\|_X^2\big(\frac{1}{2} - \frac{\lambda c_1^p}{p}\|u\|_X^{p-2} - \frac{c_2^q}{q}\|u\|_X^{q-2}\big). \end{aligned} \tag{15}$$

Since $p, q > 2$, choosing $\rho > 0$ so small that $\dfrac{\lambda c_1^p}{p}\rho^{p-2} + \dfrac{c_2^q}{q}\rho^{q-2} \leq \dfrac{1}{4}$, for any $u \in X$ with $\|u\|_X = \rho$, we have

$$E_\lambda(u) \geq \frac{1}{4}\rho^2 := \alpha > 0.$$

2. For any finite dimensional subspace $W \subset X$, there exists $R = R(W) > 0$ such that $E_\lambda(u) \leq 0$ for all $u \in W,\ \|u\| \geq R$.
For any finite dimensional subspace $W \subset X$, all the norms are equivalent. Then

$$c_3 = c_3(W) = \inf\{\|u\|_p;\ u \in W,\ \|u\|_X = 1\} > 0$$

and

$$c_4 = c_4(W) = \inf\{\|u\|_q;\ u \in W,\ \|u\|_X = 1\} > 0.$$

Thus, for $u \in W$, we have

$$E_\lambda(u) \le \frac{1}{2}\|u\|_X^2 - \frac{\lambda c_3^p}{p}\|u\|_X^p - \frac{c_4^q}{q}\|u\|_X^q \to -\infty \tag{16}$$

as $\|u\|_X \to \infty$. From (16), there exists $R = R(W) > 0$ such that $E_\lambda(u) \le 0$ for all $u \in W,\ \|u\| \ge R$. □

Remark 1. From the proof of Theorem 1, if $p = 2 < q < 2_*$, then there exists a $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, there are infinitely many critical points of E_λ in X.

In fact, in Step 1 of the proof, choose $\lambda_0 > 0$ such that $\frac{1}{2} - \frac{\lambda_0 c_1^p}{p} = 0$. Then for any $\lambda \in (0, \lambda_0)$, there exist $\alpha > 0$ and $\rho > 0$ such that $E|_{\|u\|=\rho} \ge \alpha$. The rest of proof is the same as that of Theorem 1.

3. Case $1 \le p < 2 < q < 2_*$

In this section, we study the existence of infinitely many critical points of E_λ on X in the case where $1 \le p < 2 < q < 2_*$. In this case, the geometric condition of the Mountain Pass Lemma doesn't hold due to the sublinear term. Instead of the Mountain Pass Lemma, we apply the Index theory with Kranselskii genus to a truncated functional.

From Lemma 1 and (14), it follows

$$E_\lambda(u) \ge \frac{1}{2}\|u\|_X^2 - \frac{\lambda}{p}c_1^p\|u\|_X^p - \frac{c_2^q}{q}\|u\|_X^q.$$

Let us define function $f : \mathbb{R}^+ \to \mathbb{R}$ as

$$f(x) = \frac{1}{2}x^2 - \frac{\lambda c_1^p}{p}x^p - \frac{c_2^q}{q}x^q.$$

Then for any $u \in X$, $E_\lambda(u) \ge f(\|u\|_X)$. Furthermore, there exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$, $f(x)$ attains its positive maximum at some point $R > 0$, and there are two constants $0 < R_0 < R < R_1$ such that $f(x) < 0$ for $x \in (0, R_0)$ or $x \in (R_0,\ \infty)$, $f(x) > 0$ for $x \in (R_0, R_1)$, and $f(R_0) = f(R_1) = 0$.

For $\lambda \in (0, \lambda_1)$, we make the following truncation of functional E_λ. Let $\tau : \mathbb{R}^+ \to [0, 1]$ be nonincreasing and of C^∞, such that

$$\tau(x) = 1, \quad \text{if } x \le R_0, \quad \tau(x) = 0, \quad \text{if } x \ge R_1,$$

$\varphi(u) = \tau(\|u\|_X)$. Define the truncated functional J as

$$J(u) := \frac{1}{2}\int_\Omega [u_x^2 + |D_x^{-1}\nabla_y u|^2]\, dV - \frac{\lambda}{p}\int_\Omega |u|^p\, dV - \frac{1}{q}\int_\Omega |u|^q \varphi(u)\, dV.$$

Observe that for $\|u\|_X \le R_0$, $J(u) = E_\lambda(u)$ and for $\|u\|_X > R_1$, we have

$$J(u) = \frac{1}{2}\|u\|_X^2 - \frac{\lambda c_1^p}{p}\|u\|_p^p > 0.$$

Furthermore J satisfies the following properties:

Lemma 4. 1). $J \in C^1(X, \mathbb{R})$;

2). If $J(u) \le 0$, then $\|u\|_X \le R_0$, and $J(v) = E_\lambda(v)$ for all v in a small enough neighborhood of u;

3). There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, J satisfies the $(\mathrm{PS})_c$ condition at negative level $c < 0$.

Proof. Conclusions 1) and 2) are immediate. To prove 3), observe that any $(\mathrm{PS})_c$ sequence $\{u_m\}$ of J at negative level $c < 0$ is bounded in X. In fact, from 2), for m large enough, $\|u_m\|_X < R_0$ and $J(u_m) = E_\lambda(u_m)$. Similar to Step 2 in the proof of Lemma 2, there exists a strongly convergent subsequence of $\{u_m\}$ in X. □

From 2) in Lemma 4, if we find some negative critical value of J, then we have a negative critical value of E_λ. Next, we will construct an appropriate minimax sequence of negative critical values of the truncated functional J by the Index theory involving Kransnoselskii genus γ of a nonempty close Z_2-symmetric set.

Lemma 5. For any integer k, there is $\delta = \delta(k) > 0$ such that

$$\gamma(\{u \in X;\, J(u) \le -\delta\}) \ge k.$$

Proof. For any integer k, let V_k be a k-dimensional subspace of X. For any $u \in V_k$ with norm $\|u\|_X = 1$, and $0 < \rho < R_0$, we have

$$J(\rho u) = E_\lambda(\rho u) = \frac{1}{2}\rho^2 - \frac{\lambda}{p}\rho^p\|u\|_p^p - \frac{1}{q}\rho^q\|u\|_q^q.$$

Since V_k is a space of finite dimension, all norms are equivalent. Therefore

$$c_3 = c_3(V_k) = \inf\{\|u\|_p;\, u \in V_k,\ \|u\|_X = 1\} > 0$$

and

$$c_4 = c_4(V_k) = \inf\{\|u\|_q;\, u \in V_k,\ \|u\|_X = 1\} > 0.$$

Thus,

$$J(\rho u) \le \frac{1}{2}\rho^2 - \frac{\lambda c_3^p}{p}\rho^p - \frac{c_4^q}{q}\rho^q.$$

Note that $p < 2 < q$, so we choose $\delta > 0$ (which depends on k) and $\eta < R_0$, such that $J(\eta u) \le -\delta$ for any $u \in V_k$ with $\|u\|_X = 1$. Let $S_k = \{u \in X;\, \|u\|_X = \eta\}$. Then $S_k \cap V_k \subset \{u \in X;\, J(u) \le -\delta\}$, and the monotonicity of Kransnoselskii genus γ implies that

$$\gamma(\{u \in X;\, J(u) \le -\delta\}) \ge \gamma(S_k \cap V_k) = k. \quad \square$$

From Lemmas 4 and 5, and the classical deformation lemma (e.g., Theorem 3.4 in [8]), we have the following Lemma:

Lemma 6. For any integer k, let $\Sigma_k = \{A \subset X \setminus \{0\}, A \text{ is closed}, A = -A,\ \gamma(A) \ge k\}$, and $K_c = \{u \in X,\ J'(u) = 0,\ J(u) = c\}$. If $\lambda \in (0, \lambda_1)$ with λ_1 as defined in Lemma 4, then the value

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} J(u)$$

is finite and critical, and if $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then $\gamma(K_c) \ge l + 1$. In particular, if $l > 0$, then K_c is infinite.

Proof.
1. For any integer k, c_k is negative and finite.
Denote $J_{-\delta} = \{u \in X;\, J(u) \le -\delta\}$. From Lemma 5, for any integer k, there exists $\delta(k) > 0$ such that $\gamma(J_{-\delta}) \ge k$. Since J is continuous and even, $J_{-\delta} \in \Sigma_k$. Thus $c_k \le -\delta(k) < 0$. On the other hand J is bounded from below, hence c_k is finite.
2. c_k is critical.
Assume by contradiction that c_k is regular. Since $c_k < 0$, J satisfies the $(\mathrm{PS})_c$ condition at level $c = c_k < 0$. Let $\varepsilon \in (0, 1)$ and $\Phi : X \times [0, \infty) \to X$ be determined by the classical deformation lemma (e.g., Theorem 3.4 in [8]) with $\bar{\varepsilon} = 1$, $N = \emptyset$. By definition of c_k, there exists $A \in \Sigma_k$ such that

$$\sup_{u \in A} J(u) < c_k + \varepsilon,$$

that is, $A \subset J_{c_k + \varepsilon}$. From the properties of Φ and Kransnoselskii genus γ, it follows $\Phi(A, 1) \in \Sigma_k$ and $\Phi(A, 1) \subset J_{c_k - \varepsilon}$, that is,

$$\sup_{u\in\Phi(A,1)} J(u) < c_k - \varepsilon,$$

which contradicts the definition of c_k.
3. If $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then $\gamma(K_c) \geq l + 1$.
Since $c = c_k = c_{k+1} = \cdots = c_{k+l} < 0$, J satisfies the $(\mathrm{PS})_c$ condition at level $c < 0$, hence K_c is compact and symmetric. Hence $\gamma(K_c)$ is well-defined and there exists a neighborhood N of K_c such that $\gamma(\bar{N}) = \gamma(K_c)$. For $\bar{\varepsilon} = 1$, N and c as above, let ε and $\Phi : X \times [0,\infty) \to X$ be determined by the classical deformation lemma. Since J is even, we may assume that Φ is odd. By definition of c, there exists $\gamma(A) \geq k + l$ such that

$$\sup_{u\in A} J(u) < c_k + \varepsilon,$$

that is, $A \subset J_{c_k+\varepsilon}$. From the properties of Φ and Kransnoselskii genus γ, we have $\tilde{A} = \overline{\Phi(A,1)} \in \Sigma_{k+l}$ and $\tilde{A} \subset \overline{J_{c_k-\varepsilon} \cup N}$.
On the other hand, by definition of $c = c_k$, it follows that $\gamma(\overline{J_{c_k-\varepsilon}}) < k$. Therefore

$$\begin{aligned}\gamma(K_c) &= \gamma(\bar{N}) \geq \gamma(\overline{J_{c_k-\varepsilon} \cup N}) - \gamma(\overline{J_{c_k-\varepsilon}})\\ &> \gamma(\tilde{A}) - k \geq \gamma(A) - k\\ &\geq k + l - k = l,\end{aligned}$$

which implies that $\gamma(K_c) \geq l+1$, and in particular, if $l > 0$, then K_c is infinite. □

From Lemma 6, one immediately gets the following existence of infinitely many critical points of E_λ on X with negative critical values in the case where $1 \leq p < 2 < q < 2_*$:

Theorem 2. Let $1 \leq p < 2 < q < 2_*$. For all $\lambda > 0$, there exists $\lambda_1 > 0$, such that for $\lambda \in (0, \lambda_1)$, there are infinitely many critical points of E_λ on X, hence infinitely many solutions to problem (3) or (2).

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