Boletín de Matemáticas
Nueva Serie, Volumen VIII No.1 (2001), pp. 26–30

A NOTE ON A COMMON FIXED POINT THEOREM OF B. FISHER

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Abstract. The subject of this note is to establish a common fixed point theorem in complete metric spaces which improves a well known result of B. Fisher (see [1]). Our theorem solves also the problem posed in [2].

Key words and phrases: Common fixed points in complete metric spaces.

2000 Mathematics Subject Classification. 47H10, and 54H25.

§ 1 Introduction and statement of the result

The study of common fixed points has started in the year 1936 by the well known result of Markov and Kakutani. Since this year, many works were devoted to Fixed Point Theory. Many authors have studied the existence of fixed and common fixed points and now the literature on the subject is very rich. B. Fisher has proved in his paper [1] the following result:

Theorem 1.1. [B. Fisher]: *Let (M, d) be a complete metric space. let S, T be two self-mappings of M such that*

(i) *S is continuous,*
(ii) *$d(Sx, TSy) \leq \alpha d(x, Sy) + \beta\,[d(x, Sx) + d(Sy, TSy)] + \gamma[d(x, TSy) + d(Sx, Sy)]$, for every $x, y \in M$, where $\alpha, \beta, \gamma \geq 0$ are such that $\alpha + 2\beta + 2\gamma < 1$.*

Then S and T have a unique common fixed point.

In this note, we shall prove that the assumption of continuity made on S, in Theorem 1.1, is superfluous and can be removed. In the paper [2], L. Nova has tried to remove the assumption (i) of continuity on S but she replaced it by another condition. More precisely the main result of [2] was the following:

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Theorem 1.2. [L. Nova]: *Let (M, d) be a complete metric space. Let $a, b \geq 0$ such that $a + 2b < 1$ and let $\alpha, \beta \geq 0$, such that $\beta < 1$. Let S, T be two self-mappings of M such that*

(i) $d(Sx, Sy) \leq ad(x, y) + b\left[d(x, Sx) + d(y, Sy)\right]$, *for all* $x, y \in M$,
(ii) $d(Sx, TSy) \leq \alpha d(x, Sy) + \beta\left[d(x, Sx) + d(Sy, TSy)\right]$, *for all* $x, y \in M$.

Then the following assertions are true:

(i) *there exists a unique point $z \in M$ such that $z = Tz = Sz$.*
(ii) $\lim_{n\to\infty} S^n x = z$, *for all* $x \in M$.
(iii) $\lim_{n\to\infty} TS^n x = z$, *for all* $x \in M$.

The aim of this note is to establish a theorem which refines and completes the result of B. Fisher. In a precise manner, we want to prove:

Theorem 1.3. *Let (M, d) be a complete metric space, let S, T be two self-mappings of M satisfying for all $x, y \in M$,*

$$d(Sx, TSy) \leq \alpha d(x, Sy) + \beta\left[d(x, Sx) + d(Sy, TSy)\right] + \gamma\left[d(x, TSy) + d(Sx, Sy)\right], \tag{F}$$

where $\alpha, \beta, \gamma \geq 0$ are such that $\alpha + 2\beta + 2\gamma < 1$. Then the following assertions are true:

(A) *There exists a unique point $z \in M$ such that $Fix(S) = Fix(\{S, T\}) = \{z\}$.*
(B) *For every $x_0 \in M$ the Picard sequence $\{S^n(x_0)\}$ converges to z.*
(C) *S and TS are continuous at the point z.*
(D) *For each sequence $\{x_n\}$ of elements in M, we have: $\lim_{n\to\infty} x_n = z$ if, and only if, $\lim_{n\to\infty} F_S(x_n) = 0$, where $F_S(x) := d(x, Sx)$ for all $x \in M$.*
(E) *For each sequence $\{y_n\}$ of elements in $Im(S)$ (the range of S), we have $\lim_{n\to\infty} y_n = z$, if, and only if, $\lim_{n\to\infty} F_T(y_n) = 0$.*

Moreover, if $Im(T) \subset Im(S)$ then we have $Fix(S) = Fix(T) = Fix(\{S, T\}) = \{z\}$.

§ 2 Proof of the main result

2.1 (a) Let x_0 be some point in M, and define

$$x_{2n} = Sx_{2n-1}, \quad n = 1, 2, ...$$
$$x_{2n+1} = Tx_{2n}, \quad n = 0, 1, 2, ...$$

We put $t_n := d(x_n, x_{n+1})$ for all integer n. Suppose that $n = 2m$ for some integer m. Then

$$\begin{aligned} t_n = d(x_{2m}, x_{2m+1}) &= d(Sx_{2m-1}, Tx_{2m}) = d(Sx_{2m-1}, TSx_{2m-1}) \\ &\leq \alpha d(x_{2m-1}, x_{2m}) + \beta\left[d(x_{2m-1}, x_{2m}) + d(x_{2m}, x_{2m+1})\right] \\ &+ \gamma\left[d(x_{2m-1}, x_{2m+1}) + d(x_{2m}, x_{2m})\right] \\ &\leq \alpha t_{n-1} + \beta[t_{n-1} + t_n] + \gamma\left[d(x_{2m-1}, x_{2m}) + d(x_{2m}, x_{2m+1})\right] \\ &\leq [\alpha + \beta + \gamma]t_{n-1} + [\beta + \gamma]t_n. \end{aligned}$$

From these inequalities, we deduce that

$$t_n \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right) t_{n-1}, \tag{1}$$

By similar arguments, it is easy to see that the inequality (1) remains valid for odd integers. We set $q := \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$. Then the sequence $\{t_n\}$ verifies $0 \leq t_n \leq qt_{n-1}$ for every positive integer n. Therefore $0 \leq t_n \leq q^n t_0$ for every integer n. Since $0 \leq q < 1$, the sequence $\{t_n\}$ is a strongly Cauchy sequence (i.e., Σt_n converges) and consequently $\{x_n\}$ is a Cauchy sequence. Since (M, d) is complete, this sequence must converge in M. Let z be its limit. Next, we shall prove that z is a common fixed point for S and T.

(b) Suppose that $Sz \neq z$. Then for all positive integer n, we have

$$\begin{aligned} &d(Sz, x_{2n+1}) = d(Sz, Tx_{2n}) = d(Sz, TSx_{2n-1}) \\ &\leq \alpha d(z, x_{2n}) + \beta[d(z, Sz) + d(x_{2n}, x_{2n+1})] + \gamma[d(z, x_{2n+1}) + d(Sz, x_{2n})]. \end{aligned} \tag{2}$$

By taking the limits in both sides of (2), we obtain

$$d(Sz, z) \leq [\beta + \gamma]d(Sz, z) < d(Sz, z),$$

which is a contradiction. Thus z is fixed by S. Let us show that $Tz = z$. By use of the property (F), we have

$$\begin{aligned} d(z, Tz) &= d(Sz, TSz) \\ &\leq \alpha d(z, z) + \beta[d(z, z) + d(z, Tz)] + \gamma[d(z, Tz) + d(Sz, Sz)]. \end{aligned} \tag{3}$$

(3) implies that $(1 - \beta - \gamma)d(z, Tz) = 0$. Since $\beta + \gamma < 1$, we conclude that $d(z, Tz) = 0$ and then $z \in Fix(\{S, T\})$. We deduce also that $Fix(S) \subset Fix(T)$.

(c) Suppose that there exists another point $w \neq z$ fixed by S. Then by using the property (F), we have

$$\begin{aligned} d(w,z)) &= d(Sw,TSz)) \\ &\leq \alpha d(w,z) + \beta[d(w,w) + d(z,z)] + \gamma[d(w,z) + d(w,z)] \\ &\leq [\alpha + 2\gamma]d(w,z). \end{aligned} \tag{4}$$

(4) implies that $(1 - \alpha - 2\gamma)\, d(w,z) = 0$. Since $\alpha + 2\gamma < 1$, we deduce that $w = z$. We conclude that $Fix(S) = Fix(\{S,T\}) = \{z\}$.

2.2 Let x_0 be some point in M. We consider the Picard sequence defined for every integer n, by $x_n := S^n x_0$, where S^n is the $n-$th iterate of S. We shall prove that $\{x_n\}$ converges to z. For each integer n, we set $u_n := d(x_n, z)$. Then by using the property (F), we have

$$\begin{aligned} u_{n+1} &= d(x_{n+1}, z) = d(Sx_n, TSz)) \\ &\leq \alpha d(x_n, z) + \beta d(x_n, x_{n+1}) + \gamma[d(x_n, z) + d(x_{n+1}, z)] \\ &\leq \alpha u_n + \beta[u_n + u_{n+1}] + \gamma[u_n + u_{n+1}]. \end{aligned} \tag{5}$$

(5) implies that $u_{n+1} \leq q u_n$. Therefore, $u_n \leq q^{n+1} u_0$, for all integer n. Since $q \in [0,1[$, we deduce that $\lim_{n\to\infty} u_n = 0$.

2.3 Let $x \in M$. Then by using the property (F) and the triangular property, we have

$$\begin{aligned} d(Sx,z) &= d(Sx,TSz) \\ &\leq \alpha d(x,z) + \beta d(x,Sx) + \gamma[d(x,z) + d(Sx,z)] \\ &\leq [\alpha+\gamma]d(x,z) + \beta[d(x,z) + d(Sx,z)] + \gamma d(Sx,z). \end{aligned} \tag{6}$$

(6) implies that $d(Sx,z) \leq q\, d(x,z)$. Therefore, S is continuous at z. Again, by using the property (F) and the triangular property, for every point x in M, we have

$$\begin{aligned} d(z,TSx) &= d(Sz,TSx) \\ &\leq \alpha d(z,Sx) + \beta d(Sx,TSx) + \gamma[d(z,TSx) + d(z,Sx)] \\ &\leq [\alpha+\gamma]d(Sx,z) + \beta[d(Sx,z) + d(z,TSx)] + \gamma d(z,TSx). \end{aligned} \tag{7}$$

(7) implies that $d(z,TSx) \leq q\, d(Sx,z)$. According to (6), the last inequality yields to $d(z,TSx) \leq q^2\, d(x,z)$. Therefore, TS is continuous at z.

2.4 According to (6), for every $x \in M$, we have

$$d(x,Sx) \leq d(x,z) + d(z,Sx) \leq (1+q)d(x,z).$$

Thus, if $\lim_{n\to\infty} x_n = z$ then $\lim_{n\to\infty} F_S(x_n) = 0$. Conversely, by using (6), for every $x \in M$, we get

$$d(x,z) \le d(x,Sx) + d(Sx,z) \le d(x,Sx) + qd(x,z). \tag{8}$$

From (8), we obtain $d(x,z) \le \frac{1}{1-q} d(x,Sx)$. This completes the proof of (D).

2.5 Let $w = Sx$ be an element of the range $Im(S)$. Then according to the triangular inequality and (9), we have

$$F_T(w) = d(Sx,TSx) \le d(Sx,z) + d(z,TSx) \le (1+q)d(Sx,z) = (1+q)d(w,z). \tag{9}$$

From (9) we obtain the first implication in (E). To prove the converse, let again $w = Sx$ be an element of $Im(S)$. According to (7), we have

$$\begin{aligned} d(w,z) = d(Sx,z) &\le d(Sx,TSx) + d(TSx,z) \\ &\le d(Sx,TSx) + qd(Sx,z) = F_T(w) + qd(w,z). \end{aligned} \tag{10}$$

From (10), we obtain $d(w,z) \le \frac{1}{1-q} F_T(w)$. Thus, for every sequence $\{w_n\}$ of points in $Im(S)$, if $\lim_{n\to\infty} F_T(w_n) = 0$, then we must have $\lim_{n\to\infty} w_n = z$.

2.6 Suppose in addition that $Im(T) \subset Im(S)$. From the subsection (b) of 2.1, we already know that $Fix(S) \subset Fix(T)$. It remains to prove the inverse inclusion. Let $w \in Fix(T)$. Then $w \in Im(S)$ and we can find an $u \in M$, such that $w = Tw = Su$. By using the property (F), we obtain

$$\begin{aligned} d(Sw,w) &= d(Sw,Tw) = d(Sw,TSu) \\ &\le \alpha d(w,w) + \beta[d(w,Sw) + d(w,w)] + \gamma[d(w,w) + d(Sw,w)]. \end{aligned} \tag{11}$$

(11) implies that $[1-\beta-\gamma]d(Sw,w) = 0$, which implies that $Sw = w$. This completes the proof of Theorem 1.3. □

References

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