# Prices versus Exams as Student Allocation Devices.

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**Abstract**

Under certain conditions markets and exams are identically able to generate the optimal allocation of students to schools of different quality. These conditions include the absence of liquidity constraints. But also the share of a common welfare maximizing objective by educational institutions. This paper explores the deviations from optimality implied by the strategic choice of prices and/or exams by competing schools of different quality.

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# 1 Introduction

It is generally acknowledged that liquidity constraints prevent markets from attaining the optimal allocation of students to schools of different quality. Still, since ability to learn is not observable, we need some (self) selection mechanism in order to elicit the right choices by students and/or schools. While markets make students self-select according to willingness and ability to pay, exams are used by schools to choose students according to revealed ability to learn.

Willingness to pay is a good approximation for student ability as long as the marginal return to education of higher quality is larger for higher ability students. In this case, and in absence of liquidity constraints, students make adequate choices regarding their enrolment to existing schools. When liquidity constraints exist, Fernandez and Gali (1997) and Fernandez (1998) show that exams are more efficient allocation devices for a sufficiently powerful exam technology.

Imperfect competition provides these alternative allocation devices with an additional strategic role that has been ignored in the literature. Indeed, previous work has considered atomless schools, unable to change the outcome by modifying its behavior. In this context, the only difference between prices and exams is that while the former allocates students according to their willingness to pay, the latter does according to some willingness to invest in preparing the exam. The relative efficiency of each device obviously depends on the existence of liquidity constraints and the ability for exams to identify real ability (the exam technology).

This paper explores the inefficiencies derived from the strategic choice of prices and/or exams by schools of different quality. While these two instruments generate the same, optimal, distribution of students to schools when the common objective is to maximize global welfare, the fact that schools may pursue their own objectives, or compete, changes the allocations generated by each of them.

The paper is organized as follows: section 2 presents the model and identifies the optimal allocation of students to two schools of different quality; it also shows how markets and exams may be equivalent allocation devices (i.e. yield the same, optimal, allocation of students to schools). Section 3 compares markets and exams as student allocation devices when each school pursues its own objectives and studies the possibility that both instruments be used simultaneously. Section 4 concludes.

## 2 The Model

### 2.1 Students

The utility of a student attending school $i$ depends both on the benefits of education in terms of future earnings and the costs of attending university. Assuming that future wages are the product of school quality and student ability, utility is given by

$$u = aQ_i - p_i \tag{1}$$

where $a$ is ability, with $a \in (0,1)$ and $Q_i$, $p_i$ the quality and price, respectively, of education provided at school $i$. The term ability requires some clarification. We refer here to ability to attain a degree and assume this individual characteristic to involve not only intelligence but, more importantly, willingness to devote time and effort to the task as well as other variables such as the family background. Ability, thus understood, is assumed to be uniformly distributed in the population.

Since the product $aQ_i$ is the wage, or productivity, obtained at school, we can say that individual utility is linear in lifetime earnings.

This complementarity between innate ability and school quality, standard in the literature, is not innocuous. In particular, it identifies innate ability with willingness to pay for education. Complementarity, thus understood, is easier to justify at the higher education level.

In order to decide which school to attend, individuals compare the utility they obtain from attending school H, of high quality with that obtained when attending school L, of low quality. Let $\hat{a}_H$ be the ability of the student who is indifferent among schools (alternatively, $\hat{a}_H$ is the least able individual who chooses to attend school H). From (1):

$$\hat{a}_H\ Q_L - p_L = \hat{a}_H\ Q_H - p_H$$

$$\hat{a}_H = \frac{p_H - p_L}{Q_H - Q_L}$$

It may be the case that the market is not covered. Some students prefer to remain uneducated rather than attend the low quality school. Let $\hat{a}_L$ be the ability of the individual indifferent between attending the low quality school and no school at all (alternatively $\hat{a}_L$ is the least able individual who decides to attend school).

$$\hat{a}_L Q_L - p_L = 0 \Rightarrow \hat{a}_L = \frac{p_L}{Q_L} > 0$$

## 2.2 Educational institutions

There is a high $H$ and a low $L$ quality school. Qualities are, at this stage, taken as given.[1] Schools care for the welfare or expected productivity of own students (prestige). They may also care for revenues. In order to enrol students, schools may drive an exam and fix a minimum grade over which admission is offered and/or price the education they provide. We first compute the social optimum in order to compare the role of exams and prices as alternative allocation devices. Next section studies the allocation of students generated by exams and prices, respectively, when schools care only for their own students and costs or compete with each other. We then also study the circumstances under which a mix of instruments is preferred by schools.

## 2.3 The Social Optimum

The aim of this section is to work out the optimal allocation of students to schools of given $Q_L < Q_H$. This optimal allocation, that maximizes aggregate consumption, is defined by the abilities $a_H^*$ and $a_L^*$ such that all students of ability larger than $a_H^*$ attend the high quality school and all students of ability lower than $a_H^*$ but higher than $a_L^*$ attend the low quality school. Students of ability lower than $a_L^*$ do not attend any school. Note that prices are simply transfers between individuals and thus have no effect on global welfare.

$$a_L, a_H Max \int_{a_L}^{a_H} a\, Q_L da + \int_{a_H}^{1} a\, Q_H\, da - C\left(Q_L, \overline{a}_L\right) - C\left(Q_H, \overline{a}_H\right)$$

where $a_L$ and $a_H$ are the lowest abilities at school $L$ and $H$ respectively. The cost of producing quality $C\left(Q_i, \overline{a}_i\right)$ for $i = L, H$ is assumed increasing and convex in quality. These costs fall as the average ability of enrolled students $\overline{a}_L = (a_L + a_H)/2$ and $\overline{a}_H = (a_H + 1)/2$ increase, although they fall at a decreasing rate ($C$ convex in $\bar{a}$).

The first order conditions for social welfare maximization are:

FOC($a_L$):

$$-\left(a_L^* Q_L + C_{\overline{a}_L}^1 \frac{1}{2}\right) = 0 \tag{2}$$

FOC($a_H$):

$$a_H^*\left(Q_L - Q_H\right) - \left(C_{\overline{a}_L}^1 + C_{\overline{a}_H}^2\right)\frac{1}{2} = 0 \tag{3}$$

where $C_{\overline{a}_i}^i = dC\left(Q_i, \overline{a}_i\right)/d\,\overline{a}_i$, $i = L, H$.[2]

[1] This short-run environment is considered a first approach to the problems tackled.

[2] Second order conditions for maximization are satisfied.

The social optimum is by definition directly attainable by means of exams.

**Proposition 1** *The social optimum may be attained both by exams and prices.*

**Proof.** Let

$$\begin{aligned} \hat{a}_L &= \frac{p_L}{Q_L} \\ \hat{a}_H &= \frac{p_H - p_L}{Q_H - Q_L} \end{aligned}$$

determine, respectively. the minimum ability enrolled at school $L$ and at school $H$ when the allocation of students results from their own choice. At this stage, we keep $Q_H$, $Q_L$ as given and look for prices that maximize

$$\underset{p_L, p_H}{Max} \int_{\hat{a}_L}^{\hat{a}_H} a\, Q_L da + \int_{\hat{a}_H}^{1} a\, Q_H\, da - C\left(Q_L, \overline{a}_L\right) - C\left(Q_H, \overline{a}_H\right)$$

FOC($p_L$):

$$\begin{aligned} &\left(\hat{a}_H \frac{d\hat{a}_H}{dp_L} - \hat{a}_L \frac{d\hat{a}_L}{dp_L}\right) Q_L - \hat{a}_H \frac{d\hat{a}_H}{dp_L} Q_H - C^L_{\overline{a}_L} \frac{1}{2} \frac{d\hat{a}_L}{dp_L} \\ &- C^L_{\overline{a}_L} \frac{1}{2} \frac{d\hat{a}_H}{dp_L} - C^H_{\overline{a}_H} \frac{1}{2} \frac{d\hat{a}_H}{dp_L} = 0 \end{aligned} \tag{4}$$

that can be written

$$\frac{d\,\hat{a}_H}{dp_L} \left(\hat{a}_H (Q_L - Q_H) - \frac{1}{2}\left(C^L_{\overline{a}_L} + C^H_{\overline{a}_H}\right)\right) - \frac{d\hat{a}_L}{dp_L}\left(\hat{a}_L\, Q_L + \frac{1}{2} C^L_{\overline{a}_L}\right) = 0 \tag{5}$$

On the other hand, FOC($p_H$):

$$\frac{d\hat{a}_H}{dp_H}\left(\hat{a}_H\ (Q_L - Q_H) - \frac{1}{2}\left(C^L_{\overline{a}_L} + C^H_{\overline{a}_H}\right)\right) = 0 \tag{6}$$

then, provided that $\frac{d\hat{a}_H}{dp_H} = \frac{1}{Q_H - Q_L} \neq 0$, it must be the case that

$$\hat{a}_H\ (Q_L - Q_H) - \frac{1}{2}\left(C^L_{\overline{a}_L} + C^H_{\overline{a}_H}\right) = 0$$

which is precisely the first order condition for social welfare maximization (3). Moreover, substituting this expression back into (5) it yields

$$-\frac{d\hat{a}_L}{dp_L}\left(\hat{a}_L\, Q_L + \frac{1}{2} C^L_{\overline{a}_L}\right) = 0$$

which, given that $\frac{d\hat{a}_L}{dp_L} = \frac{1}{Q_L} \neq 0$,

$$-\left(\hat{a}_L\, Q_L + \frac{1}{2} C^L_{\overline{a}_L}\right) = 0$$

is precisely the first order condition for welfare maximization (2). ■

## 3 When schools pursue their own aims

We maintain the absence of liquidity constraints assumption and modify the objectives of each university to allow for the more realistic case where each educational institution cares only for the welfare of its own students and their own costs.[3] This amounts to saying that schools compete with each other for students,.instead of agreeing on what is best for all (social optimum). The payoffs of universities $i = L, H$ is hence

$$\begin{aligned} U_L &= \int_{a_L}^{a_H} aQ_L da - C\left(Q_L, a_L\right) \\ U_H &= \int_{a_H}^{1} a\, Q_H\, da - C\left(Q_H, \overline{a}_H\right) \end{aligned}$$

With these payoffs, the allocation of students under prices or exams is no longer equivalent. Moreover, *none of them* is a social optimum. The question we tackle in the following subsections is how student allocations differ under the two alternative institutional settings.

### 3.1 Exams

Suppose first that exams are the only available instrument. Without prices, every student prefers the high quality school. Entrance exams determine the allocation of students to schools. All students with $a \geq a_H^E$ attend the high quality school. Students with $a \in (a_L^E, a_H^E)$ attend the low quality school. Finally, students with $a < a_L^E$ remain uneducated.

How do the limiting grades chosen by the university differ from those that attain the social optimum? Given $Q_H > Q_L$, the high quality university chooses $a_H$ such that

$$a_H Max \int_{a_H}^{1} a\, Q_H\, da - C\left(Q_H, \overline{a}_H\right)$$

FOC($a_H$):

$$-a_H^E Q_H - \frac{1}{2} C_{\overline{a}_H}^H = 0 \tag{7}$$

At the social optimum, from (3):

$$\begin{aligned} a_H^* \left(Q_L - Q_H\right) - \left(C_{\overline{a}_L}^L + C_{\overline{a}_H}^H\right) \frac{1}{2} &= 0 \\ \Leftrightarrow\ -\left(a_H^* Q_H + \frac{1}{2} C_{\overline{a}_H}^H\right) &= -\left(a_H^* Q_L - \frac{1}{2} C_{\overline{a}_L}^L\right) < 0 \end{aligned}$$

[3] Later, we will introduce the concern for monetary benefits.

now, instead, from (7)

$$-\left(a_H^E Q_H + \frac{1}{2} C_{\overline{a}_H}^H\right) = 0 \tag{8}$$

Since

$$\frac{d}{da_H}\left(-a_H Q_H - \frac{1}{2} C_{\overline{a}_H}^H\right) = -Q_H - \frac{1}{4} C_{\overline{a}_H, \overline{a}_H}^H < 0$$

we can conclude that $a_H^* > a_H^E$.

School $H$ is too little selective with respect to the social optimum. It accepts too many applications and incurs too high costs in order to provide enrolled students with the given $Q_H$.

The reason is that, when choosing $a_H^E$ school $H$ only considers the effects of this decision on the future earnings of its own students, failing to recognize the effect its enrolments have on the average ability of students enrolled at school $L$.

For the low quality school the problem is

$$a_L Max \int_{a_L}^{a_H} a\, Q_L da - C\left(Q_L, \overline{a}_L\right)$$

FOC($a_L^E$):

$$-a_L^E Q_L - \frac{1}{2} C_{\overline{a}_L}^L = 0$$

which coincides with (2) and is hence (conditionally) optimal. However, since $a_H^E$ is too low, the average ability of students at school $L$ is also too low [4]. Therefore $C_{\overline{a}_L}^L$ is too high and $a_L^E$ is in fact higher than at the social optimum.

**Proposition 2** *When publicly financed schools care only for their own contribution to social welfare, exams are not optimal allocation devices: the high quality school enrolls too many students, whith a negative effect on the average ability of students in both schools.*

### 3.2 Markets

Consider now the case in which school's only available instrument is prices. Recall that, at this stage, schools care for the global welfare they generate. Prices paid by students

[4] A tax per enrolled student on the high quality school can internalize the externality and achieve the social optimum.

cancel out with prices received by schools. As before, we define:

$$\begin{aligned}\hat{a}_L &= \frac{p_L}{Q_L} \\ \hat{a}_H &= \frac{p_H - p_L}{Q_H - Q_L}\end{aligned}$$

School $H$ chooses $p_H$ that maximizes its contribution to social welfare:

$$p_H Max \int_{a_H}^{1} a\, Q_H\, da - C\left(Q_H, \overline{a}_H\right)$$

FOC($p_H$):

$$-\frac{d\,\hat{a}_H(.)}{dp_H}\left(\hat{a}_H Q_H + \frac{1}{2} C^H_{\overline{a}_H}\right) = 0 \tag{9}$$

with $da_H(.)/dp_H \neq 0$, the first order condition for the maximization of the contribution to welfare coincides with that obtained when the allocation devices were exams (7). Prices are too low with respect to the social optimum. As a result, the average ability of students at the high quality school is, once again too little.

School $L$, on the other hand, chooses $p_L$ such that

$$p_L Max \int_{a_L}^{a_H} a Q_L da - C\left(Q_L, \overline{a}_L\right)$$

FOC($p_L$):

$$\frac{da_H}{dp_L}(a_H Q_L - \frac{1}{2} C^L_{\overline{a}_L}) - \frac{da_L}{dp_L}(a_L Q_L + \frac{1}{2} C^L_{\overline{a}_L}) = 0 \tag{10}$$

At the social optimum, from (3), $-\left(a^*_L Q_L + C^L_{\overline{a}_L}\frac{1}{2}\right) = 0$, now, instead (10) implies

$$-(a_L Q_L + C^L_{\overline{a}_L}\frac{1}{2}) = -\frac{\frac{da_H}{dp_L}}{\frac{da_L}{dp_L}}(a_H Q_L - \frac{1}{2} C^L_{\overline{a}_L}) > 0$$

Therefore, at school $L$, prices are too low.

**Proposition 3** *When publicly financed public schools care only about their own contribution to social welfare, prices are too low.*

### 3.3 Exams as complements to markets

Up to now, we have considered prices as mere instruments, not as part of school objectives. Clearly, prices involve profit opportunities that are being disregarded in the present setting.

Once we allow for profit maximization in the payoff of the universities, is there any reason for them to still use exams as allocative instruments in absence of liquidity constraints? This is the issue we tackle in this section.

Consider first school $H$ and suppose that, given $Q_H$, it chooses first the price, $p_H$ then a limiting admission grade $a_H^E$. Enrolments at this school equal $1 - a_H^E$ provided that all these individuals are willing to pay $p_H$. Otherwise, enrolments equal $1 - \hat{a}_H$. In other words, exams are only effectively selective if they pick students among those willing to attend school $H : a_H^E > \hat{a}_H$. Thus, the number of students at school $H$, $N_H$ is

$$N_H = min\{(1 - a_H^E), (1 - \hat{a}_H)\}$$

The objective of the school is now

$$U_H = \alpha \int_{a_H}^{1} a\, Q_H\, da + \beta\; p_H(1 - a_H) - C\,(Q_H, \overline{a}_H)$$

where $\alpha$ and $\beta$ stand for the relative weights of student welfare and profit maximization respectively in the school payoff function. Given prices, the optimal limiting grade is given by

$$-\left(\alpha a_H^E Q_H + \beta p_H + \frac{1}{2} C_{\overline{a}_H}^H\right) = 0 \tag{11}$$

In turn, the optimal price $p_H$ is given by:

$$-\frac{d\,\hat{a}_H}{dp_H}\left(\alpha \hat{a}_H\, Q_H + \beta p_H + \frac{1}{2} C_{\overline{a}_H}^H\right) + \beta(1 - \hat{a}_H) = 0 \tag{12}$$

In order to see which is larger $\hat{a}_H$ or $a_H^E$ note that, from (12):

$$-\left(\alpha a_H^E Q_H + \beta p_H + \frac{1}{2} C_{\overline{a}_H}^H\right) = -\frac{\beta(1 - a_H)}{da_H/dp_H} < 0$$

and since

$$\frac{d}{da}\left(-\left(\alpha a\, Q_H + \beta p_H + \frac{1}{2} C_{\overline{a}_H \overline{a}_H}^H\right)\right) = -\alpha Q_H - \alpha \frac{1}{4} C_{\overline{a}_H \overline{a}_H}^2 < 0$$

we can conclude that $\hat{a}_H > a_H^E$. As a result, exams are useless.

**Proposition 4** *The high quality school does not complement prices with exams (at least not in absence of liquidity constraints). Whatever $\alpha$ and $\beta$, prices allow for any desired allocation while generating some revenue.*

School $L$ also chooses prices first. Then, if $\hat{a}_L < a_1^E$ an entrance exam takes place at school $L$. The payoff of school $L$ is

$$U_L = \alpha \int_{a_L}^{a_H} a\, Q_L\, da + \beta\, p_L(a_H - a_L) - C\,(Q_L, \overline{a}_L)$$

Given prices, $a_L^E$ is such that

$$-\alpha a_L^E Q_L - \beta p_L - \frac{1}{2} C_{\overline{a}_L}^L = 0 \tag{13}$$

Prices $p_L$, on the other hand, are chosen according to

$$\begin{aligned} &\frac{d\hat{a}_H}{dp_L}(\alpha\, \hat{a}_H\; Q_L + \beta\, p_L - \frac{1}{2} C_{\overline{a}_L}^L) \\ &-\frac{d\hat{a}_L}{dp_L}(\alpha\, \hat{a}_L\; Q_L + \beta\, p_L + \frac{1}{2} C_{\overline{a}_L}^L) + \beta(\hat{a}_H\; - \hat{a}_L\;) = 0 \end{aligned} \tag{14}$$

This is true, if and only if

$$\begin{aligned} &-(\alpha\, \hat{a}_L\; Q_L + \beta\, p_L + \alpha \frac{1}{2} C_{\overline{a}_L}^L) = \\ &+ \underbrace{-\frac{d\hat{a}_H/dp_L}{d\hat{a}_L/dp_L}\left(\alpha\, a_H\; Q_L + \beta\, p_L - \frac{1}{2} C_{\overline{a}_L}^L\right)} - \underbrace{-\frac{\beta(\hat{a}_H - \hat{a}_L\;)}{da_L/dp_L}} \end{aligned} \tag{15}$$

Suppose that $\beta = 0$. Then the optimal price, $p_L$ is given, from (15) by

$$-\left(\alpha\, \hat{a}_L\; Q_L + \alpha \frac{1}{2} C_{\overline{a}_L}^L\right) = -\frac{d\hat{a}_H/dp_L}{d\hat{a}_L/dp_L}\left(\alpha\, \hat{a}_H\; Q_L - \frac{1}{2} C_{\overline{a}_L}^L\right) > 0 \tag{16}$$

Comparing (16) and (13) for $\beta = 0$, we can conclude that $\hat{a}_L < a_L^E$. Therefore exams take place. Prices are set low and the quality of the inputs maintained through selection. In other words, selection allows in this case to lower prices while maintaining standards.

Suppose now that $\alpha = 0$. Then, $dU_L/da_L < 0$ : selection through exams has a negative effect on the payoff. While exams do not take place, $p_L$ is chosen from (15) according to:

$$\frac{d\,\hat{a}_H}{dp_L}(\beta\, p_L - \frac{1}{2} C_{\overline{a}_L}^L) - \frac{d\hat{a}_L}{dp_L}(\beta\, p_L + \frac{1}{2} C_{\overline{a}_L}^L) + \beta(\hat{a}_H\; - \hat{a}_L\;) = 0$$

**Proposition 5** *If the low quality school does not care for student welfare it will not be selective at all.*

Intuitively, the low quality school has a case for selecting students through exams. It corresponds to a situation in which prices are set low in order to compete for students with the high quality school. These low prices also reduce the ability of the least able student. Exams may be used to prevent this.

## 4 Conclusion

This paper has explored the properties of exams and markets as alternative allocation devices when schools of different quality compete for students of different ability. While the social optimum may be attained by means of exams as well as prices, competition distorts the decisions of schools. The high quality school accepts too many students independently of the instrument used. As a consequence, the low quality school is left with a pool of students of lower average ability. Under exams, this lower quality school does not change the optimal admission rule, but as a result of a second order effect, it may in fact be more selective than at the social optimum. Under the market system, however, the low quality school may affect enrolments at the high quality school by its choice of prices. As a result of price competition, too many people attend school as compared to the social optimum.

By introducing a revenue maximization objective in the schools payoff functions we have analyzed the possibility for schools to choose combinations of exams and prices. We have shown that, in absence of liquidity constraints, the high quality school never uses exams. Prices allow any desired allocation while generating some profit. In contrast, the low quality school may choose a mix of instruments. It particularly does so if the weight of student's welfare in the payoff function is relatively high. Then, prices are set low in order to compete for students with the high quality school and selection takes place through exams.

## References

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[2] Fernández, R. and Gali, J.: To each according to...? Markets, tournaments and the matching problem with borrowing constraints. NBER Working Paper 5930 (1997).