Rankings of Income Distributions: A Note on Intermediate Inequality Indices*

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Abstract

The purpose of this paper is to analyze the advantages and disadvantages of several intermediate inequality measures, paying special attention to whether inequality rankings between income distributions are affected by the monetary units in which incomes are expressed. In particular, we show why one of the most popular intermediate inequality approaches, that proposed by Bossert and Pfingsten (1990), leads to measures that do not satisfy the unit-consistency axiom proposed by Zheng (2007).

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1. Introduction

There is a wide consensus in the literature about the properties an inequality measure has to satisfy when using it to compare income distributions having the same mean. Basically, it is necessary to invoke the symmetry axiom—which warranties anonymity—and the Pigou-Dalton principle of transfers—which requires a transfer of income from a richer to a poorer person to decrease inequality.[1] However, if we are interested in comparing two income distributions that have different means, we need to specify the type of mean-invariance property we want our inequality indices to satisfy. This implies the need to introduce another judgment value into the analysis, and there is no agreement among scholars with respect to this matter. Some opt to invoke the scale invariance axiom, so that the inequality of a distribution will be unaffected when all incomes increase (or decrease) by the same proportion. This is the approach followed by the relative inequality indices. Others prefer instead to call on the translation invariance axiom, under which inequality remains unaltered if all incomes are augmented (or diminished) in the same amount, thereby giving rise to the absolute inequality measures. However, as Kolm (1976) pointed out, some people may prefer an intermediate invariance approach between these two extreme views. He labeled such an inequality attitude as "centrist", against the "rightist" and "leftist" labels he used to term the aforementioned relative and absolute notions, respectively.

So far, the intermediate and absolute inequality indices have rarely been applied for ranking income distributions, since these measures are cardinally affected by the currency unit in which incomes are expressed. In a recent paper, Zheng (2007) invokes a new axiom, the unit consistency axiom, requiring that inequality rankings between income distributions remain unchanged when all incomes are multiplied by a (positive) scalar. In this new scenario, not only relative measures, but also absolute and intermediate measures that satisfy the unit consistency axiom, appear as plausible options for empirical research.

The purpose of this paper is to analyze the advantages and disadvantages of several intermediate inequality measures, paying special attention to the unit consistency

[1] Properties such as normalization, continuity, differentiability, and replication invariance are also commonly invoked, but they are of a more technical nature.

axiom. First, we demonstrate why one of the most referenced intermediate indices, that proposed by Bossert and Pfingsten (1990) (B-P hereafter),[2] is not unit-consistent. This analysis allows us to show that the problem lies in the iso-inequality criteria behind that index, which helps us explain why the decomposable intermediate inequality measures à la B-P proposed by Chakravarty and Tyagarupananda (2000) do not satisfy unit consistency either, as shown by Zheng (2007). Second, we explain why the invariance criterion proposed by Del Río and Ruiz Castillo (2000) leads instead to inequality measures that are unaffected by the currency unit. Third, we show that the intermediate measures proposed by Kolm (1976) may also violate unit consistency. Finally, we reflect on the concept of intermediateness behind the above notions.

2. Unit consistency and intermediate inequality measures

In order to ensure independence of the unit of measurement without imposing scale invariance, Zheng (2007) introduces the following property into inequality measures:[3]

Unit consistency. For any two distributions $x,\ y \in \Re^n_{++}$ and any inequality measure I ($I : \bigcup_{n \geq 2} \Re^n_{++} \to \Re_+$), if $I(x) < I(y)$, then $I(\theta x) < I(\theta y)$ for any $\theta \in \Re_{++}$.

Certainly, any relative inequality measure satisfies the above property since they are defined as those where $I(\theta x) = I(x)$ for any $\theta \in \Re_{++}$. However, we should keep in mind that there are other unit-consistent indices, apart from the scale invariant ones. In this vein, as shown by Zheng (2007), the variance and the "fair compromise" measure proposed by Krtscha (1994) are absolute and intermediate indices, respectively, that satisfy this property. In what follows, we analyze several intermediate inequality equivalence criteria by distinguishing between linear and non-linear invariances.

2.1. Linear invariance criteria

[2] See Besley and Preston (1988), Ebert and Moyes (2000), and Lambert (1993), among others.
[3] Zoli (2003) also proposes an analogous property named "weak currency independence".

The μ-inequality concept proposed by B-P is the intermediate inequality measure most frequently mentioned in the literature. According to this invariance criterion, an intermediate inequality index should satisfy the following condition for a given $\mu \in [0,\ 1]$:

$$I_{\mu}[x] = I_{\mu}\left[x + \tau(\mu x + (1-\mu)1^{n})\right]$$

for any $n \geq 2, x \in \Re^{n}_{++}$, and $\tau \in \Re$, such that $x + \tau(\mu x + (1-\mu)1^{n}) \in \Re^{n}_{++}$, where $1^{n} \equiv (1,......,1)$. As shown in Figure 1, for a given income distribution $x \in \Re^{2}_{++}$, the distributions which are μ-inequality equivalent to it are those located on the line defined by point x and vector $\mu x + (1-\mu)1^{n}$ (which represents an intermediate attitude between the relative ray given by x and total equity given by 1^{n}).[4] In particular, $I_{\mu}(x) = I_{\mu}(y)$.

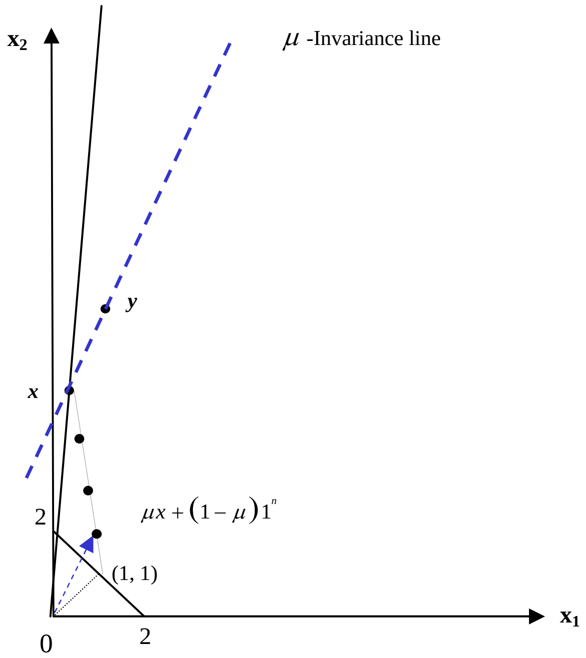

Figure 1. Invariance in B-P ($n = 2$, μ=0.25)

By using a numerical example in a five-dimensional space, Zheng (2007) showed that the decomposable intermediate inequality measures à la B-P characterized by

[4] Note that the invariance line drawn in Figure 1 should actually finish at $x_1 = 0$, but we have enlarged it in order to make the picture clearer. The same considerations apply to Figure 2.

Chakravarty and Tyagarupananda (2000) are not unit-consistent. By using the same distributions, we can prove that B-P's index also violates the above axiom.

In fact, if $\mu = 0.5$, $x = (1,2,3,4,5)$ and $y = (0.1,0.1,0.2,0.6,0.6)$, then $I_{0.5}(x) = 0.068 > 0.015 = I_{0.5}(y)$, but $I_{0.5}(10x) = 0.122 < 0.148 = I_{0.5}(10y)$, where $I_{\mu}(x) = (1+s)\left[1 - \prod_{i=1}^{n}\left(\frac{x_i + s}{\overline{x} + s}\right)^{1/n}\right]$, $s \equiv \frac{1-\mu}{\mu}$ and $\overline{x}$ represents the mean of distribution x.

In Figure 2, we illustrate why this popular intermediate inequality equivalence criterion leads to measures that do not satisfy the unit consistency axiom.

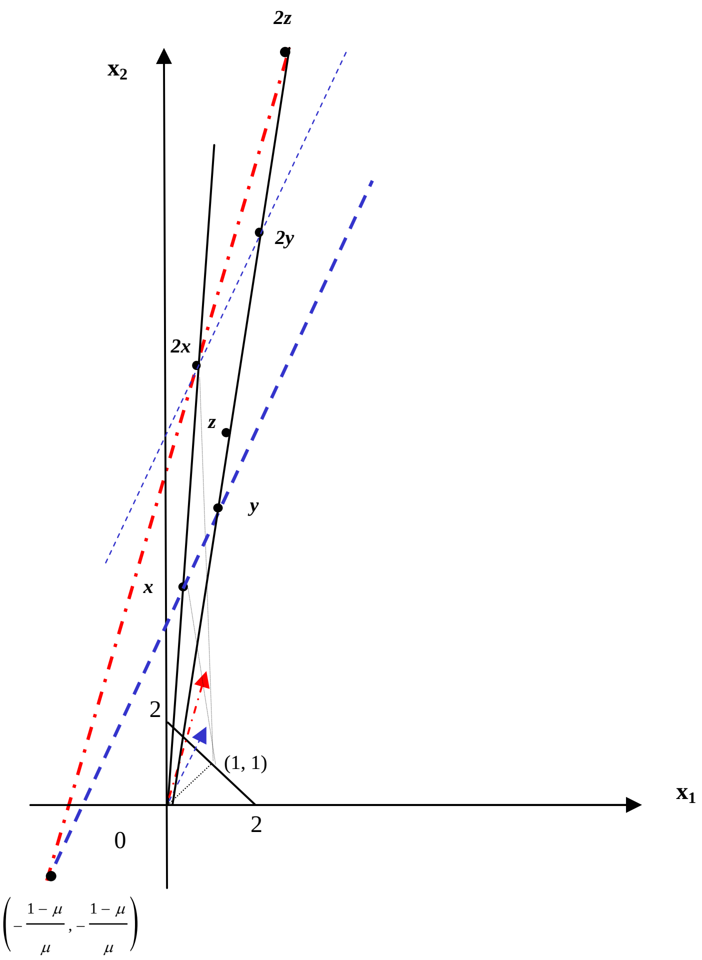

Figure 2. Unit consistency in B-P ($n = 2$, $\mu = 0.25$).

Thicker dash lines represent the two μ-invariance lines passing through points x and $2x$, that is, the set of distributions equivalent to x and $2x$, respectively. Vector

y represents an income distribution that is equivalent to x, since it is located on the invariance line of the latter. It is easy to see that any distribution between y and z has a larger inequality level than x because of the Pigou-Dalton transfer axiom. However, distributions resulting from doubling their individual incomes (which are located between $2y$ and $2z$) would have instead a lower inequality level than distribution $2x$. Therefore, changes in the currency unit do affect rankings between income distributions.

The above graphical analysis permits us to illustrate that the aforementioned five-dimensional example was not an isolated one. We have shown that, even in a two-dimension space, for any given income distribution it is possible to find an interval of distributions that violate the axiom when comparing them with the former distribution. The explanation of this behavior relies on the notion of inequality equivalence proposed by B-P. The slope of the inequality invariance line given by direction $\mu x + (1-\mu)1^n$ does depend on the total income of distribution x. In fact, keeping the relative inequality as constant, the larger the total income, the larger this slope (the slope of the invariance line corresponding to $2x$ is larger than that of x, as shown in Figure 2).[5]

This means, first, that μ may represent a different intermediate inequality attitude depending on the distribution in which the index is evaluated. Since the invariance lines are, therefore, not parallel, it is impossible to state that μ-inequality rankings are not affected by changes in the scale when comparing any two distributions. Thus, we have shown that the heart of this equivalence criterion is incompatible with the unit consistency axiom, so that any measure based on this notion violates this axiom.

Second, the μ-inequality concept approaches the "rightist" view of inequality (the invariance line becomes closer to the relative ray) when aggregate income rises (see Figure 2).[6] This means that results obtained by using this intermediate concept can be

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[5] Since the two invariance lines cut in the third quadrant (at point $\left(-\frac{1-\mu}{\mu}, -\frac{1-\mu}{\mu}\right)$, as shown by B-P), it is possible to construct inequality indices based on this invariance notion. If the two lines cut in the first quadrant, the index would not be well-defined.

[6] This tendency to the relative ray was initially pointed by Seidl and Pfingsten (1997) and Del Río and Ruiz-Castillo (2000).

quite close to those obtained with relative measures, which can be seen as unsuitable for a "centrist" measure. Moreover, in Figure 3, which shows the $\mu-$iso-inequality contours corresponding to distribution $x=(20,80)$ for two μ values, we see that the invariance line corresponding to $\mu=0.5$ is roughly indistinguishable from the ray passing by x (which defines the iso-inequality line of relative measures). In fact, to obtain an iso-inequality contour closer to the "leftist" view (i.e. closer to the absolute ray), it would be necessary to choose a parameter value extraordinary low (for example $\mu=0.005$).[7] This suggests that parameter μ, even though it takes a value between 0 and 1, has not a clear economic interpretation, since its value does not give us any idea of the invariance line location. In particular, in the above example, $\mu=0.5$ does not represent an equidistant position between the relative and absolute rays, but is instead a position close to the "rightist" view.

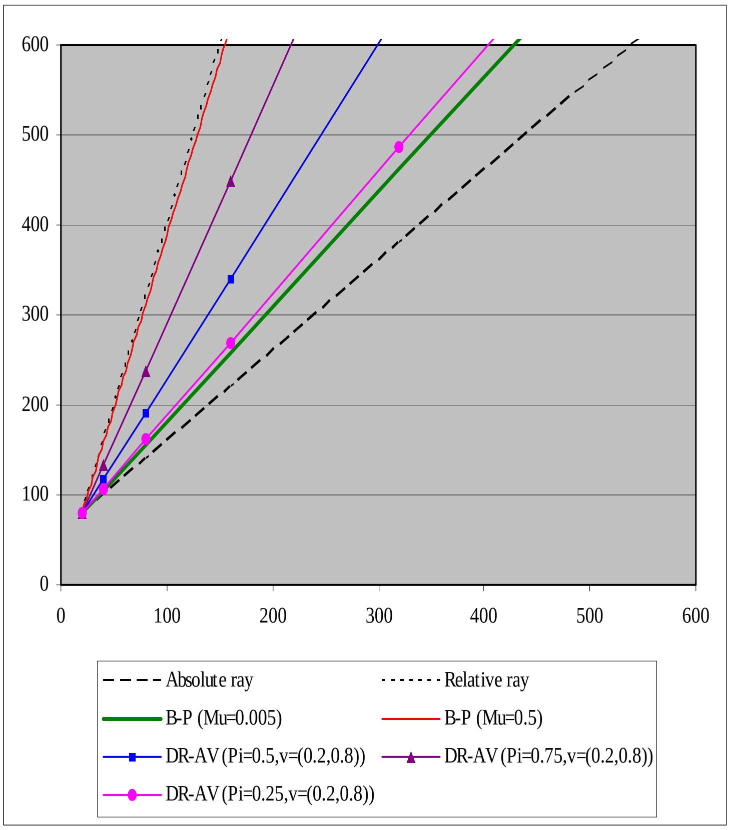

Figure 3. Iso-inequality contours corresponding to distribution $(20,80)$: Linear cases

[7] This explains why Atkinson and Bradolini (2004) find similar empirical results either by using B-P's index or relative indices, even when considering extremely low $\mu-$values $(\mu = 0.0027 \Leftrightarrow \xi = 365 \text{ dollars})$.

If one is interested in defining a linear centrist measure as a convex combination between a relative and an absolute ray, one could fix not only parameter μ, but also the reference distribution that gives rise to the "rightist" and "leftist" views. In this regard, Del Río and Ruiz-Castillo (2000) (DR-RC hereafter) propose the (v, π)-inequality, where v is a vector belonging to the n-dimensional simplex, and $\pi \in [0, 1]$. The first component fixes the distribution of reference, while the second refers to the convex combination between the relative and absolute rays associated to v.[8] Once these two components are fixed, we can calculate the n-dimensional simplex vector $\alpha = \pi v + (1-\pi)\left(\frac{1}{n}\right)1^n$, which defines the direction of the inequality equivalence ray, and the set of income distributions $\Gamma^{'}(\alpha)$ for which α represents an intermediate attitude. This set can be expressed as follows: $\Gamma^{'}(\alpha) = \left\{ x \in D:\ \pi_x v_x + (1-\pi_x)\frac{1^n}{n} = \alpha,\ \text{for some } \pi_x \in [0, 1] \right\}$, where D is the set of all possible ordered income distributions, and v_x represents the vector of income shares associated to x (it therefore belongs to the n-dimensional simplex). This means that vector $\alpha \in D$ can only be used for income distributions that are weak Lorenz-dominated by it.

In this vein, an intermediate inequality index is (v, π)-invariant in the set of income distributions $\Gamma^{'}(\alpha)$ if for any $x \in \Gamma^{'}(\alpha)$ the following expression holds:

$$I_{(v,\pi)}(x) = I_{(v,\pi)}(y),\ \text{for any } y \in P_{(v,\pi)}(x),$$

where $P_{(v,\pi)}(x) = \left\{ y \in D:\ y = x + \tau\left(\pi v + (1-\pi)\frac{1^n}{n}\right),\ \tau \in \Re \right\}$ represents the inequality invariance line. Note that this line is obtained as the convex combination, given by π, between the "leftist" and "rightist" views associated with vector v. In other words, once v and π are given, they cannot be changed: the same intermediate notion must be used when comparing any two income distributions, since otherwise we would be using different inequality measures.

[8] We have changed their original notation to make it clearer. In particular, we have switched vector x by simplex vector v, since only the income shares of the distribution of reference are required to obtain the invariance ray.

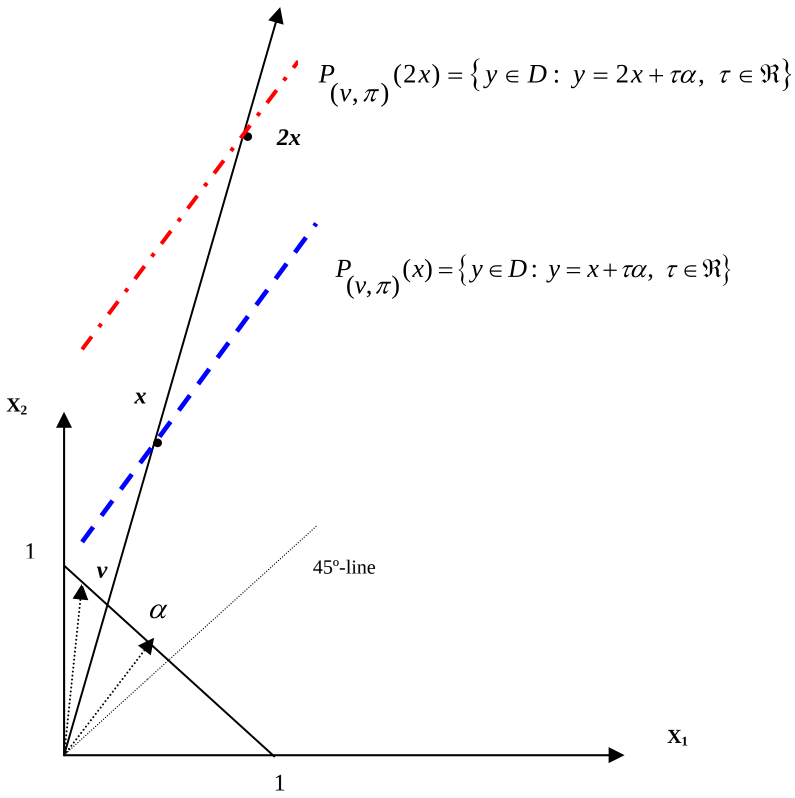

Figure 4. Invariance in DR-RC ($n=2,\ \pi=0.25$).

From the above it follows that the invariance lines passing through distributions x and $2x$ are now parallel (see Figure 4). Therefore, the (v, π)-invariance notion does not have the problem shown in Figure 2, which can help explain why the family of indices based on this approach, proposed by Del Río and Alonso-Villar (2007) (DR-AV hereafter), does satisfy the unit consistency axiom. The invariance lines corresponding to three of these indices are shown in Figure 3, where $\pi\in\{0.25,0.5,0.75\}$ and $v=(0.2,0.8)$. We can see that $\pi=0.5$ leads to an iso-inequality contour which is "equidistant" from the "rightist" and "leftist" views of distribution $(20,80)$, when choosing the vector of reference, v, as that given by the income shares of that distribution.

The distribution of reference, v, does play a very important role in this approach. So, in comparing distributions x and y (which can be assumed to have a higher mean without loss of generality) we could define vector v as the income shares of x (as in Figure 3) and choose the parameter π reflecting our inequality-invariance value judgments. By using this benchmark, we could determine whether y has a lower inequality than the distribution we would have reached if π% of the income gap had been distributed according to income shares in x and $(1-\pi)$% in equal amounts among the individuals. Note that, in doing so, the same vector of reference has to be used both for calculating the invariance line passing through distribution x and y.[9] Therefore, when studying the evolution of an economy over time, this approach allows the possibility of taking into account the starting point.

2.2. Non-linear invariance criteria

An alternative to the above intermediate notions is to assume that the iso-inequality contours are not straight lines. In this regard, Krtscha (1994) proposes an adaptive intermediate notion that gives rise to parabolas. According to his "fair compromise" notion, to keep inequality unaltered, any extra income should be allocated among individuals in the following way. The first extra dollar of income should be distributed so that 50 cents goes to the individuals in proportion to the initial income shares, and 50 cents goes in equal absolute amounts. The second extra dollar should be allocated in the same manner, starting now from the distribution reached after the first dollar allocation, and so on. This index [and the generalizations proposed by Zheng (2007)] does satisfy unit consistency, as shown by the latter. This "centrist" attitude is rather challenging since it approaches the absolute view rather soon when income increases, which makes it difficult for inequality to decrease when analyzing an economy over time. In Figure 5, we can see that, according to Krstcha's index, inequality would remain unaltered with respect to distribution $(20,80)$ if the poorer reached an income of 400 and the richer of 590, which would imply income shares of 40% and 60%, respectively. This proximity to the absolute view does not contradict, however, the tendency of this index to a relative inequality measure when income increases to infinity, while keeping inequality

[9] Both Zoli (2003) and Zheng (2004) misunderstand DR-RC's approach, since in their interpretation v does depend on distribution x. This confusion leads to the former to conclude that DR-RC invariance notion does not satisfy the path independence axiom, when it really does (see Del Río and Alonso-Villar, 2007). It also leads to the latter to conclude that DR-RC's proposal tends to an absolute notion when a given transformation that keeps inequality unaltered is performed repeatedly, while it actually tends to a relative notion (as can be easily shown). We should emphasize, however, that this tendency to a relative notion when income increases infinitely does not contradict the intermediateness of this notion, since DR-RC's approach allows to choose the vector of reference that better fits in each scenario.

constant, as shown by Zheng (2004). He shows that when moving repeatedly along an iso-inequality contour, the curve becomes eventually a straight line passing through the origin, so that the intermediate notion becomes relative. This does not mean, however, that the iso-inequality contour is close to the "rightist" view. A relative ray can be as close as wanted to the line representing total equity. In fact, if we continued our previous simulation and plotted the invariance curve for larger income levels (which are not shown in Figure 5), we would see that distribution $(81920, 84367)$, representing income shares of 49.3% and 50.7%, respectively, would have the same inequality level than distribution $(20,80)$, which seems a quite challenging attitude.

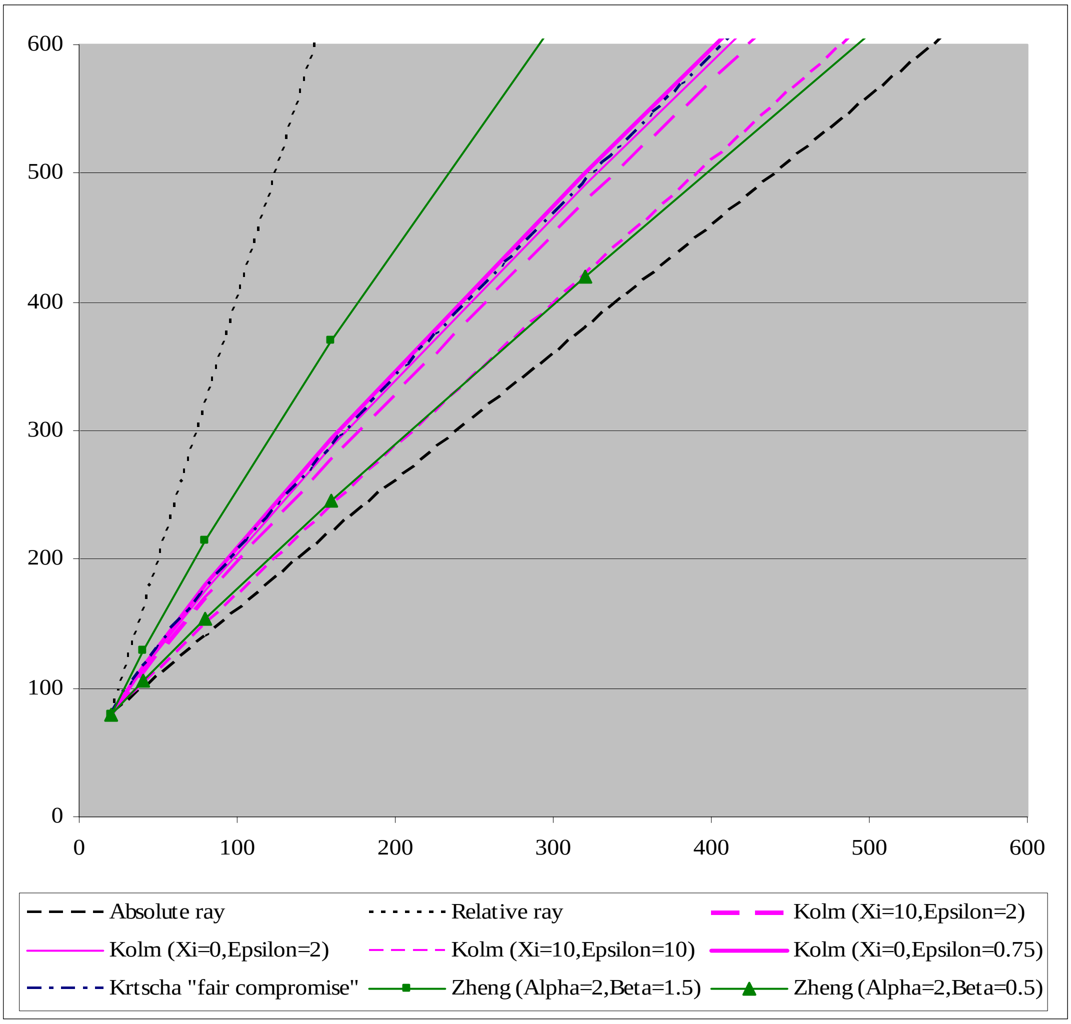

Figure 5. Iso-inequality contours corresponding to distribution $(20,80)$: Non-linear cases[10]

[10] Kolm's family of indices has iso-inequality contours that monotonically approach the absolute ray as either ξ or ε increases (if $\varepsilon > 1$). However, when $\varepsilon \in [0,1]$, there is no monotonicity with respect to

Kolm's (1976) "centrist" measures also lead to iso-inequality contours that are not straight lines.[11] As opposed to Zheng's family of indices, Kolm does not cover the whole intermediate space since, as shown in Figure 5, "centrist" attitudes are close to the "leftist" view, while those near the "rightist" view are not permitted for any parameter value.[12] On the other hand, Kolm's "centrist" measures may violate the unit consistency axiom when $\xi \neq 0$ (if $\xi = 0$ the index is homogeneous of degree 1 and, therefore, it does satisfy the axiom). In this regard, if $\xi = 10$ and $\varepsilon = 10$, for distributions $x = (2,2,6,7,7)$ and $y = (2,2,3,8,8)$, it follows that $I_{(10,10)}(x) = 1.63 < 1.66 = I_{(10,10)}(y)$ while $I_{(10,10)}(2\,x) = 4.13 > 3.94 = I_{(10,10)}(2y)$, where $I_{(\xi,\varepsilon)}(x) = \overline{x} + \xi - \left[\frac{1}{n}\left(\sum_{i=1}^{n}\left(x_i + \xi\right)^{1-\varepsilon}\right)\right]^{\frac{1}{1-\varepsilon}}$. Therefore, (ξ,ε)-inequality rankings may be affected by currency units.

3. Final remarks

The unit consistency axiom, recently invoked by Zheng (2007), guarantees that inequality rankings between income distributions remain unaffected by the unit in which incomes are expressed. This axiom does not impose such strong value judgments on inequality measurement as the scale invariance condition and, therefore, intermediate indices satisfying it appear to be plausible options for empirical research.

Intermediate measures are quite useful when comparing two income distributions, x and y (which can be assumed to have a higher mean), where the latter has at the same time a higher absolute inequality level and a lower relative inequality level than the former according to the relative and the absolute Lorenz criterion, respectively.

We have revised the centrist measures offered by the literature in order to check whether they are unit-consistent. We have shown that both the class of intermediate

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this parameter. In this example, the contour closer to the relative ray is that corresponding to $\xi = 0, \varepsilon = 0.75$.

[11] As in previous cases, Zheng (2004) proves that these curves become straight lines in the limit.

[12] Recent empirical evidence obtained by Atkinson and Brandolin (2004, p. 13) seems to support this idea: "Kolm's centrist measure basically confirms the pattern shown by Kolm's absolute measure".

inequality indices proposed by Bossert-Pfingsten (1990) and those of Kolm (1976) are affected by the currency unit. Therefore, only the “fair compromise” index proposed by Krtscha (1994), the generalizations proposed by Zheng (2007), and the indices proposed by Del Río and Alonso-Villar (2007)—which, as opposed to the others, are ray invariant—are intermediate inequality measures satisfying unit consistency. One advantage of the first two indices is that they also are decomposable, which can be very helpful for some types of empirical analysis. One advantage of the latter is that it brings a clear economic interpretation of intermediateness while emphasizing the relevance of using a distribution of reference when making income inequality comparisons.

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