Interactions inequality-polarization: an impossibility result*

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ABSTRACT

Recent literature stresses the multidimensional nature of income distribution. Two of the most relevant components are inequality and polarization. In this paper, we prove the impossibility of keeping simultaneously constant these two aspects whenever the distribution of incomes changes. Distributional change could originate from any economic policy or simply from economic growth. Hence, our result implies an effective restriction for policymakers that they cannot avoid and should not ignore. Our proof embodies a general view of polarization that includes the Wolfson and the Esteban and Ray approaches. The paper also develops other links for the case of controlling only one variable and deducing the implications for the other variable.

Key Words: polarization, inequality, economic policy.

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1. INTRODUCTION

Recent literature stresses the multidimensional nature of income distribution. Inequality, polarization, poverty, deprivation, inequality of opportunities and horizontal inequity of a tax system are some relevant aspects of income distribution considered in the literature. For decades, inequality and the principle of transfers have been the widespread summary concepts upon which the distributional effects of changes in the economic environment have been evaluated. Policy makers and researchers have usually justified their analyses of policy changes on the basis of general Lorenz-based inequality criteria (see among others, Pigou [17], Dalton [3], Kolm [15], Atkinson [2] and Shorrocks [20]).

Nowadays, many economists rely on an alternative summary concept of polarization that relates to the principle of alienation and identification between polar subgroups (Esteban and Ray [11]). It is argued that polarization is a more appropriate criterion for explaining social conflict (see for instance, Wolfson [21, 22], Esteban and Ray [11, 12], Gradín [14], D'Ambrosio [4], Montalvo [16], and Duclos *et al.* [9]). In some of these papers, the differences and the similarities of both concepts are also explored.

Under these circumstances, researchers can be tempted to introduce one or both alternative concepts as restrictions in their economic analysis. For example, the analyst may want to compute changes in polarization and inequality due to a tax reform (for instance see Hemming and Keen [13] and Dardanoni and Lambert [5]) or to control for one or both of these two variables in a simulation exercise (see Davies and Hoy [6] and Prieto *et al.* [18]).

The objective of this paper is to analyze in depth the relationship between inequality and polarization under general changes in the distribution of income (for example, due to economic policy, growth or some other environmental shocks) and to go a step beyond the existing literature.

The most striking result in this paper is the impossibility of controlling simultaneously for both polarization and inequality, if an analyst or a Government is tempted to do so. The paper develops other links for the case of controlling only one variable and deducing the implications for the other variable.

The paper proves these results for a general view of bipolarization that embodies a generalization of the Wolfson (see Rodríguez and Salas [19], henceforth RS) and Esteban and Ray approaches (henceforth ER). This general study considers not only the median income as the group separator, but also allows for mean income as separator, as proposed in Esteban *et al.* [10]. The paper provides rich, unrestricted characterizations. In other cases, the results are restricted to conditions where an upper bound for the median change, in the median-based case (or the mean quantile change, in the mean-based case), is established.

The paper has the following structure. The second section deals with the theoretical framework. The third section illustrates the inequality-neutral case. Section 4 deals with the polarization-neutral case. Section 5 proves the impossibility result and section 6 concludes.

2. THEORETICAL FRAMEWORK

A recent interest on income polarization as a different concept from inequality has emerged.[1] Whereas inequality relates to the overall dispersion of the distribution, polarization concentrates on income distribution in several focal or polar modes. According to Wolfson [21], a bipolarized income distribution is one that is spread out from the median income value, so there are fewer individuals or families with middle level incomes. In a different way, Esteban and Ray [11] adopt an identification-alienation framework. Identification relates to the notion of a within-groups feeling of identity. Meanwhile, alienation relates to the income distance between people in different groups. Besides, Esteban *et al.* [10] suggest dividing the population by the mean value, instead of the median value used by Wolfson [21]. The use of this alternative measure depends on the fact that the mean value is the income level that minimizes the average difference of income pairs within both groups and, therefore, the dispersion within each group as measured by the Gini coefficient (see Aghevli and Merhan [1] and Davies and Shorrocks [7]).

Both approaches are linked by making use of the *extended Wolfson bipolarization measure* proposed by Rodríguez and Salas [19]. Let $x \in \Re_{++}{}^{N}$ be an N-person income vector, $\bar{h}$ the upper-bound value of this distribution and $F(x)$ the income distribution function that belongs to the class of all income distribution functions, $\varphi(x)$. Then, the extended Wolfson bipolarization measure is expressed as:

[1] See for instance, Wolfson [21, 22], Esteban and Ray [11], Esteban, Gradín, and Ray [10], Gradín [14] and Rodríguez and Salas [19].

$$P_z^{RS}(F;v) = \frac{2\mu}{z}\left[G_z^B(F;v) - G_z^W(F;v)\right] \qquad (1)$$

for an inequality aversion parameter $v \in [2, 3]$ and for $z \in \{\mu, m\}$, where $\mu \in \left(0, \bar{h}\right]$ is the mean income value of the distribution F(x), $m \in \left(0, \bar{h}\right]$ is the median value, $G_z^B(F;v)$ is the between-groups component of the extended Gini coefficient (the alienation term), and $G_z^W(F;v)$ is the within-groups of the extended Gini coefficient (the identification term), computed for groups separated by the *z* value.[2,3] Under our mutually exclusive exhaustible partition, Rodríguez and Salas [19] have proved that $G(F;v) = G_z^B(F;v) + G_z^W(F;v)$, a result that is used below.

As *z* can take the mean and median income values, we take into consideration not only the identification-alienation framework, but also the recommendation by Esteban *et al.* [10] for the case of bipolarization. In the following lemma the ER bipolarization measure is simplified.

LEMMA 1: For the bipolarization case, the ER polarization index becomes:

$$P_h^{ER}(F;\alpha) = \left[q_h^{\alpha} + (1-q_h)^{\alpha}\right]G_h^B(F;2) \qquad (2)$$

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[2] In Rodriguez and Salas [19] it is proved that, given a particular income distribution, the extended Wolfson bi-polarization measure, $P_z^{RS}(F,v)$, is consistent with the *second polarization curve* if $v \in [2, 3]$. The second polarization curve plays a similar role in the context of bipolarization to that played by the Lorenz curve in the context of inequality (see Wolfson [21]). A bipolarization index is consistent with the second polarization curve if a progressive median-preserving transfer within (between) polar subgroups never reduces (increases) polarization. Note that the original Wolfson [21] bipolarization index is the particular $z = m$ and $v = 2$ case for expression (1).

[3] The extended Gini coefficient is defined by Donaldson and Weymark [8] and Yitzhaki [23] as:

$$G(F,v) = 1 - v(v-1)\int_0^1 (1-q)^{v-2} L(q)\,dq \qquad v > 1.$$

where $h \in \left(0,\bar{h}\right]$, q_h is the population quantile at the h-income value,

$$q_h = \int_0^h f(x)dx \tag{3}$$

whose complementary is $1-q_h$, and α is the identification sensitivity parameter.

Proof: The ER polarization index [11] is defined as:

$$P^{ER}(F;\alpha) = \sum_i \sum_j q_i^{1+\alpha} q_j \left|\mu_i - \mu_j\right| \tag{4}$$

where q_i and μ_i are, respectively, the population quintile and the mean income value of the income group i when there are k income groups. Therefore, whether we consider two income groups

$$P_h^{ER}(F;\alpha) = \left[q_h^{1+\alpha}(1-q_h) + (1-q_h)^{1+\alpha} q_h\right](\mu_2 - \mu_1) \tag{5}$$

the mean income values are

$$\mu_1 = \frac{L(q_h)}{q_h} \quad \text{and} \quad \mu_2 = \frac{1-L(q_h)}{1-q_h}. \tag{6}$$

Substituting (6) into (5), we obtain expression (2).

Note that one may apply the ER approach for more than two income groups and adopt any general h value. In this paper, we focus only on the bipolarization framework to make it comparable with the alternative model by Wolfson [21]. We use these two approaches to highlight the relationship between bipolarization and inequality to obtain our results (see below). First, we present some definitions.

DEFINITION 1: *A between-groups progressive/null/regressive transfer*

Given any two N-person income vectors $x, x' \in \Re_{++}^{N}$, x' is obtained from x by a between-groups progressive transfer if and only if $x' - x = \varepsilon(e_i - e_j)$ for some scalar $\varepsilon > 0$ and some $i < j$, where e_i denotes the N-tuple (0,…,0,1,0,…,0) whose only non-zero element occurs in the i-th position and $x_j > z > x'_i > x_i$, where z is the value separating the groups under consideration. If $z \geq x_j > x'_i > x_i$ or $x_j > x'_i > x_i \geq z$, then x is obtained from x by a between-groups null transfer. Conversely, x' is obtained from x by a between-groups regressive transfer if and only if x is obtained from x' by a between-groups progressive transfer.

DEFINITION 2: *A between-groups null net transfer*

For any two discrete distributions x, $x' \in \Re_{++}^{N}$, one obtains a distribution x' from x by a null (zero) net transfer from the income group strictly above the z value to the income group strictly below the z value (**NNT$_z$**). There z is the value separating the groups under consideration, if and only if it is obtained by a set of progressive/null/regressive between-groups transfers such that $\mu_z^+(x') = \mu_z^+(x)$, where $\mu_z^+(x)$ is the mean income of the truncated distribution above the z value,

$$\mu_z^+ = \int_z^{\bar{h}} x f(x) dx. \tag{7}$$

DEFINITION 3: *A between-groups regressive net transfer*

A distribution, *x*', is obtained from *x* by a *regressive (negative) net transfer* from the income group strictly above the *z* value to the income group strictly below the *z* value (**RNT$_z$**), if and only if it is obtained by a set of progressive/null/regressive between-groups transfers such that $\mu_z^+(x') > \mu_z^+(x)$.

DEFINITION 4: *A between-groups progressive net transfer*

A distribution, *x*', is obtained from *x* by a *progressive (positive) net transfer* from the income group strictly above the *z* value to the income group strictly below the *z* value (**PNT$_z$**), if and only if it is obtained by a set of progressive/null/regressive between-groups transfers such that $\mu_z^+(x') < \mu_z^+(x)$.

Now, we prove the following lemma, which is required to demonstrate the Proposition 1.

LEMMA 2:

Given any two distribution functions $F(x)$ and $F(x')$, we establish the following statements:

$$\begin{aligned} &\mathbf{NNT_z} \Leftrightarrow dL(q_z) = 0 \\ &\mathbf{RNT_z} \Leftrightarrow dL(q_z) < 0 \\ &\mathbf{PNT_z} \Leftrightarrow dL(q_z) > 0, \end{aligned} \tag{8}$$

where q_z is the population quantile at the z-income value, and $L(q_z)$ is the value of the Lorenz curve evaluated at q_z.

Proof: The Lorenz curve evaluated at q_z, $L(q_z)$, is the following transformation of the distribution function $F(x)$,

$$L(q_z) = \frac{1}{\mu}\int_0^{q_z} x dF(x) . \tag{9}$$

That is, $L(q_z)$ is the relative income accumulation up to the quantile q_z. Therefore, Lemma 1 is verified once we make use of the expression (9) and the Definitions 2, 3 and 4.

3. THE INEQUALITY-NEUTRAL CASE

We are going to show that we characterize polarization by the set of net transfers between groups (separated by the mean or the median values). First, we establish the link between the polarization and the between-groups extended Gini coefficient changes. We show that we obtain a simpler relationship for the mean-based bipolarization measurement.

PROPOSITION 1A (Prieto *et al.* [18]):

Under a $G(F;v)$-neutral change in the distribution $F(x)$, that is, for any change from distribution x to distribution x' where $x, x' \in X$ and $X = \{x \in \Re_{++}^{N} \mid G(F;v) = G_v\}$:

$$Sign(dP_m^{RS}(F;v)) = Sign(dG_m^B(F;v)) \Leftrightarrow \left|\frac{d(\mu/m)}{(\mu/m)}\right| \leq \left|\frac{d(G_m^B(F;v) - G_m^W(F;v))}{G_m^B(F;v) - G_m^W(F;v)}\right| \tag{10}$$

This means that Sign($dP_m^{RS}(F;v)$) = Sign($dG_m^B(F;v)$) for a sufficiently low $d(\mu/m)$, under an inequality-neutral distributional change. This distributional change could be due to an economic policy. For instance, in Prieto *et al*. [19] the right-hand side condition in equation (10) is proved to be weak in empirical terms under a linear tax reform. This proposition may be simplified using the analogue mean-based polarization measure, separated by the mean income value, μ.

PROPOSITION 1B:

Under a $G(F;v)$-neutral change in the distribution $F(x)$, that is, for any change from distribution x to distribution x' where $x, x' \in$ X and X = $\{x \in \Re_{++}^N \mid G(F;v) = G_v\}$:

$$Sign(dP_\mu^{RS}(F;v)) = Sign(dG_\mu^B(F;v)) = -Sign(dG_\mu^W(F;v)) \tag{11}$$

without any condition.

Proof: From expression (10) we can derive the first equality:

$$Sign(dP_\mu^{RS}(F;v)) = Sign(dG_\mu^B(F;v)) \Leftrightarrow 0 \leq \left|\frac{d(G_\mu^B(F;v) - G_\mu^W(F;v))}{G_\mu^B(F;v) - G_\mu^W(F;v)}\right|, \tag{12}$$

which is always verified. The second equality of expression (11) comes from the fact that $dG(F;v) = 0$. Note that, $P^{RS}{}_{\mu}(F;v)$ adopts the simple expression of $P^{RS}{}_{\mu}(F;v) = 2[G_{\mu}{}^{B}(F;v) - G_{\mu}{}^{W}(F;v)]$ while $G(F;v) = G_{\mu}{}^{B}(F;v) + G_{\mu}{}^{W}(F;v)$ is kept constant.

As a result, polarization change fully characterizes the between-groups inequality component. Note that Proposition 1B is markedly general in two ways. First, it does not require any condition, unlike Proposition 1A. Second, like Proposition1A, it does not only apply for a mean-constant economy, but also extends to a mean-variable economy. This property allows us to apply this result to analysis not only of a usual, mean-neutral fiscal policy, but also of a growing economy.

Analogous results for the ER polarization model when two income groups are considered follow, first, with respect to the median value and, second, considering the mean value.

PROPOSITION 2A:

Under a $G(F;2)$-neutral change in the distribution $F(x)$, that is, for any change from distribution x to distribution x' where $x, x' \in X$ and $X = \{x \in \Re_{++}{}^{N} \mid G(F;v) = G_v\}$ for $h = m$ in expression (2):

$$Sign(dP_m^{ER}(F;\alpha)) = Sign(dG_m^B(F;2))\,. \tag{13}$$

The proof of this result is straightforward from expression (2).

PROPOSITION 2B:

Under a $G(F;2)$-neutral change in the distribution $F(x)$, that is, for any change from distribution x to distribution x' where $x, x' \in X$ and $X = \{x \in \Re_{++}{}^N \mid G(F;v) = G_v\}$ for $h = \mu$ in expression (2):

$$Sign(dP_{\mu}^{ER}(F;\alpha)) = Sign(dG_{\mu}^{B}(F;2)) \Leftrightarrow \left|\frac{dT}{T}\right| \leq \left|\frac{dG_{\mu}^{B}(F;2)}{G_{\mu}^{B}(F;2)}\right| \tag{14}$$

where $T = q_{\mu}^{\alpha} + (1 - q_{\mu})^{\alpha}$.

Now we characterize the net transfers in terms of the polarization changes for a low enough change in q_{μ}. We establish the first two propositions for the Wolfson [21] polarization index. Meanwhile, Propositions 4A and 4B relate to the ER measure.

PROPOSITION 3A (Prieto *et al.* [18]):

Under a $G(F;v)$-neutral change in the distribution F, that is, for any change from distribution x to x to distribution x' where $x, x' \in X$ and $X = \{x \in \Re_{++}{}^N \mid G(F;v) = G_v\}$:

$$Sign(dP_{m}^{RS}(F;v)) = Sign(-dL(q_m)) \Leftrightarrow \left|\frac{d(\mu/m)}{(\mu/m)}\right| \leq \left|\frac{d(G_{m}^{B}(F;v) - G_{m}^{W}(F;v))}{G_{m}^{B}(F;v) - G_{m}^{W}(F;v)}\right| \tag{15}$$

PROPOSITION 3B:

Under a $G(F;v)$-neutral change in the distribution F that generates any possible combination of dq_{μ} and $dL(q_{\mu})$,

$$Sign(dP_{\mu}^{RS}(F;v)) = Sign(-dL(q_{\mu})) \Leftrightarrow \left|dq_{\mu}\right| \leq \left|dL(q_{\mu})\right| \qquad (16)$$

Proof: It is proved that $G_z^B(F;v)$ is equal to $q_z - L(q_z)$ for $z = m, \mu$ (see Rodríguez and Salas [19, pp. 74]). Given the result of Proposition 1B, then

$$Sign(dP_{\mu}^{RS}(F;v)) = Sign(dG_{\mu}^{B}(F;v)) = Sign(dq_{\mu} - dL(q_{\mu})) \qquad (17)$$

It implies that Sign($dP^{RS}_{\mu}(F;v)$) = Sign(dq_{μ} -$dL(q_{\mu})$) = Sign(-$dL(q_{\mu})$) $\Leftrightarrow | dq_{\mu} | \leq | dL(q_{\mu}) |$, for any dq_{μ} and $dL(q_{\mu})$.

For sufficiently relative low changes of m in the median-based case (or q_{μ} in the mean-based case),[4] these propositions mean that if polarization increases that will imply a regressive net transfer (**RNT**), once we make use of expression (8), and *vice versa*. The same applies to a polarization decrease with respect to progressive net transfers (**PNT**).

The last two results are replicated for the ER polarization measure, keeping the distinction between the median and the mean cases.

PROPOSITION 4A:

Under a $G(F;v)$-neutral change in the distribution F,

$$Sign(dP_m^{ER}(F;\alpha)) = Sign(-dL(0.5)) \quad (18)$$

Proof: This result comes from Proposition 2A and the fact that $G_m{}^B(F;v) = 0.5 - L(0.5)$ (see proof in Proposition 3B).

Combining Proposition 4A with Lemma 1, a straightforward link between the Esteban and Ray [11] polarization index—defined for the median value and two income groups—and the net transfers is established.

PROPOSITION 4B:

Under a $G(F;v)$-neutral change in the distribution F that generates any possible combination of dq_μ and $dL(q_\mu)$,

$$Sign(dP_\mu^{ER}(F;\alpha)) = Sign(-dL(q_\mu)) \Leftrightarrow \left|\frac{dT}{T}\right| \leq \left|\frac{dG_\mu^B(F;2)}{G_\mu^B(F;2)}\right| \quad (19)$$

where $T = q_\mu^\alpha + (1-q_\mu)^\alpha$.

Proof: This result can be obtained from Proposition 2B and the fact that $G_\mu{}^B(F;v) = q_\mu - L(q_\mu)$ (see Rodríguez and Salas [19, pp. 74]).

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[4] In Prieto *et al.* [19] these conditions to be weak in an empirical linear tax reform exercise. Besides, notice that the condition on dq_μ is always satisfied by any ranking-preserving policy.

Summarizing the results in this section, if polarization is measured using the ER index defined for the median value there is an unambiguous relationship between polarization changes and net transfers (and the between-groups Gini index). Moreover, under certain conditions, we can link changes in polarization with the set of net transfers (and the between-groups Gini index) when using the ER index defined for the mean value and the Wolfson [21] polarization measure. Furthermore, these conditions are well established.

Finally, all the propositions apply not only for a mean-constant economy, but also for a mean-variable economy. This property allows us to extend these results from the usual mean-neutral fiscal policy to a growing economy.

4. THE POLARIZATION-NEUTRAL CASE

In this case, we may characterize inequality symmetrically by the set of net transfers between groups (separated by the mean or the median value). In this exercise, we consider polarization to be the key variable to be kept constant. We also consider the two cases, the median- and the mean-based bipolarization measures.

PROPOSITION 5A:

Under a $P_m^{RS}(F;v)$-neutral change in the distribution F (for instance, due to a public policy), that is, for any change from distribution x to distribution x' where $x, x' \in X$ and $X = \{x \in \Re_{++}^N \mid P_z(F;v) = P_v\}$):

$$Sign(dG(F;v)) = Sign(dG_m^B(F;v)) \Leftrightarrow \left|\frac{dm}{m}\right| \leq \left|\frac{2dG_m^B(F;v)}{G_m^B(F;v)-G_m^W(F;v)}\right|. \quad (20)$$

Proof: Differentiating equation (1) and taking into account $dP_z(F;v) = 0$, for $z = m$, we obtain:

$$\frac{d(G_m^B(F;v)-G_m^W(F;v))}{G_m^B(F;v)-G_m^W(F;v)} = \frac{dm}{m}. \quad (21)$$

Then, given $dG(F;v) = dG_m{}^B(F;v) + dG_m{}^W(F;v)$, we can write

$$dG(F;v) = 2\, dG_m{}^B(F;v) - [G_m{}^B F;v) - G_m{}^W(F;v)]dm/m. \quad (22)$$

Then, the result straightforwardly follows.

This proposition states that Sign($dG(F;v)$) = Sign($dG_m{}^B(F;v)$) for a sufficiently low dm, under a polarization-neutral distributional change. This proposition may be simplified for the mean-based polarization measure, separated by the mean income value, μ, as stated in the following proposition:

PROPOSITION 5B:

Under a $P_\mu{}^{RS}(F;v)$-neutral change in the distribution F (for instance, due to a public policy):

$$Sign(dG(F;v)) = Sign(dG_\mu^B(F;v)) = Sign(dG_\mu^W(F;v)) \quad (23)$$

without any condition.

Proof: For $z = \mu$, expression (1) becomes

$$P_{\mu}^{RS}(F;v) = 2\left[G_{\mu}^{B}(F;v) - G_{\mu}^{W}(F;v)\right]. \tag{24}$$

If polarization is maintained fixed, $dP_{\mu}^{RS}(F;v) = 0$, then $dG_{\mu}^{B}(F;v) = dG_{\mu}^{W}(F;v)$. Moreover, it is verified that $dG(F;v) = dG_{\mu}^{B}(F;v) + dG_{\mu}^{W}(F;v)$, for our disjoint exhaustible partition. Therefore, the result is obtained, $Sign(dG(F;v)) = Sign(dG_{\mu}^{B}(F;v)) = Sign(dG_{\mu}^{W}(F;v))$.

This proposition applies not only for the usual mean-constant public reforms, but also for public reforms that achieve a non-zero mean income change, or for a growing economy. Now we characterize the net transfers in terms of the inequality change.

PROPOSITION 6A:

Under a $P_m^{RS}(F;v)$-neutral change in the distribution F (for instance, due to a public policy):

$$Sign(dG(F;v)) = Sign(-dL(q_m)) \Leftrightarrow \left|\frac{dm}{m}\right| \leq \left|\frac{2dG_m^B(F;v)}{G_m^B(F;v) - G_m^W(F;v)}\right| \tag{25}$$

Proof: It follows from the fact that $G_z^B(F;v) = q_z - L(q_z)$, for $z \in \{m,\mu\}$ (see Rodríguez and Salas, 2003). Therefore, $dG_z^B(F;v) = dq_z - dL(q_z)$. Then, the result comes straightforwardly from the fact that $dq_m = 0$ (note that $q_m = 0.5$). Again, for a sufficiently low dm the Gini-based inequality change characterizes the resulting net transfers.

PROPOSITION 6B:

Under a $P_\mu^{RS}(F;v)$-neutral change in the distribution F (for instance, due to a public policy) that generates any possible combination of dq_μ and $dL(q_\mu)$,

$$Sign(dG(F;v)) = Sign(-dL(q_\mu)) \Leftrightarrow \quad |dq_\mu| \leq |dL(q_\mu)| . \tag{26}$$

Proof: Proposition 5B ensures that Sign($dG(F;v)$) = Sign($dG_\mu^B(F;v)$). Provided that $dG_\mu^B(F;v) = dq_\mu - dL(q_\mu)$, mentioned in Proposition 6A, the result is straightforwardly obtained.

We derive analogous propositions for the ER polarization index.

PROPOSITION 7A:

Under a $P_m^{ER}(F;\alpha)$-neutral change in the distribution F:

$$Sign(dG(F;2)) = Sign(dG_m^W(F;2)) . \tag{27}$$

Proof: Differentiating equation (2), for $h = m$, we obtain $dP_m^{ER}(F;\alpha) = \left[q_m^\alpha + (1-q_m)^\alpha\right] dG_m^B(F;2)$ since $dq_m = 0$. If we fix $dP_m^{ER}(F;\alpha) = 0$, then $dG_m^B(F;2) = 0$, since the term in brackets can never be zero. Therefore, $Sign(dG(F;2)) = Sign(dG_m^W(F;2))$ is verified for our disjoint and exhaustible partition.

Moreover, this proposition establishes a clear-cut link with the net transfers occurring. If we fix $dP_m^{ER}(F;\alpha) = 0$, we have $dG_m^B(F;2) = 0$. Given the known result $G_m^B(F;v) = q_m - L(q_m)$, together with $dq_m = 0$, we obtain $dL(q_m) = 0$, which is equivalent by lemma 1 with no net transfers (NNT_m). As a conclusion, a policy controlling for $P_m^{ER}(F;\alpha)$ is equivalent to a policy constrained to zero net transfers between-groups.

PROPOSITION 7B:

Under a $P_\mu^{ER}(F;\alpha)$-neutral change in the distribution F:

$$Sign(dG(F;2)) = Sign(dG_\mu^W(F;2)) \Leftrightarrow \left|\frac{dT}{T}\right| \leq \left|\frac{dG_\mu^W(F;2)}{G_\mu^B(F;2)}\right| \tag{28}$$

where $T = q_\mu^\alpha + (1-q_\mu)^\alpha$.

Proof: Differentiating equation (2), for $h = \mu$, we obtain $dP_\mu^{ER}(F;\alpha) = T\, dG_\mu^B(F;2) + G_\mu^B(F;2)dT$.

If we fix $dP_\mu^{ER}(F;\alpha) = 0$, then $\frac{-dT}{T} = \frac{dG_\mu^B(F;2)}{G_\mu^B(F;2)}$. For our disjoint and exhaustible partition,

$Sign(dG(F;2)) = Sign(dG_\mu^B(F;2)) + Sign(dG_\mu^W(F;2))$, hence $Sign(dG(F;2)) = Sign(dG_\mu^W(F;2))$

if and only if $\left|dG_\mu^W(F;2)\right| \geq \left|dG_\mu^B(F;2)\right| = \frac{G_\mu^B(F;2)}{T}\left|dT\right|$.

Summarizing the results of this section, we provide the conditions that link changes in Gini-based inequality, the between-groups inequality index and the set of net transfers for both cases, and the median- and mean-based bipolarization indices. For the particular median-based ER bipolarization index, the conditions are neat. In this case, controlling for polarization is equivalent to controlling for the net transfers.

5. THE IMPOSSIBILITY RESULT

We are interested in controlling for the two variables, inequality and polarization, simultaneously. In this case the following lemmas and corollaries emerge that are used in the impossibility results below.

LEMMA 3: If we consider that inequality and RS polarization changes should be of the same magnitude, for $z \in \{m,\mu\}$:

$$dP_z^{RS}(F;v) = dG(F;v) \Rightarrow \quad dG_z^B(F;v) = \left(\frac{2\mu + z}{2\mu - z}\right) dG_z^W(F;v) - \left(\frac{P_z^{RS}(F;v)z^2}{\mu(2\mu - 1)}\right) d\left(\frac{\mu}{z}\right). \qquad (29)$$

Proof: The result is obtained by differentiating equation (1) and taking into account that $dG(F;v) = dG_z^B(F;v) + dG_z^W(F;v)$.

COROLLARY 1: If $z = \mu$ we obtain from above the following degenerate case:

$$dP_{\mu}^{RS}(F;v) = dG(F;v) \Rightarrow dG_{\mu}^{B}(F;v) = 3dG_{\mu}^{W}(F;v) \tag{30}$$

Alternatively,

$$dG_{\mu}^{B}(F;v) \neq 3dG_{\mu}^{W}(F;v) \Rightarrow dP_{\mu}^{RS}(F;v) \neq dG(F;v) \tag{31}$$

It is impossible to implement a policy that intends to change both variables, polarization and inequality, measured by these indices, by the same magnitude unless we are in the unlikely case of $dG_{\mu}^{B}(F;v) = 3dG_{\mu}^{W}(F;v)$.

LEMMA 4: If we consider that inequality and ER polarization changes should be of the same magnitude, for $z \in \{m,\mu\}$:

$$dP_{z}^{ER}(F;\alpha) = dG(F;2) \Rightarrow \quad dG_{z}^{B}(F;2) = \frac{1}{T-1}\left[dG_{z}^{W}(F;2) - P_{z}^{ER}(F,\alpha)\frac{dT}{T}\right]. \tag{32}$$

Proof: The result is obtained by differentiating equation (2) and taking into account that $dG(F;v) = dG_{z}^{B}(F;v) + dG_{z}^{W}(F;v)$.

COROLLARY 2: If $z = m$ we obtain from above the following degenerate case:

$$dP_m^{ER}(F;\alpha) = dG(F;2) \Rightarrow dG_m^B(F;2) = \lambda dG_m^W(F;2) \tag{33}$$

where $\lambda = \dfrac{1}{2(0.5)^{\alpha} - 1}$.

Again, it is impossible to implement a policy that intends to change both variables, polarization and inequality, measured by these indices, by the same magnitude unless we are in the unlikely case of $dG_m^B(F;2) = \lambda dG_m^W(F;2)$. [5]

Finally, we present the impossibility results.

PROPOSITION 8: *An impossibility result (RS polarization case)*

Under a $P_\mu^{RS}(F;v)$ and G(F;v)-neutral changes in the distribution F (for instance, due to a public policy), that is, for any change from distribution x to distribution x' where x, $x' \in$ X and X = $\{x \in \Re_{++}^N \mid G(F;v) = G_v$ and $P_\mu^{RS}(F;v) = P_v\}$), for $z = \mu$:

$$dP_\mu^{RS}(F;v) = dG(F;v) = 0 \Rightarrow dG_\mu^B(F;v) = dG_\mu^W(F;v) = 0 \tag{34}$$

Proof: If $dP_{\mu}^{RS}(F;v) = 0$ then $2\left(dG_{\mu}^{B}(F;v) - dG_{\mu}^{W}(F;v)\right) = 0$, that is, $dG_{\mu}^{B}(F;v) = dG_{\mu}^{W}(F;v)$. However, we know from corollary 1 that $dG_{\mu}^{B}(F;v) = 3dG_{\mu}^{W}(F;v)$. Therefore, both results are compatible if and only if $dG_{\mu}^{B}(F;v) = dG_{\mu}^{W}(F;v) = 0$.

PROPOSITION 9: *An impossibility result (ER polarization case)*

Under a $P_m^{ER}(F;\alpha)$ and G(F;2)-neutral changes in the distribution F, that is, for any change from distribution x to distribution x' where $x, x' \in$ X and X = $\{x \in \Re_{++}^{N} \mid G(F;2) = G$ and $P_m^{ER}(F;\alpha) = P\}$), for $z = m$:

$$dP_m^{ER}(F;\alpha) = dG(F;2) = 0 \Rightarrow dG_m^{B}(F;2) = dG_m^{W}(F;2) = 0 \qquad (35)$$

Proof: If $dP_m^{ER}(F;\alpha) = 0$ and $dG(F;2) = 0$, by corollary 2, $dG_m^{B}(F;2) = \lambda dG_m^{W}(F;2)$. Notice that for disjoint and exhaustible partitions $dG_m^{B}(F;2) = -dG_m^{W}(F;2)$. Then $\lambda = -1$, which requires $2(0.5)^{\alpha} = 0$, what is impossible. Therefore, both results are compatible if and only if $dG_m^{B}(F;2) = dG_m^{W}(F;2) = 0$.

It is impossible to implement a policy that intends to keep constant both variables, polarization and inequality, measured by these indices, unless we are in the unlikely (degenerate) case where

[5] The reader will notice that these degenerate cases (corollaries 1 and 2) also apply for the $z = m$ median-based RS bipolarization and for the $z = \mu$ mean-based ER bipolarization versions, respectively, though the expressions are not as simple as the ones deduced above.

the change in the between-groups Gini index is exactly equal to the change in the within-groups Gini index in either the RS and the ER cases.

Note that proposition 8 relates to the mean-based polarization measure for the RS case, while proposition 9 relates to the median-based polarization for the ER case. However, the impossibility result is more general, as it also applies for the inverted cases, that is, for the median-based RS and for the mean-based ER cases, although the expressions are not as simple as the ones deduced above.[6]

6. CONCLUDING REMARKS

Following the current interest in polarization, we make use of a compromised measure (the extended Wolfson bipolarization index) and the general Esteban and Ray polarization measure to analyze the relationship between two of the most relevant income distribution variables in the literature: inequality and polarization.

In this paper, we prove the impossibility of controlling, at the same time, for inequality and polarization. This result implies an effective restriction on economic policy that is inescapable.

[6] For the median-based ($z = m$) RS polarization measure, it will be impossible to implement a policy which intends to keep polarization and inequality constant at the same time, unless

$$dG_m^B(F;v) = -dG_m^W(F;v) = -\frac{1}{4}\left(\frac{m}{\mu}\right)^2 P_m^{RS}(F;v)\,d\left(\frac{\mu}{m}\right)$$

And for the mean-based ($z = \mu$) ER polarization measure, unless

Furthermore, we show some of the relationships between these two concepts and analyze the economic policy implications of fixing one of these variables separately. In particular, we analyze the implications of extreme inequality- and polarization-neutral distributional changes.

Under the inequality-neutral case, polarization change is fully characterized by the between-groups inequality alienation component in the mean-separating value context. It does not only apply for a mean-preserving economy, but may also be extended to a mean-variable economy. This property allows us to apply this result not only to analyze a usual mean-neutral fiscal policy, but also to a growing economy. In the median-separating context, this link is also true for certain conditions. Besides, we characterize the set of net transfers in terms of polarization change under certain conditions, unless for the ER median-separating case (whose characterization is unconditional).

Under the polarization-neutral case, we provide the conditions that link changes in the Gini-based inequality with the set of net transfers for both cases, the median- and mean-based bipolarization and the RS and ER polarization frameworks. Tables 1A and 1B summarize all characterizations and conditions found. Over the 14 possible results, 5 characterizations are unrestricted. For the other 9 conditional cases, we observe that the conditions always refer to the requirement of a sufficiently low change in the median value, in the median-base measures, and/or of a sufficiently low change in the mean quantile, in the mean-based case. In fact, we have derived the specific upper-bound values (see the legend to Tables 1A and 1B) behind these conditions.

$$dG_{\mu}^{B}(F;2) = P_{\mu}^{ER}(F;\alpha)\frac{dT}{T}$$

Further research could be initiated to explore the links of these variables with some other relevant components of the income distribution, such as deprivation, poverty and so on.

References

[1] Aghevli, B.B. and F. Mehran (1981): "Optimal grouping of income distribution data", *Journal of the American Statistical Association*, 76, 22-26.

[2] Atkinson, A.B. (1970): "On the Measurement of Inequality", *Journal of Economic Theory*, 2, 244-263.

[3] Dalton, H. (1920): "Measurement of the Inequality of Incomes", *The Economic Journal*, 30, 348-361.

[4] D'Ambrosio, C. and E. Wolff (2001): "Is Wealth Becoming More Polarized in the United States?" Working Paper 330, Levy Economics Institute, Bard College.

[5] Dardanoni, V. and P.J. Lambert (1988): "Welfare Ranking of Income Distribution: A Role for the Variance and some Insights for Tax Reform", *Social Choice and Welfare*, 5, 1-17.

[6] Davies, J.B. and M. Hoy (2002): "Flat rate taxes and inequality measurement", *Journal of Public Economics*, 84, 33-46.

[7] Davies, J.B. and A.F. Shorrocks (1989): "Optimal grouping of income and wealth data", *Journal of Econometrics*, 42, 97-108.

[8] Donaldson, D. and J.A. Weymark (1980): "A Single Parameter Generalization of the Gini Index and Inequality", *Journal of Economic Theory*, 22, 67-86.

[9] Duclos, J-Y., Esteban J.M. and D. Ray (2004): "Polarization: concepts, measurement, estimation", *Econometric*a, forthcoming.

[10] Esteban, J.M., Gradín C. and D. Ray (1999): "Extensions of a measure of polarization, with an application to the income distributions of five OECD countries", *Documentos de Traballo*, Universidade de Vigo.

[11] Esteban, J.M. and D. Ray (1994): "On the measurement of polarization", *Econometric*a, 62, 4, 819-851.

[12] Esteban, J.M. and D. Ray (1999): "Conflict and distribution", *Journal of Economic Theory*, 87, 379-415.

[13] Hemming, R. and M. J. Keen (1983): "Single crossing conditions in comparisons of tax progressivity", *Journal of Public Economics*, 20, 373-380.

[14] Gradín, C. (2000): "Polarization by sub-populations in Spain, 1973-91", *Review of Income and Wealth*, 46 (4), 457-474.

[15] Kolm, S.C. (1976a): "Unequal Inequality: I", *Journal of Economic Theory*, 12, 416-442.

[16] Kolm, S.C. (1976b): "Unequal Inequality: II", *Journal of Economic Theory*, 13, 88-111.

[17] Pigou, A.C. (1912): *Wealth and Welfare*, Macmillan, Londres.

[18] Prieto-Rodríguez, J., Rodríguez, J.G. and R. Salas (2004): "Is the inequality-neutral flat tax reform really neutral?", *Documentos de Trabajo*, Fundación CentrA E2004/43. Electronic version at http://www2.fundacion-centra.org/pdfs/E200443.pdf

[19] Reynal-Querol, M. (2002): “Ethnicity, Political Systems, and Civil Wars,” *Journal of Conflict Resolution* **46**, 29–54.

[20] Rodríguez, J.G. and R. Salas (2003): “Extended bi-polarization and inequality measures”, *Research on Economic Inequality*, Vol. 9, 69-83.

[21] Shorrocks, A.F. (1983): “Ranking Income Distribution”, *Economica*, 50, 3-17.

[22] Wolfson, M.C. (1994): “When inequalities diverge”, *American Economic Review*, 84, 353-358.

[23] Wolfson, M.C. (1997): “Divergent inequalities: theory and empirical results”, *Review of Income and Wealth*, 43, 401- 421.

[24] Yitzhaki, S. (1983): “On an extension of the Gini inequality index”, *International Economic Review*, 24, *3, 617-28.*

Table 1A: Inequality-neutral case

Equivalence ($\Leftrightarrow$)	dP_z^{RS}		dP_z^{ER}	
	z=m	z=μ	z=m	z=μ
dG_z^B	Cond 1 (P-1A)	Uncond (P-1B)	Uncond (P-2A)	Cond 2 (P-2B)
$dL(q_z)$	Cond 1 (P-3A)	Cond 3 (P-3B)	Uncond (P-4A)	Cond 2 (P-4B)

Table 1B: Polarization-neutral case

Equivalence ($\Leftrightarrow$)	RS-neutral case		ER-neutral case	
	dG		dG	
	z=m	z=μ	z=m	z=μ
dG_z^B	Cond 1' (P-5A)	Uncond (P-5B)	-	-
$dL(q_z)$	Cond 1' (P-6A)	Cond 3 (P-6B)	-	-
dG_z^w	-	-	Uncond (P-7A)	Cond 2'(P-7B)

In brackets: proposition number, i.e., (P-1A) means Proposition 1A.

Legend to Tables 1A and 1B:

Uncond: unconditional with no restrictions.

Cond 1 : $\left|\frac{d(\mu/m)}{(\mu/m)}\right| \leq \left|\frac{d(G_m^B(F;v)-G_m^W(F;v))}{G_m^B(F;v)-G_m^W(F;v)}\right|$

Cond 1' : $\left|\frac{dm}{m}\right| \leq \left|\frac{2dG_m^B(F;v)}{G_m^B(F;v)-G_m^W(F;v)}\right|$

} relate to sufficiently low ***dm***

Cond 2 : $\left|\frac{dT}{T}\right| \leq \left|\frac{dG_\mu^B(F;2)}{G_\mu^B(F;2)}\right|$

Cond 2' : $\left|\frac{dT}{T}\right| \leq \left|\frac{dG_\mu^W(F;2)}{G_\mu^B(F;2)}\right|$

Cond 3 : $\left|dq_\mu\right| \leq \left|dL(q_\mu)\right|$

} relate to sufficiently low $\boldsymbol{dq_\mu}$

where $T = q_\mu^\alpha + (1-q_\mu)^\alpha$.