Measuring polarization, inequality, welfare and poverty

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ABSTRACT

This paper analyses the relationship between polarization and inequality, welfare and poverty measurements. Firstly, the Wolfson polarization measure is generalized in terms of the between-groups and within-groups Gini components for income groups separated by any z income value. Secondly, it is proved that polarization is the difference between the rich and the poor income groups welfare once the identification feeling of individuals is based on their utility function. Thirdly, the proposed polarization measure is a function of the Sen poverty index, its extension due to Shorrocks (1995) and the normalized poverty deficit index whether the z income value is the poverty line. Finally, those results are linked to the Esteban and Ray (1994) and Esteban et al. (1999) polarization measures.

JEL Classification: D39, D63, H30.
Key Words: polarization, inequality, welfare, poverty.

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Acknowledgements: This paper has benefited from support of the Spanish Ministry of Science and Technology [Project #SEC2003-08397] and Fundación BBVA. The usual disclaimer applies.

1. Introduction

Polarization measurement has been recently proposed as a relevant variable to characterize income distributions.[1] Nowadays, polarization is widely accepted as a distinct concept from inequality. In fact, polarization concentrates the income distribution on several focal or polar modes, whereas inequality relates to the overall dispersion of the distribution. A more bipolarized income distribution is one that is more spread out from the middle, so there are fewer individuals or families with middle level incomes (Wolfson, 1994 and 1997). Therefore, polarization measures can be used to complement the analysis of an income distribution. Whether a researcher is interested in making income distribution comparisons it is useful to study not only the amount of inequality, poverty and welfare but also polarization.

However, relationships between inequality, poverty and welfare measures have been the main focus of a huge amount of distributional research (see, for instance, Lambert, 2001 and the references therein) meanwhile connections between those concepts and the one of income polarization have almost not been analyzed yet. It is well known what are the similarities and the differences between welfare, inequality and poverty measures but, we know very few about the meaning of income polarization in terms of welfare, poverty and inequality. This is the main drawback of using polarization measurement as a complementary tool for income distribution analysis.

In this paper, polarization measurement is put into connection with the other main three faces of an income distribution: inequality, welfare and poverty.

Firstly, it is formally established a general relationship between the Wolfson polarization index and the Gini-based inequality measurement. The Wolfson polarization measure in terms of the between-groups and within-groups Gini components for income groups separated by any z income value is obtained. Then, polarization expressions for the median and the mean income values found in the literature (see Rodríguez and Salas, 2003 and Prieto *et al.*, 2004a) are viewed as particular cases of the Generalized Wolfson polarization measure. Polarization (for any z income value) and inequality are viewed within the same framework, with subtraction

[1] See, among others, Foster and Wolfson (1992), Esteban and Ray (1994 and 1999), Wolfson (1994 and 1997), Esteban *et al.* (1999), Tsui and Wang (2000), Gradín (2000), Zhang and Kanbur (2001), D'Ambrosio and Wolff (2001), Chakravarty and Majumder (2001), Rodríguez and Salas (2003), Prieto *et al.* (2004a and 2004b) and Duclos *et al.* (2004).

and addition of the within-groups dispersion corresponding to polarization and inequality, respectively. Moreover, it is proved that the generalized Wolfson bipolarization measure is a function of the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization measures. The proposed polarization measure is a function of the Esteban and Ray (1994) polarization index when only two groups are considered. The generalized Wolfson bipolarization measure is also a function of the Esteban *et al.* (1999) polarization index whether two groups are considered and the measurement error weight β is equal to 1. Therefore, the relationships between the Generalized Wolfson bipolarization measure and the welfare and poverty measurements (developed below) can be linked to the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization measures as well.

Secondly, a nice relationship between polarization and welfare measurements arises when envy between people is considered in the utility function. It is proved that polarization is the difference between the richer income group welfare and the poorer income group welfare whether people utility depends not only on their own income but also on their group incomes. As a result, polarization increases whenever the welfare of the richer income group goes up or/and the welfare of the poorer income group goes down. Besides, the identification feeling of individuals is based on their utility function under the proposed framework of analysis. This seems to be closer to the original motivation of the identification-alienation framework (see Esteban and Ray, 1994 and Duclos *et al.*, 2004) than just relying the identification term on the density function value.

Thirdly, polarization and poverty measurements are closed related measures whether the z income value according to which the income groups are separated is the poverty line. In that case, polarization between poor people and the rest of the income distribution explicitly considers the value of a poverty index. In particular, the Generalized Wolfson polarization measure can be written as a function of the Sen's poverty index (see Sen, 1976), its extension due to Shorrocks (1995) and also as a function of a member of the Foster-Greer-Thorbecke family of poverty measures, the so-called normalized poverty deficit (see Foster *et al.*, 1984). It is proved that more poverty (due to the increase on the proportion of poor people and/or the income gap ratio) and/or more richness (measured by the normalized richness surplus index) in the society means more polarization.

The paper is organized as follows. The Generalized Wolfson polarization index for any z income value is obtained in Section 2. In section 3 the relationship between polarization and welfare is analyzed. Poverty and polarization measurements are put into connection in section 4 and section 5 concludes.

2. Polarization and inequality: the Generalized Wolfson polarization index

Let $F \in \Re^n$ be an income distribution of *n* individuals, families or households, with a mean income value μ and a median income value *m*.

The Wolfson's index of bipolarization was originally proposed for a population divided in two groups by the median value:

$$P_m^W(F) = 4\frac{\mu}{m}P_1 = 4\frac{\mu}{m}\left[T_m - \frac{G(F)}{2}\right] = 2\frac{\mu}{m}\left[2(0.5 - L(0.5)) - G(F)\right] \qquad (1)$$

where P_1 is the lightly shaded area in Figure 1, $G(F)$ is the Gini coefficient of the income distribution *F* and T_m is the trapezoid area delimitated by the diagonal line and the tangent to the Lorenz curve (L) at the 50th population percentile. This trapezoid area is equal to the vertical distance between the 45-degree line and the Lorenz curve at the median percentile, $L(0.5)$. The larger the shaded area P_1 is, the fewer individuals or households with middle level incomes are, so the higher the polarization is.

Furthermore, if we divide the population in two groups by the mean income value (instead of the median one) the average difference of income pairs within both groups, that is, the dispersion within each group measured by the Gini coefficient is minimized (see Aghevli and Merhan, 1981 and Davies and Shorrocks, 1989). In this case, expression (1) becomes:

$$P_\mu^W(F) = 2\left[2(q_\mu - L(q_\mu)) - G(F)\right] \qquad (2)$$

where q_μ is the population percentile at the mean income value and $L(q_\mu)$ is the value of the Lorenz curve evaluated at q_μ. Note that the trapezoid area is easy to calculate when the income groups are separated by the median or the mean incomes. However, much more difficulties arise when a different income value is considered (see theorem 1 below).

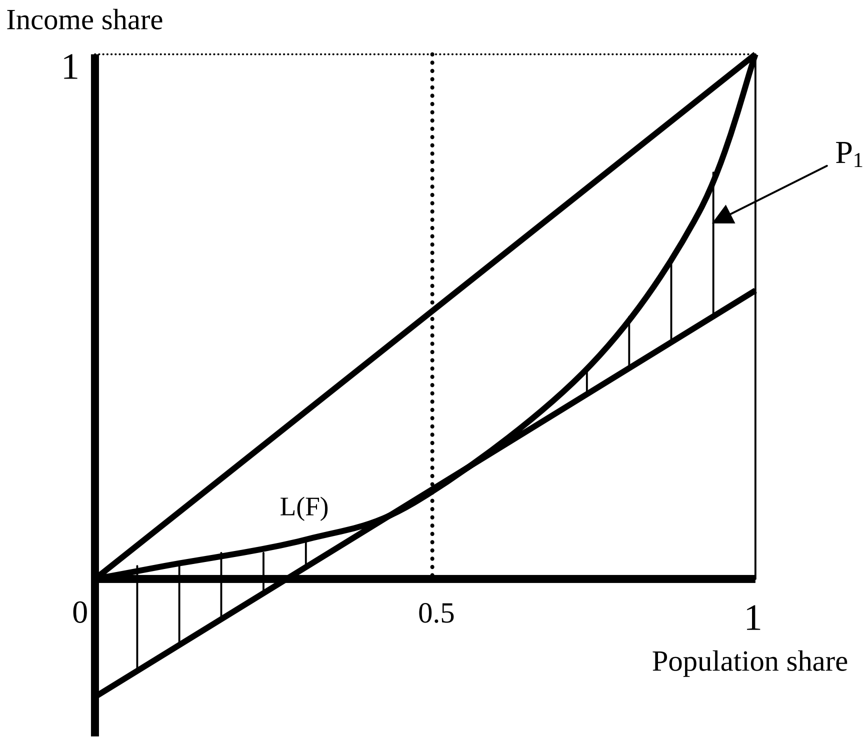

Figure 1. The Wolfson bipolarization Measure

The Wolfson index of polarization has been reformulated in terms of the Gini components. The additively decomposition of the Gini coefficient by groups of population (see, for instance, Bhattacharya and Mahalanobis, 1967, Pyatt, 1976 or Lambert and Aronson, 1993) is:

$$G(F) = G^B(F) + G^W(F) = G^B(F) + 2\left[\int_0^1 L_B(q)dq - \int_0^1 L(q)dq\right] = G^B(F) + \sum_k t_k r_k G_k \qquad (3)$$

whether incomes groups do not overlap. $G^B(F)$ is the between-groups Gini coefficient, $G^W(F)$ is the within-groups Gini coefficient, L_B is the between-groups Lorenz curve, t_k is the proportion of population in group k, r_k is the share in total income of group k and G_k is the Gini coefficient of group k.

The Wolfson index of polarization has been reformulated in terms of the between-groups Gini coefficient and the within-groups Gini coefficient in the following way (see Rodríguez and Salas, 2003 and Prieto *et al.*, 2004a):

$$P_m^W(F) = 2\frac{\mu}{m}\left[G_m^B(F) - G_m^W(F)\right] \tag{4}$$

$$P_\mu^W(F) = 2\left[G_\mu^B(F) - G_\mu^W(F)\right] \tag{5}$$

where the income groups are separated by the median and the mean income values, respectively. Therefore, polarization and inequality are explicitly viewed within the same framework, with subtraction and addition of the within-groups dispersión corresponding to polarization and inequality, respectively. In other words, progressive income transfers between groups reduce inequality and polarization, while progressive income transfers within groups reduce inequality but increase polarization.

Another advantage of the above reformulations is the connection between the Wolfson concept of polarization and the polarization model of Esteban and Ray (1994) and Duclos *et al.* (2004) that is established. The polarization measures in Esteban and Ray (1994) and Duclos *et al.* (2004) rely almost exclusively on the identification-alienation framework. Alienation relates to the accentuation of polarization by inter-group heterogeneity while identification relates to the accentuation of polarization by intra-group homogeneity. Hence, in our framework, $G^B(F)$ can represent feelings of alienation between dissimilar individuals and $\mathrm{G^W(F)}$ can represent feelings of identification between similar individuals. A different interpretation of this identification-alienation framework in terms of the individuals utility function and the difference between the mean income values is proposed in section 3, see below.

Now we generalize the Wolfson bipolarization index in terms of the between-groups and within-groups Gini components for any z income value.

Theorem 1 (*The Generalized Wolfson polarization index*): Let $F \in \Re^n$ be an income distribution separated in two groups by any income value z. Then, the Generalized Wolfson polarization index ($GP_z(F)$ henceforth) in terms of the between-groups and within-groups Gini components is:

$$GP_z(F) = 2\frac{\mu}{z}\left[G_z^B(F) - G_z^W(F)\right] + \frac{2}{z}(1-2q_z)(\mu - z) \tag{6}$$

Proof: Let us consider a z income value below the median (without losing generality) in what follows. We obtain the following expression for bipolarization when the Wolfson methodology (mutatis mutandi) is applied:

$$GP_z(F) = 2\frac{\mu}{z}\left[2T_z - G(F)\right] \tag{7}$$

where T_z is the trapezoid area delimitated by the 45-degree line and the tangent to the Lorenz curve at the z population percentile. This area is equal to the vertical distance between the 45-degree line and the tangent value at the median population percentile (see Figure 2).

The vertical distance between the Lorenz curve value at the z population percentile, $L(q_z)$, and the 45-degree line, is equal to the between groups Gini coefficient by construction (see Figure 2):

$$B = q_z - L(q_z) = G_z^B(F) \tag{8}$$

Therefore, we need to obtain T_z as a function of B to generalize the Wolfson bipolarization index in terms of the between-groups and within-groups Gini components for any z income value.

Let us consider the trapezoid delimitated by the diagonal line with slope 1, the tangent to the Lorenz curve at q_z with slope z/μ and the vertical distances B and T_z in Figure 2. We change the coordinates (see Figure 3) and apply some geometric results.

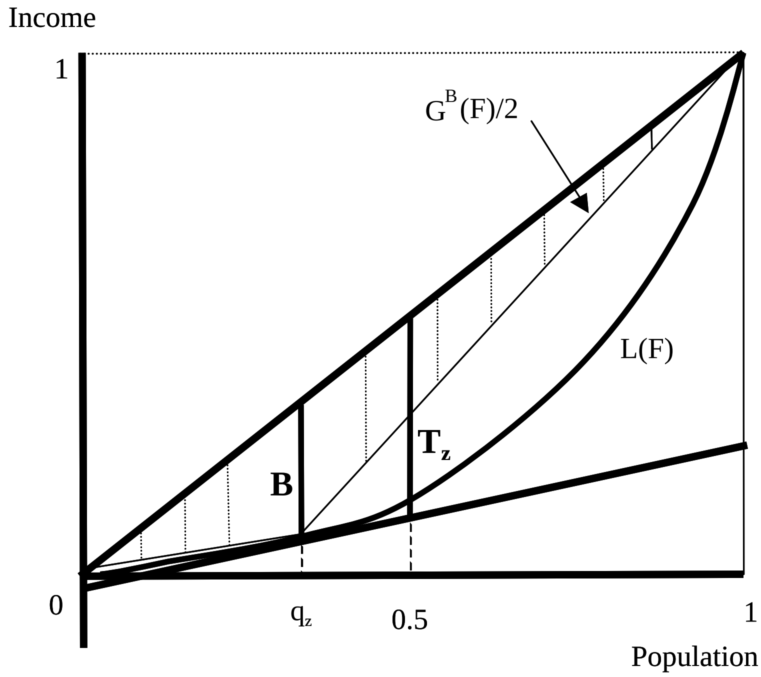

Figure 2. Bipolarization according to a *z* income value

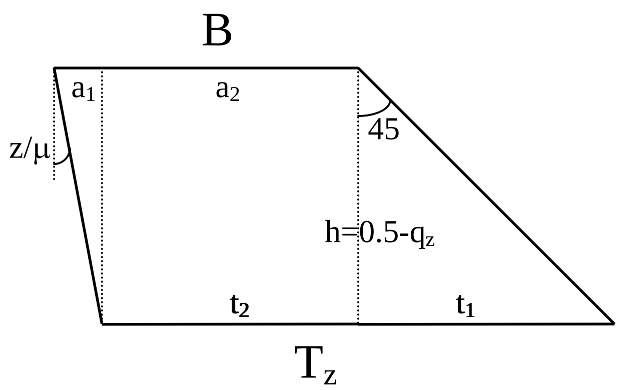

Figure 3. The A-T_z trapezoid area

The slope of the diagonal line is 1, therefore the segment t_1 is equal to the height $0.5\text{-}q_z$. Moreover, if we apply the *straight-line equation*[2] it is easy to prove that segment a_1 is equal to $(z/\mu)\cdot(0.5\text{-}q_z)$. As a result, $B=(z/\mu)\cdot(0.5\text{-}q_z)+a_2$ and $T_z=(0.5\text{-}q_z)+t_2$. We know that $a_2=t_2$ so:

[2] Recall that the point-slope form of the straight-line equation is: $(y_1\text{-}y_0)=\delta\cdot(x_1\text{-}x_0)$, where δ is the slope.

$$T_z = (0.5 - q_z) + B - \frac{z}{\mu}(0.5 - q_z) = B + (0.5 - q_z)\left(1 - \frac{z}{\mu}\right) \tag{9}$$

Now we substitute expressions (8) and (9) in equation (7):

$$GP_z(F) = 2\frac{\mu}{z}\left[2G_z^B(F) + 2(0.5 - q_z)\left(1 - \frac{z}{\mu}\right) - G(F)\right] \tag{10}$$

Hence, the general expression (6) for the Wolfson index of polarization is obtained. The proof is similar whether we consider a *z* income above the median value and expression (6) does not change. The following two corollaries are immediate.

Corollary 1: Let $F \in \Re^n$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. If $z = m$ then $GP_m(F) = 2\frac{\mu}{m}\left[G_m^B(F) - G_m^W(F)\right]$.

Corollary 2: Let $F \in \Re^n$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. If $z = \mu$ then $GP_\mu(F) = 2\left[G_\mu^B(F) - G_\mu^W(F)\right]$.

The polarization expressions for the median and the mean income values found in the literature (see expressions (4) and (5)) are viewed as particular cases of the Generalized Wolfson polarization measure.

To end with this section it is proved that the generalized Wolfson bipolarization measure is a function of the Esteban *et al.* (1999) polarization index when two income groups are considered and the measurement error weight β is equal to 1 and it is also a function of the Esteban and Ray (1994) polarization measure when only two groups are considered. Those relationships are used later on to generalize some of the found results to the Esteban *et al.* (1999) and the Esteban and Ray (1994) polarization measures.

Theorem 2 (*the Generalized Wolfson polarization measure as a function of the Esteban, Gradín and Ray (1999) polarization index*): Let $F \in \Re^n$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. Then,

$$GP_z(F) = \frac{2\mu}{zT} P_z^{EGR}(F;\alpha;1) + \frac{2}{z}\left[(1-2q_z)(\mu - z) - \mu G_z^W(F)\left(1-\frac{1}{T}\right)\right] \qquad (11)$$

where $P_z^{EGR}(F;\alpha;\beta)$ is the Esteban, Gradín and Ray (1999) polarization index for the income distribution F separated in two groups by the z income value, α is the identification sensitivity parameter, β is the measurement error weight and T is $q_z^\alpha + (1-q_z)^\alpha$.

Proof: The Esteban, Gradín and Ray (1999) polarization index is

$$P_z^{EGR}(F;\alpha;\beta) = P_z^{ER}(F;\alpha) - \beta\varepsilon(F;\ell) \qquad (12)$$

where $P_z^{ER}(F;\alpha)$ is the Esteban and Ray (1994) polarization index for two income groups separated by the z income value and $\varepsilon(F;\ell)$ is the measurement error that occurs when we consider ℓ (where data is gathered) the relevant income distribution instead of *F*.

The Esteban and Ray (1994) polarization index is

$$P^{ER}(F;\alpha) = \sum_i \sum_j q_i^{1+\alpha} q_j \left|\mu_i - \mu_j\right| \qquad (13)$$

where q_i and μ_i are, respectively, the population quintile and the mean income value of the income group *i*. Therefore, whether we consider two income groups

$$P_z^{EGR}(F;\alpha;\beta) = \left[q_z^{1+\alpha}(1-q_z) + (1-q_z)^{1+\alpha} q_z\right](\mu_2 - \mu_1) - \beta\left[G(F) - G(\ell)\right] \quad (14)$$

The mean income values are

$$\mu_1 = \frac{L(q_z)}{q_z} \quad \text{and} \quad \mu_2 = \frac{1-L(q_z)}{1-q_z} \qquad (15)$$

so

$$P_z^{EGR}(F;\alpha;\beta) = \left[q_z^{\alpha} + (1-q_z)^{\alpha}\right] G_z^{B}(F) - \beta\, G_z^{W}(F) \tag{16}$$

Whether we consider expression (16) for β = 1 and result (6) together, the theorem 2 is proved.

Corollary 3: Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the z income value. Whether the identification sensitivity parameter α and the measurement error weight β are equal to 1,

$$GP_z(F) = \frac{2\mu}{z} P_z^{EGR}(F;1;1) + \frac{2}{z}(1-2q_z)(\mu - z) \tag{17}$$

Corollary 4: Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the m income value. Whether the identification sensitivity parameter α and the measurement error weight β are equal to 1,

$$GP_m(F) = \frac{2\mu}{m} P_m^{EGR}(F;1;1) \tag{18}$$

Corollary 5: Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the μ income value. Whether the identification sensitivity parameter α and the measurement error weight β are equal to 1,

$$GP_\mu(F) = 2P_\mu^{EGR}(F;1;1) \tag{19}$$

Theorem 3 (*the Generalized Wolfson polarization measure as a function of the Esteban and Ray (1994) polarization index*): Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the *z* income value. Then,

$$GP_z(F) = \frac{2\mu}{zT} P_z^{ER}(F;\alpha) + \frac{2}{z}\left[(1-2q_z)(\mu - z) - \mu G_z^W(F)\right] \tag{20}$$

where α is the identification sensitivity parameter and T is $q_z^\alpha + (1-q_z)^\alpha$.

Proof: Whether we consider

$$P_z^{ER}(F;\alpha) = \left[q_z^\alpha + (1-q_z)^\alpha\right] G_z^B(F) \tag{21}$$

in expression (6) the result is obtained.

Nevertheless, notice that the Esteban and Ray (1994) and the Esteban *et al.* (1999) polarization indexes can be applied to any number of income groups, instead of the Generalised Wolfson polarization measure which can only be applied to two income groups.

In the next section we use abbreviated welfare functions containing the Gini coefficient to interpret polarization in terms of welfare.

3. Polarization and welfare

An interesting relationship between polarization and welfare measurements arises when envy between people is considered in their utility function. In fact, it is proved that polarization increases whenever the welfare of the richer income group goes up or/and the welfare of the poorer income group goes down.

It is well known that the rankings induced on any two income distributions with the same mean income value by a symmetric, increasing and individualistic abbreviated welfare function *W* and by –*G* are not necessarily the same (see Newbery, 1970). Nevertheless, it is still possible to justify the use of an abbreviated welfare function containing the Gini coefficient whether *W* is non-individualistic (see Sheshinski, 1972, Kakwani, 1980 and 1986, for example).

Let $D(x;y)$ be the relative deprivation felt by an individual with income x in respect of an individual with income y where

$$
\begin{aligned}
D(x;y) &= y - x && \text{if } x \le y \\
D(x;y) &= 0 && \text{if } x \ge y
\end{aligned}
\tag{22}
$$

(see Runciman, 1966). Then, the overall deprivation felt by an individual with income x is

$$D_F(x) = \int D(x;y) f(y) dy \tag{23}$$

Now let $U^D(x, F)$ be the utility function of an individual with income x where

$$U^D(x, F) = ax - bD_F(x) \qquad a, b > 0 \tag{24}$$

The individual cares not only about his own income but also about the distribution he inhabits. In particular, the higher deprivation he feels the lower utility he enjoys.

The following result justifies the use of an abbreviated welfare function containing the Gini coefficient whether *W* is non-individualistic.

Result 1 (Lambert, 2001, pp. 123-124)[3]: when $U^D(x,F) = ax - bD_F(x)$, $W_F^D = \int U^D(x,F) f(x) dx = \mu_F(a - bG_F)$ for every income distribution *F*.

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[3] A similar result, $W_F = \mu_F[a - 0.5b(1+G_F)]$, is obtained whether the altruistic utility function $U(x,F) = x[a - bF(x)]$ $\quad a, b > 0$ is used, where the arguments are the own income level and the proportion of people less well-off than himself.

We use this result (for $a = b = 1$) latter on to put into connection the overall bipolarization suffered by a society with the rich income group welfare (see below).

A parallel result arises when a new concept is introduced: the *relative abundance*.
Let $A(x;y)$ be the relative abundance felt by an individual with income x in respect of an individual with income y where

$$A(x;y) = x - y \qquad \text{if } x \geq y$$
$$A(x;y) = 0 \qquad \text{if } x \leq y \tag{25}$$

The overall abundance felt by an individual with income x is

$$A_F(x) = \int A(x;y) f(y) dy \tag{26}$$

Now let $U^A(x, F)$ be the utility function of an individual with income x where

$$U^A(x,F) = ax + bA_F(x) \qquad a, b > 0 \tag{27}$$

In this case the sentiment of envy is different: an individual with income x is more well-off whether more people have less income than himself. People care for status. As a consequence, the more relative abundance an individual with income x feels, the more utility he enjoys.
The following result allows the use of an abbreviated welfare function (containing the Gini coefficient) whether W is non-individualistic in a different way than the result 1 does.

Theorem 4 (*a welfare function based on the relative abundance concept*)[4]: Let $F \in \Re^n$ be an income distribution, $A_F(x)$ the relative abundance function and $U^A(x,F) = ax + bA_F(x)$ for $a, b > 0$, then $W_F^A = \int U^A(x,F) f(x)dx = \mu_F(a + bG_F)$.

Proof: Whether we substitute equations (25), (26) and (27) in the welfare function definition we obtain the following expression:

$$W_F^A = \int_0^\infty U^A(x,F) f(x)dx = a\mu + b\int_0^\infty \left[\int_0^x (x-y) f(y)dy \right] f(x)dx \tag{28}$$

As q=$F(x)$ and $L(q) = \frac{1}{\mu}\int_0^x y f(y)dy$, we derive

$$\mu L(q) = \int_0^x y f(y)dy \tag{29}$$

$$L'(q) = \frac{dL(q)}{dx}\frac{dx}{dq} = \frac{xf(x)}{\mu}\frac{1}{f(x)} = \frac{1}{\mu}x \tag{30}$$

Now we substitute (29) and (30) in (28):

$$W_F^A = \mu\left[a + b\int_0^1 [qL'(q) - L(q)]dq \right] \tag{31}$$

From the definition of the Gini inequality index we know that

$$\int_0^1 L(q)dq = \frac{1 - G(F)}{2} \tag{32}$$

[4] A similar result, $W_F = \mu_F[a + 0.5b(1 + G_F)]$, is obtained whether the utility function $U(x,F) = x[a + bF(x)]$ $a,\ b > 0$ is used, where the arguments are the own income level and the proportion of people less well-off than himself.

We integrate by parts expression (32):

$$\int_0^1 qL'(q)dq = \frac{G(F)+1}{2} \tag{33}$$

We only need to substitute expressions (32) and (33) in equation (31) to finish the proof.

We use this theorem (for $a = b = 1$) latter on to put into relationship the overall polarization suffered by a society with the welfare's poor income group (see below). Nevertheless, we need to prove the following lemma before a connection between both abbreviated welfare functions and economic polarization is established.

In the next lemma we decompose the Generalized Wolfson polarization measure in two terms, each of them corresponding with the two transformed areas (below and above $L(q_z)$) that define polarization (see Figure 1).

Lemma 1 (*the Generalized Wolfson polarization measure decomposition*): Let $F \in \Re^n$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. Then,

$$\begin{aligned} GP_z(F) = {} & 4\frac{\mu}{z}\left[\frac{1}{2}\left(q_z^2\left(1+\frac{z}{\mu}\right)-2q_zL(q_z)\right)-\int_0^{q_z}(q-L(q_z))dq\right]+ \\ & +4\frac{\mu}{z}\left[\frac{1}{2}\left(1+q_z-2L(q_z)-\frac{z}{\mu}(1-q_z)\right)(1-q_z)-\int_{q_z}^{1}(q-L(q_z))dq\right] \end{aligned} \tag{34}$$

where each term at the right side of the equation (34) correspond with the two transformed areas (below and above $L(q_z)$) that define polarization (shaded areas in Figure 1).

Proof: we know from theorem 1 that

$$GP_z(F) = 2\frac{\mu}{z}\left[G_z^B(F) - G_z^W(F)\right] + \frac{2}{z}(1-2q_z)(\mu - z) =$$

$$= 4\frac{\mu}{z}\left[G_z^B(F) - \frac{G(F)}{2}\right] + 4\frac{\mu}{z}(\frac{1}{2} - q_z)\left(1 - \frac{z}{\mu}\right) \tag{35}$$

Now we substitute expression (8) in equation (35),

$$= 4\frac{\mu}{z}\left[\frac{1}{2} - L(q_z) - \frac{z}{2\mu} + \frac{z}{\mu}q_z - \frac{G(F)}{2}\right] =$$

$$= 4\frac{\mu}{z}\left[\left(q_z^2\left(1+\frac{z}{\mu}\right) - 2q_z L(q_z) + (1-q_z) + q_z(1-q_z) - 2(1-q_z)L(q_z) - \frac{z}{\mu}(1-q_z)^2\right)\frac{1}{2}\right] -$$

$$-4\frac{\mu}{z}\left[\int_0^{q_z}(q - L(q))dq + \int_{q_z}^{1}(q - L(q))dq\right] \tag{36}$$

Then, we only need to reorder terms in (36) to obtain expression (34).

Note that the two terms at the right side of the equation (34) are actually the two areas below and above $L(q_z)$ which define polarization (see Figure 1):

The term

$$\frac{1}{2}\left(q_z^2\left(1+\frac{z}{\mu}\right) - 2q_z L(q_z)\right) = \frac{1}{2}\left[\left[q_z - L(q_z)\right] + \left[0 - \left(L(q_z) - \frac{z}{\mu}q_z\right)\right]\right]q_z \tag{37}$$

corresponds with the trapezoid area below q_z. $\left(L(q_z) - \frac{z}{\mu}q_z\right)$ is the negative vertex of the trapezoid and it is calculated applying the point-slope form of the straight-line equation (see footnote 2).

The term,

$$\frac{1}{2}\left(1+q_z-2L(q_z)-\frac{z}{\mu}(1-q_z)\right)(1-q_z)=\frac{1}{2}\left[\left[q_z-L(q_z)\right]+\left[1-L(q_z)-\frac{z}{\mu}(1-q_z)\right]\right](1-q_z)$$

(38)

corresponds with the trapezoid area above q_z. The vertex $\left[L(q_z)-\frac{z}{\mu}(1-q_z)\right]$ is calculated applying the point-slope form of the straight-line as well (see footnote 2).

Once the Lemma 1 is proved and the welfare functions discussed above (see result 1 and theorem 4) are considered we achieve the following result:

Theorem 5 (*the Generalized Wolfson polarization measure as a function of the income groups welfare*): Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure, W_P^A be the welfare achieved by the poor group and W_R^D be the welfare achieved by the rich group. Then,

$$GP_z(F)=2q_z^2\left[1-\frac{W_P^A}{z}\right]+2(1-q_z)^2\left[\frac{W_R^D}{z}-1\right] \tag{39}$$

Polarization increases whether the welfare of the rich income group goes up or/and the welfare of the poor income group goes down, and vice versa.

Polarization is viewed as a function of people welfare which depends not only on the own income but also on the feeling of envy with respect to the own incomes group. In particular, people in the rich income group feel envy (relative deprivation) from individuals with higher income meanwhile people in the poor income group feel envy (relative abundance) from individuals with lower income.

On the one hand, income polarization increases whether the mean income value of the rich income group increases (that is, μ_R moves away from z) while polarization decreases whether the mean income value of the poor income group increases (that is, μ_P moves closer to z). On the other hand, income polarization increases whether relative deprivation in the rich income group and/or relative abundance in the poor income group decrease. This polarization behavior has a clear interpretation under the identification-alienation framework:

a) Whether μ_R moves away from z, alienation (between both income groups) increases; whether μ_P moves closer to z, alienation decreases.

b) Whether relative deprivation in the rich income group decreases, identification (within the rich income group) increases; Whether relative abundance in the poor income group decreases, identification (within the poor income group) increases.

Alienation is determined by the difference between μ_R and μ_P. Identification depends negatively on the magnitude of envy, the relative deprivation and relative abundance that individuals feel.

A relevant question arises. The polarization model of Esteban and Ray (1994) and Duclos *et al.* (2004) consider that the identification term is the density function value. However, there is no reason to believe that grouping of income distribution data conveniently conform to the psychological demands of group identification as the own authors recognized. At this respect, the proposed framework of analysis seems to be closer to their original motivation of the identification-alienation framework. In fact, the identification feeling is based on the individuals utility function where not only the own income but also their envy sentiment matters.

Proof: Let μ_P be the mean income value of the poorer group (the one below the z income value) and μ_R be the mean income value of the richer group (above the z income value), then

$$L_B(q) = \frac{1}{\mu}\int_0^q \mu_P dF = q\frac{\mu_P}{\mu} \qquad \forall\ q \in [0, q_z] \tag{40}$$

and

$$L_B(q) = \frac{1}{\mu}\int_0^{q_z} \mu_P dF + \frac{1}{\mu}\int_{q_z}^{q} \mu_R dF = q_z\frac{\mu_P}{\mu} + (q - q_z)\frac{\mu_R}{\mu} \qquad \forall\ q \in (q_z, 1] \tag{41}$$

We derive from (3):

$$\int_0^{q_z} L_B(q)dq - \int_0^{q_z} L(q)dq = \frac{1}{2} q_z \left(q_z \frac{\mu_P}{\mu} \right) G_P \tag{42}$$

$$\int_{q_z}^{1} L_B(q)dq - \int_{q_z}^{1} L(q)dq = \frac{1}{2}(1-q_z)\left((1-q_z)\frac{\mu_R}{\mu} \right) G_R \tag{43}$$

Therefore, whether we consider expressions (40), (41), (42) and (43) we obtain:

$$\int_0^{q_z} (q - L(q_z))dq = \left[\int_0^{q_z} q dq - \int_0^{q_z} L_B(q)dq \right] + \left[\int_0^{q_z} L_B(q)dq - \int_0^{q_z} L(q)dq \right] = \frac{1}{2} q_z^2 \left(1 - \frac{\mu_P}{\mu} \right) + \frac{1}{2} q_z^2 \frac{\mu_P}{\mu} G_P \tag{44}$$

$$\int_{q_z}^{1} (q - L(q))dq = \left[\int_{q_z}^{1} q dq - \int_{q_z}^{1} L_B(q)dq \right] + \left[\int_{q_z}^{1} L_B(q)dq - \int_{q_z}^{1} L(q)dq \right] =$$

$$= \frac{1}{2}\left(1-q_z^2\right) - \frac{1}{2}\left(1-q_z^2\right)\frac{\mu_P}{\mu} + q_z(1-q_z)\frac{\mu_R - \mu_P}{\mu} + \frac{1}{2}(1-q_z)^2 \frac{\mu_R}{\mu} G_R \tag{45}$$

Now, we substitute expressions (44) and (45) in equation (34) and take into account that $L(q_z) = L_B(q_z) = q_z \dfrac{\mu_P}{\mu}$,

$$GP_z(F) = 4\frac{\mu}{z}\left[\frac{1}{2} q_z^2 \frac{z}{\mu} - \frac{1}{2} q_z^2 \frac{\mu_P}{\mu} - \frac{1}{2} q_z^2 \frac{\mu_P}{\mu} G_P \right] +$$

$$+ 4\frac{\mu}{z}\left[q_z \frac{z}{\mu} - \frac{z}{2\mu} - \frac{1}{2} q_z^2 \frac{z}{\mu} + \frac{1}{2}\left(1+q_z^2\right)\frac{\mu_R}{\mu} - q_z \frac{\mu_R}{\mu} - \frac{1}{2}(1-q_z)^2 \frac{\mu_R}{\mu} G_R \right] \qquad \Leftrightarrow$$

$$GP_z(F) = \left[2q_z^2 - 2q_z^2 \frac{\mu_P}{z} - 2q_z^2 \frac{\mu_P}{z} G_P \right] +$$

$$+ \left[4q_z - 2 - 2q_z^2 + 2(1+q_z^2)\frac{\mu_R}{z} - 4q_z \frac{\mu_R}{z} - 2(1-q_z)^2 \frac{\mu_R}{z} G_R \right] \qquad \Leftrightarrow$$

$$GP_z(F) = 2q_z^2\left[1 - \frac{\mu_P}{z}(1+G_P)\right] + 2(1-q_z)^2\left[\frac{\mu_R}{z}(1-G_R) - 1\right] \quad (46)$$

Finally, we only need to consider result 1 and theorem 2 for a = b = 1 to obtain the expression (39) in theorem 3.

Note that the welfare of the rich income group is higher than the welfare of the poor income group in general, nevertheless, this is not guaranteed for all cases. It can happen that the magnitude of relative abundance and/or deprivation feelings suffered by people in the poor and rich income groups, respectively, can more than compensate the superiority of the mean value's rich income group over the mean value's poor income group.
A straightforward result derived from theorem 3 is:

Corollary 6: Let $F \in \Re^n$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. Then, if $z = m$

$$GP_m(F) = \frac{1}{2m}\left[W_R^D - W_P^A\right] \quad (47)$$

Polarization becomes just half the difference between the normalized welfare (by the median income) of the richer income group and the poorer income group, whether we consider income groups separated by the median income value.

In this case, it is guaranteed that the welfare of the richer income group is higher than the welfare of the poorer income group as polarization can not be negative. Hence, polarization decreases whether the welfare's poorer income group takes closer to the welfare's richer income group.

Now we generalize the found connections between polarization and welfare to the Esteban *et al.* (1999) polarization index.

Theorem 6 (*The Esteban et al. (1999) polarization index as a function of the income groups welfare*): Let $F \in \Re^n$ be an income distribution, $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization measure for two income groups separated by the *z* income value, W_P^A be the welfare achieved by the poor group and W_R^D be the welfare achieved by the rich group. Then,

$$P_z^{EGR}(F;1;1) = \frac{z}{\mu}\left[q_z^2\left(1-\frac{W_P^A}{z}\right)+(1-q_z)^2\left(\frac{W_R^D}{z}-1\right)\right]-\frac{1}{\mu}(1-2q_z)(\mu-z) \qquad (48)$$

Polarization according to the Esteban *et al.* (1999) polarization index is also a function of people welfare whether the identification sensitivity parameter α and the parameter β are equal to 1 and there are only two income groups.

Proof: Once we consider expressions (17) and (39) together, the proof of this result is straightforward.

Corollary 7: Let $F \in \Re^n$ be an income distribution, $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization measure for two income groups separated by the *m* income value, W_P^A be the welfare achieved by the poor group and W_R^D be the welfare achieved by the rich group. Then,

$$P_m^{EGR}(F;1;1) = \frac{1}{4\mu}\left[W_R^D - W_P^A\right] \qquad (49)$$

whether α and β are equal to 1.

Corollary 8: Let $F \in \Re^n$ be an income distribution, $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization measure for two income groups separated by the μ income value, W_P^A be the welfare achieved by the poor group and W_R^D be the welfare achieved by the rich group. Then,

$$P_{\mu}^{EGR}(F;1;1) = q_{\mu}^{2}\left(1-\frac{W_{P}^{A}}{\mu}\right)+(1-q_{\mu})^{2}\left(\frac{W_{R}^{D}}{\mu}-1\right) \tag{50}$$

whether α and β are equal to 1.

4. Polarization and poverty

Polarization and poverty measurements can be related whether the z income value according to which the income groups are separated is the poverty line. In this case, polarization between poor people and the rest of the income distribution explicitly considers the value of a poverty index. In the next three results the Generalized Wolfson polarization measure is written as a function of the Sen poverty index (Sen, 1976), its extension due to Shorrocks (1995) and also, as a function of a member of the Foster-Greer-Thorbecke family of poverty measures, the normalized poverty deficit (see Foster *et al.*, 1984). It is proved that more poverty due to the increase on the proportion of poor people and/or the income gap ratio means more income polarization in the society. Furthermore, polarization increases whether richness measured by a new concept, the normalized richness surplus index, grows up.

Let us recall some concepts before the results are presented. The Sen poverty index is

$$S_{z}^{S}(F) = H_{z}(F)\left[I_{z}(F)+(1-I_{z}(F))G_{P}\right] \tag{51}$$

where z is the poverty line, $H_z(F) = q_z$ is the headcount ratio or proportion of the population who are poor in F and $I_{z}(F) = 1-\frac{\mu_P}{z}$ is the income gap ratio (see Sen, 1976).[6]

Shorrocks (1995) have proposed the following generalization of the Sen poverty index:

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[5] This is the *official* replication invariant version of the original Sen poverty index $S_{z}^{S}(F) = H_{Z}(F)I_{Z}(F)+[r/r+1](1-I_{Z}(F))G_{P}(F)$ where r is the number of poor persons.

[5] This is the *official* replication invariant version of the original Sen poverty index $S_{z}^{S}(F) = H_{Z}(F)I_{Z}(F)+[r/r+1](1-I_{Z}(F))G_{P}(F)$ where r is the number of poor persons.

$$S_z^{SH}(F) = (2 - H_z(F))H_z(F)I_z(F) + H_z(F)(1 - I_z(F))G_P \tag{52}$$

This poverty index is not only replication invariant, but also continuous and consistent with the progressive transfer axiom.

The family of poverty indices introduced by Foster, Greer and Thorbecke (1984) is

$$S_z^{FGT}(F;\gamma) = \int_0^z \Gamma(x)^\gamma f(x)dx \tag{53}$$

where $\Gamma(x) = \max\left\{\frac{z-x}{z}, 0\right\}$ and $\gamma \geq 0$. Notice that $D_z(F) = \int_0^z (z-x)f(x)dx$ is the poverty deficit index so the Foster-Greer-Thorbecke family of poverty measures becomes the normalized poverty deficit index or product of the headcount and income gap ratios, $D_z(F)/z = H_z(F)I_z(F)$, when $\gamma = 1$.

We define, in an analogous way to the normalized poverty deficit index, the *normalized richness surplus index*:

$$R_z(F) = \overline{H}_z(F)O_z(F) = (1 - q_z)\left[\frac{\mu_R}{z} - 1\right] \tag{54}$$

It is the product of the proportion of the population who are not poor in *F*, $\overline{H}_z(F)$, and the *income overabundance gap ratio,* $O_z(F)$, which is the average richness gap μ_R-z normalized by the poverty threshold. This concept can be interpreted as a richness index whether people with income above the poverty line are considered rich. Therefore, there are only two kind of individuals in this society: poor and rich people.

Theorem 7 (*the Generalized Wolfson polarization measure as a function of the Sen poverty index*): Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $S_z^S(F)$ be the Sen poverty measure. Then,

$$GP_z(F) = 2\left[q_z\left(S_z^S(F) - q_z \frac{\mu_P}{z} G_P \right) + (1-q_z)R_z(F) - \frac{\mu}{z} G_z^W(F) \right] \tag{55}$$

where $R_z(F)$ is the normalized richness surplus index and $G_z^W(F)$ is the within-groups Gini coefficient.

Proof: Let us consider equation (46) and expression (51) together

$$GP_z(F) = 2q_z^2\left[1-(1-I_z(F))(1+G_P)\right] - 2(1-q_z)^2 \frac{\mu_R}{z} G_R + 2(1-q_z)^2 \left[\frac{\mu_R}{z} - 1 \right] \tag{56}$$

That is,

$$\begin{aligned} GP_z(F) &= 2q_z^2\left[I_z(F) - (1-I_z(F))G_P\right] - 2(1-q_z)^2 \frac{\mu_R}{z} G_R + 2(1-q_z)R_z(F) \\ &= 2q_z\left[S_z^S(F) - 2q_z \frac{\mu_P}{z} G_P \right] - 2(1-q_z)^2 \frac{\mu_R}{z} G_R + 2(1-q_z)R_z(F) \end{aligned} \tag{57}$$

Once it is considered the within-groups Gini coefficient,

$$G_z^W(F) = q_z^2 \frac{\mu_P}{\mu} G_P + (1-q_z)^2 \frac{\mu_R}{\mu} G_R \tag{58}$$

the result is proved.

Corollary 9 (*the Generalized Wolfson polarization measure as a function of the Shorrocks poverty index*): Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $S_z^{SH}(F)$ be the Shorrocks poverty measure. Then,

$$GP_z(F) = 2\left[\frac{q_z}{2-q_z}\left(S_z^{SH}(F) - q_z^2 \frac{\mu_P}{z} G_P\right) + (1-q_z)R_z(F) - \frac{\mu}{z} G_z^W(F)\right] \tag{59}$$

where $R_z(F)$ is the normalized richness surplus index. This corollary is straightforward once the proof of theorem 7 and the expression (52) are considered (mutatis mutandis).

Bipolarization between poor people and the rest of the income distribution explicitly considers the value of a poverty index: the Sen poverty index or its extension due to Shorrocks (1995). Moreover, polarization is an increasing function of richness according to the normalized richness surplus index. However, polarization depends negatively on the dispersion within the income groups according to the Gini coefficient. As we show in section 2 progressive transfers within groups increase polarization. As a result, whether the proportion of poor people and/or the income gap ratio change, polarization and poverty variations have the same sign (in this case more poverty means more polarization). However, whether the Gini coefficient for the poor people group changes, polarization and poverty variations have the opposite sign.[7] Therefore, the proposed bipolarization measure is a non-increasing function of the Sen and Shorrocks poverty indexes.

Corollary 10 (*the Generalized Wolfson polarization measure as a function of the normalized poverty deficit index*)[8]: Let $F \in \Re^n$ be an income distribution, $GP_z(F)$ be the Generalized Wolfson polarization measure and $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then,

[7] Notice that whether only the Gini coefficient for the poor people group changes $dGP_z(F) = -2q_z^2 \frac{\mu_P}{z} dG_P$.

[8] It can be shown that the area below the first polarization curve (see Wolfson 1994, 1997) for incomes below z is equal to the normalized poverty deficit index.

$$GP_z(F) = 2\left[q_z S_z^{FGT}(F;1) + (1-q_z)R_z(F) - \frac{\mu}{z} G_z^W(F)\right] \qquad (60)$$

where $S_z^{FGT}(F;1)$ is the normalized poverty deficit index and $R_z(F)$ is the normalized richness surplus index.

In this case, bipolarization between poor people and the rest of the income distribution is an increasing function of poverty, according to the normalized poverty deficit index, and also an increasing function of richness, according to the normalized richness surplus index. More poverty and/or richness whatever the source increases polarization. In this case, the distribution of income amongst the poor does not matter when measuring poverty. Inequality and poverty are considered to be different issues (this view is justified for instance in Lewis and Ulph, 1988).

In what follows we generalize the found relationship between poverty and polarization to the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes.

Theorem 8 (*The Esteban and Ray (1994) polarization measure as a function of the Sen poverty index*): Let $F \in \Re^n$ be an income distribution, $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the z income value and $S_z^S(F)$ be the Sen poverty measure. Then,

$$P_z^{ER}(F;\alpha) = \frac{zT}{\mu}\left[q_z\left(S_z^S(F) - q_z\frac{\mu_P}{z}G_P\right) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) \qquad (61)$$

where $R_z(F)$ is the normalized richness surplus index. Note that the negative second term on the right hand of equation (61) vanishes whether z is equal to m or μ.

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[9] Notice that whether only the Gini coefficient for the poor people group changes $dGP_z(F) = -2q_z^2 \frac{\mu_P}{z} dG_P$.

Proof: Once we consider expressions (20) and (55) together the proof of theorem 8 is straightforward.

Corollary 11 (*The Esteban and Ray (1994) polarization measure as a function of the Shorrocks poverty index*): Let $F \in \Re^n$ be an income distribution, $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the z income value and $S_z^{SH}(F)$ be the Shorrocks poverty measure. Then,

$$P_z^{ER}(F;\alpha) = \frac{zT}{\mu}\left[\frac{q_z}{2-q_z}\left(S_z^{SH}(F) - q_z^2\frac{\mu_P}{z}G_P\right) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) \tag{62}$$

where $R_z(F)$ is the normalized richness surplus index. Note that the negative second term on the right hand of equation (62) vanishes whether z is equal to m or μ.

Corollary 12 (*The Esteban and Ray (1994) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the z income value and $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then,

$$P_z^{ER}(F;\alpha) = \frac{zT}{\mu}\left[q_z S_z^{FGT}(F;1) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) \tag{63}$$

where $R_z(F)$ is the normalized richness surplus index.

Corollary 13 (*The Esteban and Ray (1994) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the *m* income value and $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then,

$$P_m^{ER}(F;\alpha) = \frac{mT}{2\mu}\left[S_m^{FGT}(F;1) + R_m(F)\right] \tag{64}$$

where $R_z(F)$ is the normalized richness surplus index.

Corollary 14 (*The Esteban and Ray (1994) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the μ income value and $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then,

$$P_\mu^{ER}(F;\alpha) = T\left[q_\mu S_\mu^{FGT}(F;1) + (1-q_\mu)R_\mu(F)\right] \tag{65}$$

where $R_z(F)$ is the normalized richness surplus index.

In the last two results, bipolarization between poor people and the rest of the income distribution is just an increasing function of poverty, according to the normalized poverty deficit index, and richness, according to the normalized richness surplus index.

Theorem 9 (*The Esteban et al. (1999) polarization measure as a function of the Sen poverty index*): Let $F \in \Re^n$ be an income distribution, $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et* *al.* (1999) polarization index for two income groups separated by the z income value and $S_z^S(F)$ be the Sen poverty measure. Then,

$$P_z^{EGR}(F;\alpha;1)=\frac{T}{\mu}\left[zq_z\left(S_z^S(F)-q_z\frac{\mu_P}{z}G_P\right)+z(1-q_z)R_z(F)-\frac{\mu}{T}G_z^W(F)-(1-2q_z)(\mu-z)\right] \tag{66}$$

where $R_z(F)$ is the normalized richness surplus index.

Proof: Once we consider expressions (11) and (55) together the proof of theorem 9 is straightforward.

Corollary 15 (*The Esteban et al. (1999) polarization measure as a function of the Shorrocks poverty index*): Let $F \in \Re^n$ be an income distribution, $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the z income value and $S_z^{SH}(F)$ be the Shorrocks poverty measure. Then,

$$\begin{aligned}P_z^{EGR}(F;\alpha;1)=\frac{T}{\mu}&\left[\frac{zq_z}{2-q_z}\left(S_z^{SH}(F)-q_z^2\frac{\mu_P}{z}G_P\right)+z(1-q_z)R_z(F)\right]\\&-G_z^W(F)-\frac{T}{\mu}(1-2q_z)(\mu-z)\end{aligned} \tag{67}$$

where $R_z(F)$ is the normalized richness surplus index.

Corollary 16 (*The Esteban et al. (1999) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the z

income value and $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then,

$$P_z^{EGR}(F;\alpha;1) = \frac{zT}{\mu}\left[q_z S_z^{FGT}(F;1) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) - \frac{1}{T}G_z^W(F) \quad (68)$$

where $R_z(F)$ is the normalized richness surplus index.

Therefore, as a conclusion of this section, it can be said that more poverty (due to the increase on the proportion of poor people and/or the income gap ratio) and/or more richness (measured by the normalized richness surplus index) in the society means more polarization.

5. Concluding remarks

The relationship between polarization measurement and inequality, welfare and poverty issues is analyzed in this paper. Firstly, the Wolfson polarization measure is generalized in terms of the between-groups and within-groups Gini components for income groups separated by any z income value. Moreover links between the Generalized Wolfson polarization measure and the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes are proposed. Secondly, it is proved that polarization according to the Generalized Wolfson polarization index and the Esteban *et al.* (1999) polarization measure are the difference between the rich and the poor income groups welfare once the identification feeling of individuals is based on their utility function. Thirdly, the proposed polarization measure and the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization measures are a function of the Sen poverty index, its extension due to Shorrocks (1995) and the normalized poverty deficit index whether the z income value is the poverty line.

The main drawback of the paper is also the main road for future research: the generalization to more than two income groups.

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