Compensating and Equivalent Variations associated with Quantitative Constraints

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ABSTRACT

The concepts of compensating and equivalent variation are widely used in Public Economics.They derive from the expenditure function and are applied to price changes. In this paper we enlarge the field of application of these concepts to situations involving quantity changes. Using the constrained expenditure function, we study the compensating and equivalent variations associated with changes in quantitative constraints on labour supply and credit demand.

1. Introduction

The concepts of compensating and equivalent variation are widely used in Public Economics.They derive from the expenditure function and are applied to price changes. In this paper we enlarge the field of application of these concepts to situations involving quantity changes. Using the constrained expenditure function, we study the compensating and equivalent variations associated with changes in quantitative constraints on labour supply and credit demand.

2. Compensating and equivalent variations

The terms compensating and equivalent variations, of such a frequent use in Public Economics, are typically applied to price changes in consumer choice models, where the consumer does not face any quantitative restriction on his decision variables. In this paper we propose an application of these same concepts to ration changes imposed by quantitative constraints on some of the consumer's decision variables. Our discussion will be framed in a deterministic, static consumer choice model.

In this section, after briefly recalling the compensating and equivalent variations concepts in their conventional price change use, we define their counterparts in a rationing setting. So, consider a consumer with preferences defined over two commodity vectors x and y of n and m elements, respectively, by the well-behaved direct utility function $U(x, y)$. Let p and q be the prevailing prices of those commodities, and R the consumer's exogenous income.

CHANGES IN PRICES

From a *primal* perspective, the consumer's *indirect utility function*

$$V(p, q, R) \equiv U[x^M(p, q, R), y^M(p, q, R)] = \max_{x,y} U(x, y) \quad \text{s.t.} \quad p\,x + q\,y = R. \tag{1}$$

summarises the consumer's choice problem. Recall that $x^M(\cdot)$ and $y^M(\cdot)$ are the *ordinary* demand functions for x and y, solution to the maximisation programme.

From a *dual* perspective, the consumer's *expenditure function*

$$e(p, q, u) \equiv [p\,x^H(p, q, u) + q\,y^H(p, q, u)] = \min_{x,y} p\,x + q\,y \quad \text{s.t.} \quad U(x, y) = u \tag{2}$$

provides also an alternative summary of the consumer's choice problem. Now $x^H(\cdot)$ and $y^H(\cdot)$ are the *compensated* demand functions for x and y, solution to the minimisation programme.

Both programmes lead to the same equilibrium, provided that $u = V(p, q, R)$. In fact, the following identities hold:

$$\begin{aligned} &[1] \quad x^M(p, q, R) \equiv x^H(p, q, u) \\ &[2] \quad y^M(p, q, R) \equiv y^H(p, q, u) \\ &[3] \quad R \equiv e(p, q, u) \end{aligned} \tag{3}$$

Consider now that q changes to q^1. For interpretative purposes and without loss of generality suppose a price increase so that $q^1 \geqslant q$, with at least one component strictly greater. Let $u^1 = V(p, q^1, R) < u$ be the utility level attained after the price change. In this context, the *compensating variation* (CV) associated with the price change is defined as the amount of income the consumer should receive to get, at the new prices, the same welfare level as before the price change. The indirect utility function translates this requirement into formal language

$$V(p, q^1, R + CV) = V(p, q, R) = u. \tag{4}$$

(4) provides an *implicit* definition of CV. The expression (3.3) tells us directly that R is the minimal expenditure required to reach u at the old prices: $R \equiv e(p, q, u)$. (3.3) allows us also to deduce that $R + \text{CV}$ is the minimal expenditure required to reach u at the new prices. i.e. a $R + \text{CV} = e(p, q^1, u)$. Eliminating R between these two equalities, yields an *explicit* expression for CV, namely

$$CV = e(p, q^1, u) - e(p, q, u). \tag{5}$$

As we all know, the *equivalent variation* (EV) associated with the price change is defined as the maximum amount of income the consumer is willing to pay so as to be free of the price change. Of course this maximum amount of income has to do with the welfare level attained by our consumer after the price increase. Using again the IUF, the EV is implicitly defined by

$$V(p, q, R - EV) = V(p, q^1, R) = u^1. \tag{6}$$

Again, the use of (3.3) permits to deduce from (6) an explicit definition of EV, to wit,

$$EV = e(p, q^1, u^1) - e(p, q, u^1). \tag{7}$$

Both (5) and (7) gives exact measures of CV and EV as the difference of the expenditure function evaluated at two price (and utility!) levels.

CHANGES IN QUANTITATIVE CONSTRAINTS

Consider now the situation where besides a budget constraint, the consumer suffers a binding quantitative restriction $\bar{y}$ on the y commodities he can buy. By binding we mean that at the prevailing prices (p, q) and income R our consumer would be willing to buy more y than what he is allowed to, that is $y^M(p, q, R) \geq \bar{y}$. From the *primal* perspective, the presence of $\bar{y}$ leads to the *constrained indirect utility function* (CIUF):

$$V_c(p, q, R, \bar{y}) \equiv U[x_c^M(p, q, R, \bar{y}), \bar{y}] = \max_x U(x, \bar{y}) \; s.t. \; p\,x + q\,\bar{y} = R, \tag{8}$$

which now summarises the consumer's choice problem. Notice that $x_c^M(\cdot)$ is the constrained vector of *ordinary* demand functions for the free decision variables, x, solution of the maximisation programme. Observe also that both $x_c^M(\cdot)$ and $V_c(\cdot)$ internalise the constraint $\bar{y}$. The subscript c stands for constrained.

From a *dual* perspective, the consumer's *constrained expenditure function* (CEF)

$$e_c(p, q, u, \bar{y}) \equiv [p\,x_c^H(p, u, \bar{y}) + q\,\bar{y}] = \min_x p\,x + q\,\bar{y} \quad \text{s.t. } U(x, \bar{y}) = u \tag{9}$$

gives an alternative summary of the consumer's choice problem. Now $x_c^H(p, u, \bar{y})$ is the constrained vector of *compensated* demand functions for the free decision variables, x, solution of the minimisation programme. Notice also that $x_c^H(\cdot)$ is independent of the rationed commodity prices q. Finally observe that both $x_c^H(\cdot)$ and $e_c(\cdot)$ internalise the constraint $\bar{y}$.

Both programmes lead to the same equilibrium, provided $u = V(p, q, R, \bar{y})$. In fact, the following identities hold:

$$\begin{aligned} &[1] \quad x_c^M(p, q, R, \bar{y}) \equiv x_c^H(p, u, \bar{y}) \\ &[2] \quad R \equiv e_c(p, q, u, \bar{y}) \end{aligned} \tag{10}$$

Consider now that $\bar{y}$ changes to $\bar{y}^1$. For interpretative purposes and without loss of generality suppose a ration decrease so that $\bar{y}^1 \leq \bar{y}$, with at least one component strictly lower. Let $u^1 = V(p, q, R, \bar{y}^1) < u$ be the utility level attained after the ration change. In this context, the *compensating variation* (CV) associated with the *ration change* is defined as the amount of income the consumer should receive to get, at the new rations, the same welfare level as before the ration change. The constrained indirect utility function translates this requirement into formal language

$$V_c(p, q, R + CV, \bar{y}^1) = V_c(p, q, R, \bar{y}) = u \qquad (11)$$

(11) gives an *implicit* definition of CV. The expression (10.2) directly tells us that R is the minimal expenditure required to reach u at the old ration: $R \equiv e_c(p, q, u, \bar{y})$, and also allows us to deduce that $R + \mathrm{CV}$ is the minimal expenditure required to reach the same welfare u at the new ration, that is $R + \mathrm{CV} = e_c(p, q, u, \bar{y}^1)$. Eliminating R between these two equalities, yields an *explicit* expression for CV , namely

$$CV = e_c(p, q, u, \bar{y}^1) - e_c(p, q, u, \bar{y}). \qquad (12)$$

The *equivalent variation* (EV) associated with the *ration change* is defined as the maximum amount of income the consumer is willing to pay so as to be free of the ration change. Of course this maximal income has to do with the lower welfare level, u^1 , attained by our consumer after the ration decrease. Using again the CIUF, the EV is implicitly defined by

$$V_c(p, q, R - EV, \bar{y}) = V_c(p, q, R, \bar{y}^1) = u^1 \qquad (13)$$

Again, the use of (10.2) allows us to deduce from (13) an explicit definition of EV, namely

$$EV = e_c(p, q, u^1, \bar{y}^1) - e_c(p, q, u^1, \bar{y}). \qquad (14)$$

As in the case of price changes, both (12) and (14) gives exact measures of CV and EV as the difference now of the *constrained* expenditure function evaluated at two *ration* (and utility!) levels.

REMARK. If we let the initial constraint, $\bar{y}$, be such that $\bar{y} = y^M(p, q, R) \equiv y^H(p, q, u)$, (12) and (14) can be interpreted as the compensating and equivalent variations associated with the *introduction* of the quantiative constraint $\bar{y}^1$, in a previously unconstrained setting.

VIRTUAL PRICES

Analitically the CV and EV measures just proposed are well defined and can be computed either through the CIUF or through the CEF. The use of *virtual* prices, i.e. prices permitting the free choice of $\bar{y}$, provides alternative computation methods. The primal approach requires computing virtual prices and incomes. The dual approach, more desirable whenever two or more prices change ($m \geq 2$, in our case) since it is independent from the order in which the prices change, only requires computing virtual prices.

Following the dual approach, Neary and Roberts (1980) derived the properties of both $x_c^H(p, u, \bar{y})$ and $e_c(p, q, u, \bar{y})$ from their unconstrained counterparts, using as link a vector of *virtual* prices, $\bar{q}$, allowing the free choice of the constrained vector $\bar{y}$. In the present context, $\bar{q}$ is implicitly defined by the equality

$$y^H(p, \bar{q}, u) = \bar{y}. \qquad (15)$$

This implies

$$x_c^H(p, u, \bar{y}) = x^H(p, \bar{q}, u) \qquad (16)$$

and

$$e_c(p, q, u, \bar{y}) = e(p, \bar{q}, u) + (q - \bar{q})\,\bar{y}. \qquad (17)$$

In the following two sections, we apply all these methods to compute the compensating and equivalent variations associated with a credit constraint, section 3, and a labour constraint, section 4. In the first case we restrict our analysis to a two-commodity environment ($n = m = 1$). In the second case, we begin with two commodities ($n = m = 1$) and finish with $n + 1$ commodities ($n > m = 1$). Interesting examples with more than one constrained commodity ($m \geq 2$) will have to remain in the agenda of future research. This, of course, does not limit the interest of our proposal which is rather general.

3. Rationing credit demand

In this section we examine in some depth the compensating and equivalent variations associated with the imposition and the change of a quantitative constraint on the demand for credit, in a standard, deterministic, static model. A detailed exposition of the constrained and unconstrained relationships from both the primal and the dual perspectives are presented and discussed.

PRELIMINARIES. For illustrative purposes we use the following specific example:

Cobb-Douglas utility function: $U(Y_1 + D, C_2) = (Y_1 + D)^{1/2}\, C_2^{1/2}$
Parameters: $(p_1, Y_1, Y_2) = (1.1, 5, 100)$, $\Rightarrow r = 10\,\%$.

PRIMAL PROBLEM. Suppose that our consumer is a borrower that lives during two periods. He chooses the consumption plan that adapts best to his pattern of income perception, given the interest rate. More formally, he solves the problem:

$$\max_{C_1, C_2} \mathcal{U}(C_1, C_2)\ s.t. \tag{18}$$

$$C_1 = Y_1 + D \tag{19}$$

$$C_2 = Y_2 - D(1 + r), \tag{20}$$

where C_t (resp. Y_t) denotes consumption (resp, exogenous income) in period t, $t = 1, 2$, and D stands for debt or credit (minus savings).

Since the quantitative constraint will bear on debt, it is convenient to reformulate the problem so as to make D a decision variable. This is done by substituting (19) into (18). The previous problem reduces to choosing D and C_2 so as to

$$\max_{D, C_2} U(D, C_2) \equiv \mathcal{U}(Y_1 + D, C_2)\ s.t.\ (3) \tag{21}$$

With the price of future consumption normalised to unity and denoting $p_1 = (1 + r)$ the price of present consumption , (21) solves for an ordinary demand for debt and an ordinary demand for second period consumption $\{D^M(p_1, Y_1, Y_2), C_2^M(p_1, Y_1, Y_2)\}$, which, replaced in the objective function, gives the indirect utility function

$$V(p_1, Y_2) \equiv U(D^M(p_1, Y_2), C_2^M(p_1, Y_2)). \tag{22}$$

In our example, we have

$$\begin{aligned}
&(1)\; D^M(p_1, Y_1, Y_2) = \frac{1}{2}\left(\frac{Y_2}{p_1} - Y_1\right),\\
&(2)\; C_2^M(p_1, Y_1, Y_2) = \frac{p_1\, Y_1 + Y_2}{2},\\
&(3)\; V(p_1, Y_1, Y_2) = \frac{Y_1\, p_1 + Y_2}{2\, p_1^{1/2}},
\end{aligned} \tag{23}$$

where Y_1 will play no active role in the subsequent analysis. It has been chosen low enough to force a borrower behaviour to our consumer. Notice that $D^M(\cdot) > 0$ provided $Y_2 > p_1\, Y_1$. For the chosen parameters, $(p_1, Y_1, Y_2) = (1.1, 5, 100)$, this requirement is perfectly well satisfied. Observe also that both commodities are normal with respect to second period income.

For the given parameters, equations (23) yield the *initial equilibrium and utility level*:

$$\{D_0, C_{2\,0}, u_0\} = \left\{\frac{945}{22}, \frac{211}{4}, \left(\frac{222605}{88}\right)^{1/2}\right\} \simeq \{42.95, 52.75, 50.29\} \tag{24}$$

This initial equilibrium corresponds to point A in Figure 1, which portraits the primal problem. The thick curve, denoted u_0, represents the indifference curve $U(D, C_2) \equiv (Y_1 + D)^{1/2}\, C_2^{1/2} = u_0$, whereas the thick inclined line illustrates the budget constraint (20). Besides this, we have drawn a thick vertical line for the constraint $\overline{D}_1 = 20$, and two points, B and C, representing the equilibria with and without compensation, respectively, attainable after the imposition of $\overline{D}_1$. Point C is crossed by a lower indifference curve, denoted u_1.

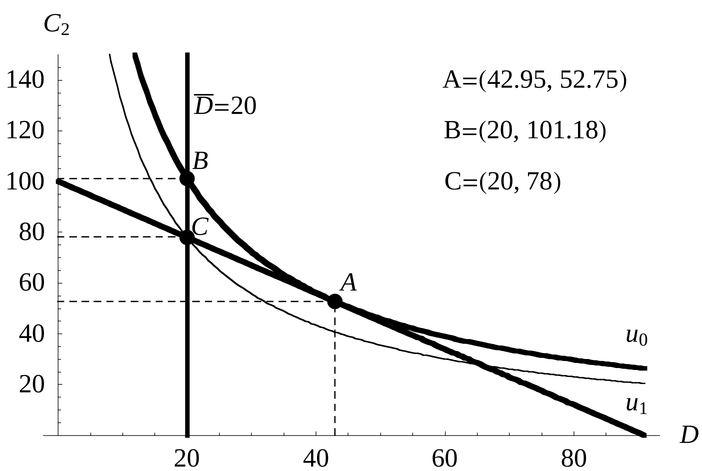

Figure 1. Borrower's equilibria.

CONSTRAINED PRIMAL PROBLEM Suppose now that our consumer faces a restriction on the maximum amount he can borrow, say $D \leq \overline{D}$ assumed to be binding. The consumer chooses C_2 so as to

$$\max_{C_2}\; U(\overline{D}, C_2)\; s.t.\; C_2 = Y_2 - p_1\, \overline{D} \tag{25}$$

The budget constraint solves for the constrained ordinary demand function for second period consumption $C_{2c}^M(p_1, Y_2, \overline{D}) = Y_2 - p_1\, \overline{D}$, which, replaced in the objective function, gives the constrained indirect utility function (CIUF)

$$V_c(p_1, Y_1, Y_2, \overline{D}) \equiv U(\overline{D}, C^M_{2c}(p_1, Y_2, \overline{D})) \tag{26}$$

In our example, we have

$$\begin{aligned} &(1)\ C^M_{2c}(p_1, Y_2, \overline{D}) = Y_2 - p_1 \overline{D} \\ &(2)\ V_c(p_1, Y_1, Y_2, \overline{D}) = (Y_1 + \overline{D})^{1/2} (Y_2 - p_1 \overline{D})^{1/2} \end{aligned} \tag{27}$$

The utility level reached by the borrower corresponds to the constrained equilibrium and is therefore lower than that attained without the credit constraint (see Figure 2 below). Thus taking $\overline{D} = \overline{D}_1 = 20 < D_0$, leads to the *constrained equilibrium* and *utility level* (see point C in Figure 1):

$$\{D_1, C_{21}, u_1\} = \{20, 78, 1950^{1/2} \simeq 44.16\} \tag{28}$$

For the given parameters, the CIUF $V_c(11/10, 5, 100, \overline{D})$ is a concave function of $\overline{D}$, attaining a maximum at $\overline{D}_0 \equiv 945/22$ of $u_0 = \sqrt{222605/88}$. It ceases to be real-valued in the interval $\overline{D} \in (90, 91]$, explaining why it does not reach the $\overline{D}$ axis.

In Figure 2, V_c is shown together with the points:

$$\begin{aligned} &(\overline{D}_0, u_0) = \left(945/22, \sqrt{222605/88}\right), \\ &(\overline{D}_1, u_1) = \left(20, \sqrt{1950}\right) \\ &(\overline{D}_2, u_2) = \left(30, \sqrt{2345}\right) \end{aligned}$$

where the latter will be used below. Notice that the point $(\overline{D}_0, u_0)$ would be attained by the borrower in the absence of credit constraints. In other words, $(\overline{D}_0, u_0) = (D_0, u_0) \equiv (D^M(p_1, Y_1, Y_2), V(p_1, Y_1, Y_2))$. As a consequence, the points of V_c to the right of $(\overline{D}_0, u_0)$ are not valid since they violate the rationing condition $D_0 \geq \overline{D}$.

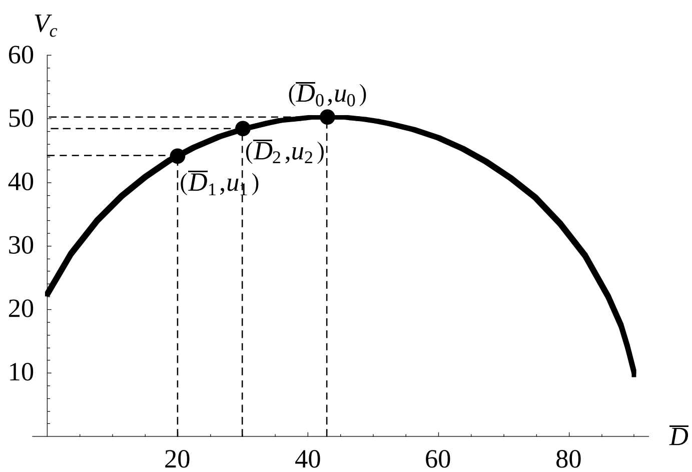

Figure 2. The CIUF as a function of the credit constraint. The lower the ration the lower the utility reached.

DUAL PROBLEM. The consumer chooses (D, C_2) so as to minimise expenditure and keep utility at an exogenously given utility level, u:

$$\min_{D, C_2}\ p_1 D + C_2\ \ s.t.\ U(D, C_2) \geq u \tag{29}$$

Notice that we are using second period income to compensate (see the budget constraint (20)). (29) solves for the compensated demands for debt and second period consumption $\{D^H(p_1, u), C_2^H(p_1, u)\}$, which replaced in the objective function yields the *expenditure function*

$$e(p_1, u) \equiv p_1 D^H(p_1, u) + C_2^H(p_1, u) \tag{30}$$

In our example, we have

$$\begin{aligned}&(1)\ D^H(p_1, Y_1, u) \equiv p_1^{-1/2} u - Y_1\\&(2)\ C_2^H(p_1, u) = p_1^{1/2} u\\&(3)\ e(p_1, Y_1, u) \equiv 2 p_1^{1/2} u - p_1 Y_1\end{aligned} \tag{31}$$

For the given parameters and $u = u_0$ one obtains the following identities (see point A in Figure 1):

$$\begin{aligned}&(1)\ D^H(p_1, u_0) \equiv D^M(p_1, Y_1, Y_2) = 945/22 \simeq 42.95\\&(2)\ C_2^H(p_1, u_0) \equiv C_2^M(p_1, Y_1, Y_2) = 211/4 \simeq 52.75\\&(3)\ e(p_1, u_0) \equiv Y_2 = 100\end{aligned} \tag{32}$$

CONSTRAINED DUAL PROBLEM. Suppose now that our consumer faces a restriction on the maximum amount he can borrow, say $D \leq \overline{D}$ assumed to be binding. The consumer chooses C_2 so as to

$$\min_{C_2}\ p_1 \overline{D} + C_2 \ \text{ s.t. } U(\overline{D}, C_2) = u \tag{33}$$

The utility constraint determines the constrained compensated demand function for second period consumption, $C_{c2}^H(u, \overline{D})$, which replaced in the objective function gives the *constrained expenditure function* (CEF)

$$e_c(p_1, u, \overline{D}) \equiv p_1 \overline{D} + C_{2c}^H(u, \overline{D}) \tag{34}$$

In our example we have

$$\begin{aligned}&(1)\ C_{2c}^H(u, Y_1, \overline{D}) = u^2/(Y_1 + \overline{D})\\&(2)\ e_c(p_1, Y_1, u, \overline{D}) = p_1 \overline{D} + u^2/(Y_1 + \overline{D})\end{aligned} \tag{35}$$

The CEF is increasing and convex in u, and convex in $\overline{D}$. In the example, and for the given (p_1, Y_1), $e_c(p_1, Y_1, u, \overline{D})$, attains a minimum at $\overline{D}_{\min}(u) = \left(\sqrt{110}\, u - 55\right)/11$, see Figure 4 below.

By evaluating (35) for the given parameters, constraint $\overline{D} = \overline{D}_1 = 20$ and utility levels u_0 and u_1 we get the equilibria (see points B and C in Figure 1) and constrained expenditures:

$$\begin{aligned}&(1)\ \{\overline{D}_1, C_{2c}^H(u_0, \overline{D}_1), e_c(p_1, u_0, \overline{D}_1)\} = \left\{20, \frac{44521}{440}, \frac{54201}{440}\right\} \simeq \{20, 101.18, 123.18\}\\&(2)\ \{\overline{D}, C_{2c}^H(u_1, \overline{D}), e_c(p_1, u_1, \overline{D})\} \simeq \{20, 78, 100\}\end{aligned} \tag{36}$$

The comparison of (28) and (36.2) provides us with the identities

$$\begin{aligned}&(1)\ C_{2c}^H(u_1, \overline{D}_1) \equiv C_{2c}^M(p_1, Y_2, \overline{D}_1) = 78\\&(2)\ e_c(p_1, u_1, \overline{D}_1) \equiv Y_2 = 100\end{aligned} \tag{37}$$

Note, however, that

$$\begin{aligned}&(1)\ C_{2c}^H(u_0, \overline{D}_1) \simeq 101.18 \neq 78 = C_{2c}^M(p_1, Y_2, \overline{D}_1)\\&(2)\ e_c(p_1, u_0, \overline{D}_1) \simeq 123.18 \neq Y_2\end{aligned} \tag{38}$$

What does hold is the identity

$$C^H_{2c}(u_0, \overline{D}_1) \equiv C^M_{2c}(p_1, e_c(p_1, u_0, \overline{D}_1), \overline{D}_1) \qquad (39)$$

In effect, from (27.1) $C^M_{2c}(p_1, e_c(p_1, u_0, \overline{D}_1), \overline{D}_1) = e_c(p_1, u_0, \overline{D}_1) - p_1 \overline{D}_1 = \frac{54201}{440} - \frac{11}{10} 20 = \frac{44521}{440}$ which is $C^H_{2c}(u_0, \overline{D}_1)$.

The previous identity is important as it permits to obtain one of the central results in rationing theory, to wit, a kind of "Slutsky" equation for an infinitesimal change in the ration at $(u_0, \overline{D}_1)$:

$$\frac{\partial C^M_{2c}}{\partial \overline{D}} = \frac{\partial C^H_{2c}}{\partial \overline{D}} - (p_1 - \overline{p}_1) \frac{\partial C^M_{2c}}{\partial Y_2} \qquad (40)$$

where $\overline{p}_1$ is a virtual price verifying $D^H(\overline{p}_1, u_0) = \overline{D}_1$. According to (40), the total effect of a ration change is made up of a substitution effect and an income effect.

VIRTUAL PRICES

Primal perspective. Our borrower would freely choose the constrained equilibrium $(D, C_2) = \{\overline{D}, Y_2 - p_1 \overline{D}\}$ if he faced the *virtual price and* (second period) *income* $\{\tilde{p}_1, \tilde{Y}_2\}$ implicitly defined by

$$\begin{aligned} &(1)\ D^M(\tilde{p}_1, Y_1, \tilde{Y}_2) = \overline{D} \\ &(2)\ C^M_2(\tilde{p}_1, Y_1, \tilde{Y}_2) = Y_2 - p_1 \overline{D} \end{aligned} \qquad (41)$$

In our example these equations are

$$\begin{aligned} &(1)\ \frac{1}{2}\left(\frac{\tilde{Y}_2}{\tilde{p}_1} - Y_1\right) = \overline{D} \\ &(2)\ \frac{\tilde{p}_1 Y_1 + \tilde{Y}_2}{2} = Y_2 - p_1 \overline{D} \end{aligned} \qquad (42)$$

and solve for

$$\begin{aligned} &(1)\ \tilde{p}_1 = \left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right) \\ &(2)\ \tilde{Y}_2 = \left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right)(2\overline{D} + Y_1) \end{aligned} \qquad (43)$$

Plugging (43) into the IUF yields the CIUF, that is

$$V(\tilde{p}_1, Y_1, \tilde{Y}_2) = V_c(p_1, Y_1, Y_2, \overline{D}) \qquad (44)$$

Proof. See Appendix.

For the given parameters and $\overline{D} = \overline{D}_1 = 20$, we obtain $\{\tilde{p}_1, \tilde{Y}_2\} = \{78/25, 702/5\} \simeq \{3.12, 140.4\}$. Both indirect utility functions in (44) lead to $u_1 = \sqrt{1950}$ for $\overline{D} = \overline{D}_1 = 20$. We are getting, therefore, point C in Figure 1. See also Figure 3 below.

Dual perspective. Our borrower would freely choose the constrained equilibrium $(D, C_2) = \{\overline{D}, Y_2 - p_1 \overline{D}\}$, providing the utility level u, if he faced the *virtual price* $\overline{p}_1$ implicitly defined by

$$D^H(\overline{p}_1, u) = \overline{D} \qquad (45)$$

In our example this equation is, from (31.1), $\overline{p}_1^{-1/2}\, u - Y_1 = \overline{D}$, and solves for

$$\overline{p}_1 = \left(\frac{u}{\overline{D} + Y_1}\right)^2 \tag{46}$$

For the given parameters and $\overline{D} = \overline{D}_1 = 20$, we have $\overline{p}_1(u_0, \overline{D}_1) = 44521/11000 \simeq 4.04736$ and $\overline{p}_1(u_1, \overline{D}_1) = 78/25 = 3.12$. Notice that $\overline{p}_1(u_1, \overline{D}_1) = \tilde{p}_1$. See Figure 3 below.

Adding $(p_1 - \overline{p}_1)\overline{D}$ to the expenditure function (31.3), using (46), proves for our particular example a general relationship between the constrained and unconstrained expenditure functions established by Neary and Roberts. In the present application to credit rationing reads:

$$e_c(p_1, u, \overline{D}) = e(\overline{p}_1, u) + (p_1 - \overline{p}_1)\overline{D}. \tag{47}$$

with $\overline{p}_1$ implicitly defined by (45).

Proof. See appendix.

What makes (47) interesting is the possibility of deriving the properties of its LHS from its RHS. In particular,

$$\frac{\partial e_c}{\partial \overline{D}} = \left(\frac{\partial e}{\partial \overline{p}_1} - \overline{D}\right)\frac{\partial \overline{p}_1}{\partial \overline{D}} + (p_1 - \overline{p}_1) = (p_1 - \overline{p}_1), \tag{48}$$

where the second equality obtains using Shephard's lemma and (45). This permits to compute the difference $e_c(p_1, u, \overline{D}_1) - e_c(p_1, u, \overline{D}_0)$ as the integral

$$\begin{aligned} &e_c(p_1, u, \overline{D}_1) - e_c(p_1, u, \overline{D}_0) \\ &= \int_{\overline{D}_0}^{\overline{D}_1} (p_1 - \overline{p}_1(u, \overline{D}))\, d\overline{D} \\ &= \int_{\overline{D}_1}^{\overline{D}_0} \overline{p}_1(u, \overline{D})\, d\overline{D} - p_1(\overline{D}_0 - \overline{D}_1) \end{aligned} \tag{49}$$

where $\overline{p}_1(u, \overline{D})$ is the inverse of the compensated demand curve for credit.

Figure 3 illustrates the primal and dual approaches to the search of virtual prices for a credit constraint of $\overline{D} = \overline{D}_1 = 20$. Notice that $D^M(p_1, Y_1, \tilde{Y}_2) = D^H(p_1, u_1)$ at $p_1^* = 78/25 = 3.12$. As previously mentioned, this is precisely the price satisfying $\overline{p}_1(u_1, \overline{D}_1) = \tilde{p}_1$.

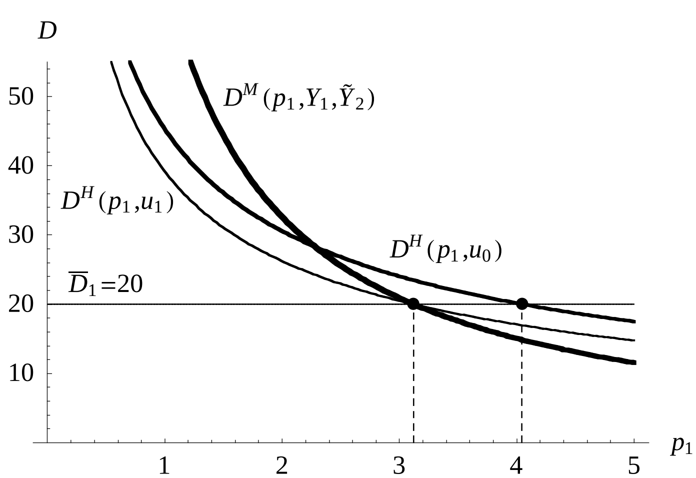

Figure 3. Virtual prices for $\overline{D} = \overline{D}_1 = 20$.

COMPENSATING AND EQUIVALENT VARIATIONS

We have now a a wealth of methods to compute the compensating and equivalent variations associated with the imposition or the change of a quantitative constraint on credit demand. To that effect, we consider two cases. In the first case, called *introduction*, the credit constraints decreases from the initial value $\overline{D}_0 = 945/22 \simeq 42.95$ to the final value $\overline{D}_1 = 20$. As $\overline{D}_0$ is also the amount of credit D_0 freely chosen by our borrower for the given prices and incomes, the corresponding compensating and equivalent variations can be interpreted as those associated with the imposition of the binding credit constraint $\overline{D}_1$ to a previously unconstrained borrower. In the second case, called *change*, we take as the *initial* constraint, $\overline{D} = \overline{D}_2 \equiv 30$, and ask for the compensating and equivalent variations associated with a decrease in $\overline{D}$ from $\overline{D}_2$ to the same final constraint $\overline{D}_1$. Both measures can be computed indistinctly as follows:

- Via constrained indirect utility function.

Introduction

The *compensating variation* is implicitly defined by $V_c(p_1, Y_1, Y_2, \overline{D}_0) = V_c(p_1, Y_1, Y_2 + \text{CV}, \overline{D}_1) = u_0$. As we now that $V_c(p_1, Y_1, Y_2, \overline{D}_0) = u_0$, CV obtains as the solution of the second equality, namely $V_c(11/10, 5, 100 + \text{CV}, 20) = \sqrt{222605/88}$, which yields $\boxed{\text{CV} = 10201/440} \simeq 23.1841$.

The *equivalent variation* is implicitly defined by $V_c(p_1, Y_1, Y_2, \overline{D}_1) = V_c(p_1, Y_1, Y_2 - \text{EV}, \overline{D}_0) = u_1$. As we now that $V_c(p_1, Y_1, Y_2, \overline{D}_1) = u_1$, EV obtains as the solution of the second equality, namely $V_c(11/10, 5, 100 - \text{EV}, 945/22) = \sqrt{1950}$, which gives $\boxed{\text{EV} = 10201/844} \simeq 12.0865$.

Change

The *compensating variation* is implicitly defined by $V_c(p_1, Y_1, Y_2, \overline{D}_2) = V_c(p_1, Y_1, Y_2 + \text{CV}, \overline{D}_1) = u_2$. Now we have to compute, via CIUF, the utility level corresponding to the new initial debt constraint $\overline{D}_2$, namely $u_2 = V_c(11/10, 5, 100, 30) = \sqrt{2345} \simeq 48.4252$. Then we use u_2 to solve the second equality, $V_c(11/10, 5, 100 + \text{CV}, 20) = \sqrt{2345}$, for CV. This gives $\boxed{\text{CV} = 79/5} = 15.8$.

The *equivalent variation* is implicitly defined by $V_c(p_1, Y_1, Y_2, \overline{D}_1) = V_c(p_1, Y_1, Y_2 - \text{EV}, \overline{D}_2) = u_1$. Since $V_c(p_1, Y_1, Y_2, \overline{D}_1) = u_1$, EV obtains as the solution of the second equality, namely $V_c(11/10, 5, 100 - \text{EV}, 30) = \sqrt{1950}$. This gives $\boxed{\text{EV} = 79/7} \simeq 11.2857$

- Via constrained expenditure function.

Introduction

$\text{CV} = e_c(p_1, Y_1, u_0, \overline{D}_1) - e_c(p_1, Y_1, u_0, \overline{D}_0) = 54201/440 - 100 = \boxed{10201/440} \simeq 23.1841$.

$\text{EV} = e_c(p_1, Y_1, u_1, \overline{D}_1) - e_c(p_1, Y_1, u_1, \overline{D}_0) = 100 - 74199/844 = \boxed{10201/844} \simeq 12.0865$.

Notice that $e_c(p_1, Y_1, u_0, \overline{D}_0) = e_c(p_1, Y_1, u_1, \overline{D}_1) = 100 = Y_2$

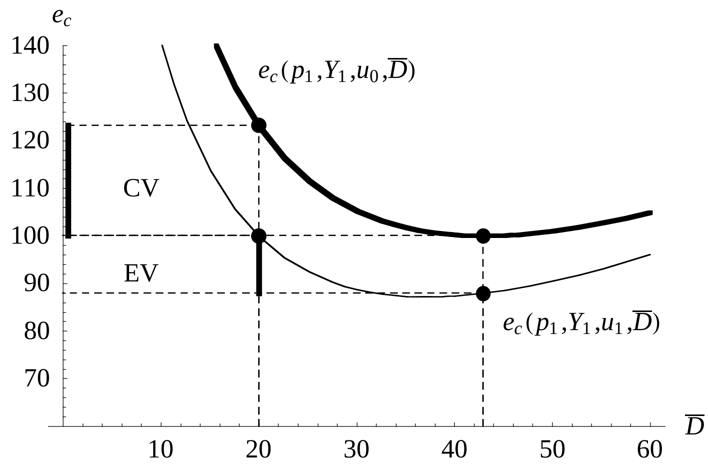

Figure 4. Two CEFs evaluated at u_0 and u_1, respectively, as a function of the credit constraint. The CV is the vertical distance between $e_c(p_1, Y_1, u_0, \overline{D}_1)$ and $e_c(p_1, Y_1, u_0, \overline{D}_0)$. The EV is the vertical distance between $e_c(p_1, Y_1, u_1, \overline{D}_1)$and $e_c(p_1, Y_1, u_1, \overline{D}_0)$.

Change

$\text{CV} = e_c(p_1, Y_1, u_2, \overline{D}_1) - e_c(p_1, Y_1, u_2, \overline{D}_2) = 579/5 - 100 = \boxed{79/5} = 15.8$

$\text{EV} = e_c(p_1, Y_1, u_1, \overline{D}_1) - e_c(p_1, Y_1, u_1, \overline{D}_2) = 100 - 621/7 = \boxed{79/7} \simeq 11.2857.$

Notice that $e_c(p_1, Y_1, u_2, \overline{D}_2) = e_c(p_1, Y_1, u_1, \overline{D}_1) = 100 = Y_2$.

- Via virtual prices from a dual perspective (introduction case)

$$\begin{aligned} \text{CV} &= \int_{\overline{D}_1}^{\overline{D}_0} \overline{p}_1(u_0, \overline{D})\, d\overline{D} - p_1(\overline{D}_0 - \overline{D}_1) \\ &= \int_{\overline{D}_1}^{\overline{D}_0} \left(\frac{u_0}{\overline{D} + Y_1}\right)^2 d\overline{D} - p_1(\overline{D}_0 - \overline{D}_1) \\ &= \int_{20}^{945/22} \left(\frac{222605}{88\,(5 + \overline{D})^2}\right) d\overline{D} - \frac{11}{10}\left(\frac{945}{22} - 20\right) = \frac{10201}{440} \simeq 23.1841 \end{aligned} \tag{50}$$

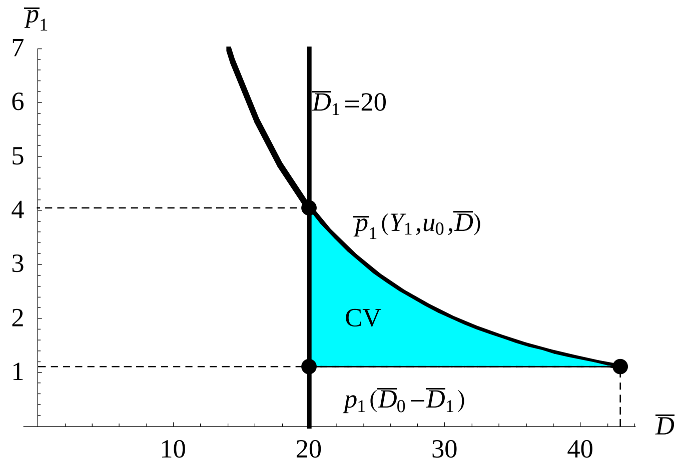

Figure 5.1. CV as a portion of the area under the inverse compensated demand curve for debt.

$$\begin{aligned} \text{EV} &= \int_{\overline{D}_1}^{\overline{D}_0} \overline{p}_1(u_1, \overline{D})\, d\overline{D} - p_1(\overline{D}_0 - \overline{D}_1) \\ &= \int_{\overline{D}_1}^{\overline{D}_0} \left(\frac{u_1}{\overline{D}+Y_1}\right)^2 d\overline{D} - p_1(\overline{D}_0 - \overline{D}_1) \\ &= \int_{20}^{945/22} \left(\frac{1950}{(5+\overline{D})^2}\right) d\overline{D} - \frac{11}{10}\left(\frac{945}{22} - 20\right) = \frac{10201}{844} \simeq 12.0865 \end{aligned} \tag{51}$$

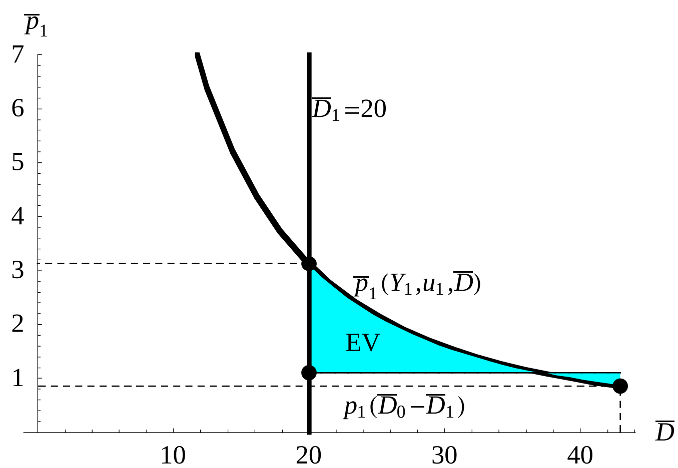

Figure 5.2. EV as an irregular portion of the area under the inverse compensated demand curve for debt.

4. Rationing labour supply

In this section we develop an exact measure of underemployment compensation allowing an underemployed individual to enjoy the same welfare level as an employed one. A similar analysis applies to the compensation to be given to individuals involved in a reduction of a compulsory working time.

Consider a consumer-worker with preferences defined over n goods and work by the well behaved utility function $U(x, \ell)$. Let (p, w) denote the vector of prices of the n goods and the scalar wage rate, respectively. His *expenditure function* is defined as follows:

$$\begin{aligned} e(p, w, u) &\equiv p\, x^H(p, w, u) - w\, \ell^H(p, w, u) \\ &= \min_{x,\ell}\; p\, x - w\, \ell \;\; s.t.\;\; U(x, \ell) = u, \end{aligned} \qquad (52)$$

where $x^H(\cdot)$ represents the vector of Hicksian or compensated demands for the n goods and $\ell^H(\cdot)$ the compensated supply of labour.

Suppose now that at the prevailing prices (p, w) our consumer-worker is unable to sell as much labour as he wants to keep utility at the level u and can only work for $\bar{\ell}$ units of time. That is, $\ell^H(p, w, u) \geq \bar{\ell}$. His *constrained expenditure function* internalises the labour ration $\bar{\ell}$ and becomes

$$\begin{aligned} e_c(p, w, u, \bar{\ell}) &\equiv p\, x_c^H(p, u, \bar{\ell}) - w\, \bar{\ell} \\ &= \min_{x}\; p\, x - w\, \bar{\ell} \;\; s.t.\;\; U(x, \bar{\ell}) = u, \end{aligned} \qquad (53)$$

where subscript c stands for constrained. Notice that $x_c^H(\cdot)$ is independent of w.

A relationship between both expenditure functions can be established by invoking the notion of *virtual* wage rate. This term, coined by Rothbarth (1940-41), refers to that wage rate which would induce an unconstrained individual to supply the ration level $\bar{\ell}$. Denoting it by $\overline{w}$, the *virtual* wage rate is implicitly defined by

$$\ell^H(p, \overline{w}, u) = \bar{\ell} \qquad (54)$$

and implies

$$x^H(p, \overline{w}, u) = x_c^H(p, u, \bar{\ell}) \qquad (55)$$

Notice that $\overline{w}$ is an implicit function of p, u and $\bar{\ell}$, and that (54) provides an easy way to compute it.

Using (54) and (55) in (53) leads to the announced relationship between both expenditure functions, namely

$$e_c(p, w, u, \bar{\ell}) = e(p, \overline{w}, u) - (w - \overline{w})\, \bar{\ell} \qquad (56)$$

The properties of the constrained expenditure function, $e_c(\cdot)$, may be derived either as a direct application of the envelope theorem (see appendix) or better [1] from the properties of the unconstrained expenditure function, $e(\cdot)$, by using equation (56). They are

$$\partial e_c / \partial p_i = x_{ic}^H, \quad i = 1, \ldots, n \qquad (57)$$

$$\partial e_c / \partial w = -\bar{\ell} \qquad (58)$$

$$\partial e_c / \partial u = \lambda_c^H \qquad (59)$$

$$\partial e_c / \partial \bar{\ell} = -(w - \overline{w}) \qquad (60)$$

Our comments concentrate on (60) which gives a precise measure of the *benefit* (resp. cost) to the household of an increase (resp. decrease) in $\bar{\ell}$: a small increase in the amount of $\bar{\ell}$ reduces the expenditure required to attain the same utility level u by the difference between the virtual and the actual price of ℓ. The fact that $w > \overline{w}$ reflects *involuntary underemployment*. Integrating over (60) provides an exact measure of "true" un(der)employment compensation.

Let $\bar{\ell}^*$ be the number of units of time our worker would freely choose to supply at (p, w, u), that is $\bar{\ell}^* = \ell^H(p, w, u)$. The *underemployment compensation*, $b(\bar{\ell})$, is obtained as the difference of the constrained expenditure function evaluated at $\bar{\ell} \in (0, \bar{\ell}^*)$ and $\bar{\ell}^*$, and can be written, in view of (60), successively as follows:

$$
\begin{aligned}
b(\bar{\ell}) &= e_c(p, w, u, \bar{\ell}) - e_c(p, w, u, \bar{\ell}^*) \\
b(\bar{\ell}) &= \int_{\bar{\ell}^*}^{\bar{\ell}} \frac{\partial e_c(p, w, u, \bar{\ell})}{\partial \bar{\ell}}\, d\bar{\ell} \\
b(\bar{\ell}) &= \int_{\bar{\ell}}^{\bar{\ell}^*} (w - \overline{w}(\bar{\ell}))\, d\ell \\
b(\bar{\ell}) &= \int_{\bar{\ell}}^{\bar{\ell}^*} w\, d\bar{\ell} - \int_{\ell}^{\ell_0} \overline{w}(\bar{\ell})\, d\bar{\ell} \\
b(\bar{\ell}) &= w(\bar{\ell}^* - \bar{\ell}) - \int_{\bar{\ell}}^{\bar{\ell}^*} \overline{w}(\bar{\ell})\, d\bar{\ell}
\end{aligned}
\qquad (61)
$$

where $\overline{w}(\bar{\ell})$ results from (54).

The *unemployment compensation* obtains evaluating (61) at $\bar{\ell} = 0$, which leads to

$$
\begin{aligned}
b(0) &= e_c(p, w, u, 0) - e_c(p, w, u, \bar{\ell}^*) \\
b(0) &= w\,\bar{\ell}^* - \int_0^{\bar{\ell}^*} \overline{w}(\bar{\ell})\, d\bar{\ell}
\end{aligned}
\qquad (62)
$$

The un(der)employment compensation can thus be seen as a *compensating variation* associated *not* with a price change, but with a change of the ration level $\bar{\ell}$.

EXAMPLE

Suppose $n = 1$ and normalize p to unity. Change notation $Y = x$ so as to get the income(Y)-labor(ℓ) primal model [max $U(Y, \ell)$ s.t. $Y = w\,\ell + R$], where R stands for non-wage income. Assume $U(Y, \ell) \equiv 4\,Y^{1/2} + H - \ell$, with H denoting time endowment (and $H - \ell$ leisure time).

The primal model [$\max_{Y,\ell}$ $4\,Y^{1/2} + H - \ell$ s.t. $Y = w\,\ell + R$] leads to the ordinary demand for income, the ordinary supply of labor, and indirect utility function

$$
Y^M(w) = 4\,w^2, \;\; \ell^M(w, R) = 4\,w - R/w, \;\; V(w, R) = H + 4\,w + R/w \qquad (63)
$$

In what follows we take the parameters $\{H, R, w\} = \{52, 18, 9\}$, leading to the equilibrium and utility level $\{Y^*, \ell^*, u^*\} = \{324, 34, 90\}$.

The dual problem $\left[\min_{Y,\ell} \; Y - w\,\ell \;\; \text{s.t.}\; 4\,Y^{1/2} + H - \ell = u\right]$ leads to the compensated demand for income, the compensated supply of labor, and the expenditure function:

$$
\begin{aligned}
&Y^H(w) = 4\,w^2, \;\; \ell^H(w, u) = 8\,w + H - u, \\
&e(w, u) \equiv Y^H(w) - w\,\ell^H(w, u) = w\,(u - 4\,w - H)
\end{aligned}
\qquad (64)
$$

The dual problem $\left[\min_{Y} \; Y - w\,\bar{\ell} \;\text{ s.t. }\; 4\,Y^{1/2} + H - \bar{\ell} = u\right]$ leads to compensated constrained demand for income and CEF:

$$Y_c^H(w,\,u,\,\bar{\ell}) = \frac{1}{16}\,(u - H + \bar{\ell})^2$$
$$e_c(w,\,u,\,\bar{\ell}) \equiv Y_c^H(w,\,u,\,\bar{\ell}) - w\,\bar{\ell} = \frac{1}{16}\,(u - H + \bar{\ell})^2 - w\,\bar{\ell} \tag{65}$$

The CEF for $(w,\,u) = (9,\,90)$ becomes $e_c(9,\,90,\,\bar{\ell}) = \frac{1}{16}\,(\bar{\ell} + 38)^2 - 9\,\bar{\ell}$ and it is shown in Figure 6.

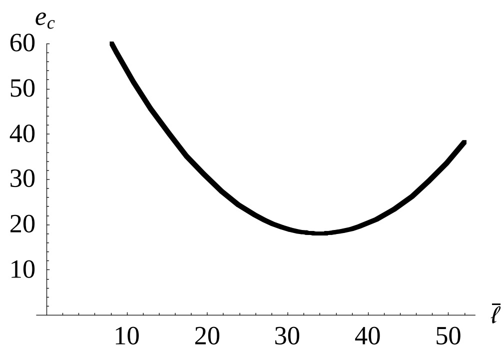

Figure 6. The CEF in terms of the labour ration $\bar{\ell}$.

The *virtual wage rate function* is defined implicitly by $\ell^H(w,\,u) = \bar{\ell}$ and explicitly by $\overline{w}(u,\,\bar{\ell}) \equiv (-H + u + \bar{\ell})/8 = (38 + \bar{\ell})/8$, the latter evaluated at $(H,\,u) = (52,\,90)$. Denoted $\overline{w}(\bar{\ell})$ in the un(der)employment compensation formulas, it appears in Figures 7 and 8 below as the increasing straight line.

The *un(der) employment compensation* expressed as $b(\bar{\ell}) = \int_{\bar{\ell}}^{\bar{\ell}^*} (w - \overline{w}(\bar{\ell}))\, d\,\bar{\ell}$. In Figure 2 we take the ration $\bar{\ell} = 21 \in [0,\,34]$. Recall that $(w,\,\bar{\ell}^*) = (9,\,34)$, we have $b(21) = \int_{21}^{34} w\,d\,l - \int_{21}^{34} \overline{w}(l)\,d\,l = 10.5625$.

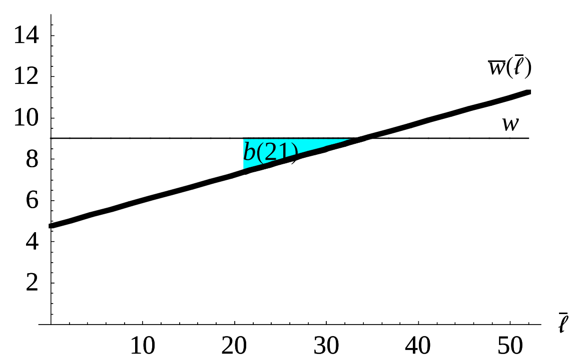

Figure 7. The underemployment compensation function $b(\bar{\ell})$ is the shaded area.

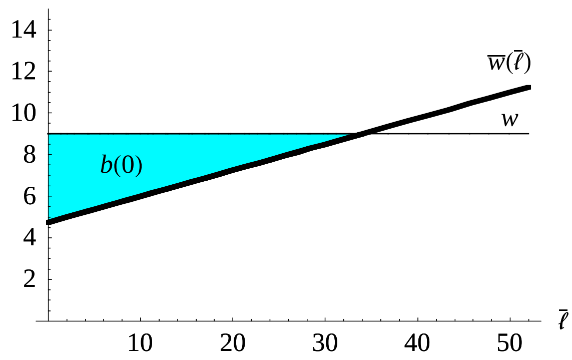

Figure 8. The unemployment compensation $b(0)$ is the shaded area.

The *un*(*der*) *employment compensation* computed as the difference between the CEF evaluated at any $\overline{\ell} \in [0, \overline{\ell}^*]$ and $\overline{\ell} = \overline{\ell}^*$: $b(\overline{\ell}) = e_c(w, u, \overline{\ell}) - e_c(w, u, \overline{\ell}^*) \ = \frac{1}{16}(\overline{\ell} - 34)^2$ for $(w, u, \overline{\ell}^*) = (9, 90, 34)$. This is shown in Figure 9.

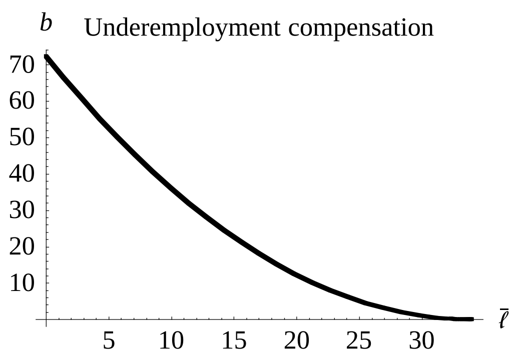

Figure 9. The underemployment compensation function $b(\overline{\ell})$.

([1]) The envelope theorem leads to the expression $\partial e_c / \partial \overline{\ell} = -(w - \tilde{w})$, where $\tilde{w} \equiv -\lambda_c^H (\partial U / \partial \overline{\ell})$ stands for a *reservation* wage rate. Clearly $\tilde{w}$ must equal $\overline{w}$, but this is not clear at all without making (54) and (55) explicit.

References

J.M Neary and K. J. Roberts (1980), "The theory of household behaviour under rationing", *European Economic Review*.

Appendix

Deriving (44). Plugging (43) into (27.2) yields successively:

$$V(\tilde{p}_1, Y_1, \tilde{Y}_2) = \frac{\tilde{p}_1 Y_1 + \tilde{Y}_2}{2\,\tilde{p}_1^{1/2}}$$

$$= \frac{\left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right) Y_1 + (2\overline{D} + Y_1)\left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right)}{2\left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right)^{1/2}}$$

$$= \frac{2(\overline{D} + Y_1)\left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right)}{2\left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right)^{1/2}}$$

$$= \frac{Y_2 - p_1 \overline{D}}{\left(\frac{Y_2 - p_1 \overline{D}}{Y_1 + \overline{D}}\right)^{1/2}}$$

$$= (Y_2 - p_1 \overline{D})^{1/2} (Y_1 + \overline{D})^{1/2}$$

$$= V_c(p_1, Y_1, Y_2, \overline{D})$$

Deriving (47). Adding $(p_1 - \overline{p}_1)\overline{D}$ to (35.2) yields successively:

$$\begin{aligned} & e(\overline{p}_1, u) + (p_1 - \overline{p}_1)\overline{D} \\ & = (2\,\overline{p}_1^{1/2} u - \overline{p}_1 Y_1) + (p_1 - \overline{p}_1)\overline{D} \\ & = p_1 \overline{D} + [2\,\overline{p}_1^{1/2} u - \overline{p}_1 (Y_1 + \overline{D})] \quad \text{and using } \overline{p}_1 = [u/(\overline{D} + Y_1)]^2 \\ & = p_1 \overline{D} + \left[2\left(\frac{u}{\overline{D} + Y_1}\right) u - \left(\frac{u}{\overline{D} + Y_1}\right)^2 (Y_1 + \overline{D})\right] \\ & = p_1 \overline{D} + \frac{2u^2}{\overline{D} + Y_1} - \frac{u^2}{\overline{D} + Y_1} \\ & = p_1 \overline{D} + \frac{u^2}{Y_1 + \overline{D}} \\ & = e_c(p_1, u, \overline{D}) \end{aligned} \tag{67}$$