APPROXIMATIONS TO THE AREA OF THE ELLIPSOID

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When, teaching the Calculus, I had to discuss the applications of Taylor's series, I often chose some approximation to the area of the ellipsoid as an instructive illustration, restricting myself to ellipsoids of revolution in more elementary classes. Little by little, I gathered various results of which I present a survey in the following, reserving the proofs for a more detailed publication. The excellent pedagogue to whom these pages are dedicated might be interested by some of these remarks which, as I said, originated in my classroom activity.

1. *The form of the approximations.* Let $E = E(a, b, c)$ denote the area of the surface of the ellipsoid with semiaxes a, b, c. Evidently, the function $E(a, b, c)$ has the following properties.

(I) It is defined, continuous, and non-negative for $a \geqq 0$, $b \geqq 0$, $c \geqq 0$, positive for $a > 0$, $b > 0$, $c > 0$.

(II) It is homogeneous of degree 2 in a, b, c.

(III) It is symmetric in a, b, c.

(IV) For $a = b = c$, its value is $4\pi a^2$.

It is reasonable to approximate E by functions of a, b, c, which share with E these properties. Such a function is

$$(1.1) \qquad P_{\lambda\mu\nu} = P_{\lambda\mu\nu}(a, b, c) =$$

$$= \frac{4\pi}{6}\left(a^\lambda b^\mu c^\nu + a^\lambda b^\nu c^\mu + a^\mu b^\lambda c^\nu + a^\mu b^\nu c^\lambda + a^\nu b^\lambda c^\mu + a^\nu b^\mu c^\lambda\right)$$

if the real numbers λ, μ, ν are supposed to satisfy the conditions

(1.2) $$\lambda+\mu+\nu=2,$$

(1.3) $$\lambda \geqq \mu \geqq \nu \geqq 0.$$

Three functions $P_{\lambda\mu\nu}(a, b, c)$ deserve particular attention and special notation. They are the following

(1.4) $$P_{2,0,0}=4\pi \frac{a^2+b^2+c^2}{3} = A,$$

(1.5) $$P_{1,1,0}=4\pi \frac{bc+ca+ab}{3} = F,$$

(1.6) $$P_{2/3,\,2/3,\,2/3}=4\pi (abc)^{2/3}=G.$$

A is the arithmetic mean of the areas of three spherical surfaces; the spheres have as radii the semiaxes a, b, c of the ellipsoid. F is also the arithmetic mean of three spherical surfaces; any great circle of the first sphere has the same area πbc as the intersection of the ellipsoid with the plane that passes through its center and is perpendicular to its axis of length $2a$; the two other spheres are similarly related to the two other axes of the ellipsoid. Finally, G is the geometric mean of the three spherical areas whose arithmetic mean is A; it is also the geometric mean of the three other spherical areas whose arithmetic mean is F; and it is also the area of the surface of the sphere whose volume is equal to the volume of the ellipsoid.

We may consider linear combinations of a finite number of $P_{\lambda\mu\nu}$ as

(1.7) $$kP_{\lambda\mu\nu}+k'P_{\lambda'\mu'\nu'}+k''P_{\lambda''\mu''\nu''}+\dots .$$

It is understood that if we replace λ, μ, ν in (1.2) (1.3) by λ', μ', ν' these conditions remain satisfied, and the same is true of λ'', μ'', ν'', ... If this is so, and the coefficients k, k', k'', ... satisfy the conditions

$$k+k'+k''+\ldots=1, \tag{1.8}$$

$$k>0,\ k'>0,\ k''>0,\ \ldots, \tag{1.9}$$

the linear combination (1.7) has the same properties (I) (II) (III) (IV) as E. We may and shall consider (1.1) as a special case of (1.7), admitting that the latter may consist of just one term.

Particular approximations of this kind have been considered before. Boussinesq [1] proposed as approximations to E

$$4\pi\left(\frac{a+b+c}{3}\right)^2=\frac{1}{3}A+\frac{2}{3}F, \tag{1.10}$$

$$4\pi\left(\frac{4}{5}\frac{a+b+c}{3}+\frac{1}{5}(a\,b\,c)^{1/3}\right)^2= \tag{1.11}$$

$$=\frac{16}{75}A+\frac{32}{75}F+\frac{3}{75}G+\frac{24}{75}P_{4/3,\,1/3,\,1/3}\,.$$

Peano [2] considered F, see (1.5), and

$$4\pi\left(\frac{bc+ca+ab}{3}+\frac{(b-c)^2+(c-a)^2+(a-b)^2}{30}\right)=\frac{1}{5}A+\frac{4}{5}F. \tag{1.12}$$

Let us introduce the following terminology. The expression (1.7) will be termed a *regular* approximation if the conditions (1.2) (1.3) (1.8) (1.9) are satisfied. If only the conditions (1.2) and (1.8) are supposed to be satisfied and (1.3) and (1.9) are dropped (so that some of the numbers $\lambda, \mu, \nu,\ \lambda', \mu', \nu',\ \ldots\ k, k', \ldots$ may be negative), the expression (1.7) is called a *semi-regular* approximation. Semi-regular approximations have the properties (II), (III), (IV), but not necessarily (I), whereas regular approximations have all properties (I) to (IV). An approximation which has not the form (1.7) is neither regular nor semi-regular, and may be called *irregular*.

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(1) J. Boussinesq, *Cours d'analyse infinitésimale*, Paris, 1890, vol. 2, 74*-77*.

(2) G. Peano, *Valori approssimati dell' area dell' ellissoide.* Atti della R. Accademia dei Lincei, Rendiconti, 4 series, vol. 6, 2nd semester (1890) p. 317-321.

Let P be any approximation to E. If $E \geqq P$ for $a \geqq 0$, $b \geqq 0$, $c \geqq 0$, P is called a *lower* approximation to E, if $E \leqq P$ for $a \geqq 0$, $b \geqq 0$, $c \geqq 0$, P is called an *upper* approximation. Lower and upper approximations are called *unilateral* approximations. $P-E$ is called the *error* of the approximation. $(P-E)/E$ is called the *relative error*. If an approximation is not unilateral, its error is positive for certain values of a, b, c, and negative for certain other values.

2. *Inequalities*. I list here the most important unilateral approximations to E; they stand on the right hand side of the following inequalities.

$$E > F, \tag{2.1}$$

$$E < A, \tag{2.2}$$

$$E < \frac{1}{2} A + \frac{1}{2} F. \tag{2.3}$$

(2.1) is valid unless $a=b=c$ or two semiaxes vanish; (2.2) is valid unless $a=b=c$; (2.3) is valid unless $a=b=c$, or two semiaxes are equal and the third vanishes (as $a=b$, $c=0$), In the cases excluded, the inequality concerned is turned into equality.

The right hand sides of the foregoing inequalities are regular approximations; the right hand side of the next one is a semi-regular approximation.

$$(2.4)\quad E < F + \frac{4\pi}{30} \frac{a^{12/7}(b-c)^2 + b^{12/7}(c-a)^2 + c^{12/7}(a-b)^2}{(a\,b\,c)^{4/7}} =$$

$$= P_{1,1,0} + \frac{1}{5} P_{10/7,\,8/7,\,-4/7} - \frac{1}{5} P_{8/7,3/7,3/7}$$

holds unless $a=b=c$.

The following inequalities (2.6) to (2.10) suppose that

$$a \geqq b \geqq c > 0. \tag{2.5}$$

We consider various irregular approximations.

$$E > 2\pi a b \left(1 + \frac{c^2}{a^2}\right), \tag{2.6}$$

$$E < F + \frac{4\pi}{6}\, a b \left(1 - \frac{c^2}{a^2}\right)^2 . \tag{2.7}$$

These two inequalities are valid unless $a = b = c$, or $c = 0$, in which cases the right hand side gives the exact value of E.

The following inequalities compare the area of the surface of a general ellipsoid with the areas of oblate spheroids.

$$2E(a, b, c) > ab\,[E(1, 1, c/a) + E(1, 1, c/b)], \tag{2.8}$$

$$E(a, b, c) < ab\,E(1, 1, [1 - (1 - c^2/a^2)^{1/2}(1 - c^2/b^2)^{1/2}]^{1/2}) \tag{2.9}$$

are valid unless $a = b$, in which case the equality is obvious.

The following inequality supposes $c < 1$, $c' < 1$ and is valid unless $c = c'$.

$$E(1, 1, c) + E(1, 1, c') > 2E(1, 1, (c\,c')^{1/2}). \tag{2.10}$$

The proofs for the foregoing inequalities cannot be presented here; some are based on the representation of E by an integral, others on its representation by a series. Geometric considerations may also lead to inequalities for E. If the volume of any solid which is not a sphere is equal to the volume of a sphere then the area of the surface of the solid is larger than that of the sphere. Applying this to the ellipsoid, we obtain that

$$E > G \tag{2.11}$$

unless $a = b = c$. A related theorem, discovered by H. A. Schwarz (3) leads to the inequality

$$E(a, b, c) > E((ab)^{1/2}, (ab)^{1/2}, c) \tag{2.12}$$

(3) H. A. Schwarz, *Werke*, vol. 2, p. 327-340.

valid unless $a=b$. Inequalities (2.11) (2.12) do not need the assumption (2.5).

(2.11) is an easy consequence of (2.1) (since the geometric mean is less than the arithmetic mean). We can deduce (2.12) from (2.8) and (2.10) under the restriction (2.5) but not without restriction.

I add the following inequality which is valid unless $a=b=c$, or two semiaxes vanish

$$E<4\pi\left(\frac{b^2c^2+c^2a^2+a^2b^2}{3}\right)^{1/2}. \tag{2.13}$$

Inequalities (2.1) and (2.11) are due to Peano, the others are new so far as I know. For the first complete proof of (2.3) I am indebted to Professor G. Szegö.

3. *Unilateral regular approximations.* In this section, we restrict ourselves to the consideration of regular approximations. Let P denote such an approximation; then P is represented by (1.7) (which contains (1.1) as a special case), the conditions (1.2) (1.3) (1.8) (1.9) are fulfilled, and $P=P(a,b,c)$ has the properties (I) (II) (III) (IV) stated at the beginning of section 1. Our problem is to characterize those regular approximations which are unilateral. This problem can be completely solved and its solution is stated in the following two theorems of which the first deals with lower approximations and the second with upper approximations.

I. *P being a regular approximation, the inequality*

$$E\geqq P$$

is valid for $a\geqq 0$, $b\geqq 0$, $c\geqq 0$ if and only if

$$\lambda\leqq 1,\ \lambda'\leqq 1,\ \lambda''\leqq 1,\ \ldots\ . \tag{3.1}$$

If this condition is fulfilled then, for $a>0$, $b>0$, $c>0$,

$$F>P$$

unless P happens to be F, or $a=b=c$.

II. *Let P be a regular approximation, and let h_0 denote the sum of the coefficients of the $P_{\lambda\mu\nu}$ with $\nu \neq 0$, h_1 the sum of the coefficients of the $P_{\lambda\mu\nu}$ with $\nu = 0$, $\mu > 0$, and h_2 the coefficient of $P_{2,0,0} = A$. The inequality*

$$E \leqq P$$

is valid for $a \geqq 0$, $b \geqq 0$, $c \geqq 0$ if and only if

$$4h_2 + 2h_1 \geqq 3. \tag{3.2}$$

If this condition is fulfilled then, for $a > 0$, $b > 0$, $c > 0$,

$$\frac{1}{2}A + \frac{1}{2}F < P$$

unless P happens to be $(A+F)/2$ or $a = b = c$.

In order to understand clearly Theorem II, let us observe that h_0 is the sum of certain of the coefficients $k, k', k'', \ldots$ arising in (1.7) and so is h_1, whereas h_2 is either $=0$, namely in the case in which $P_{2,0,0} = A$ does not arise in (1.7), or it is the coefficient of this particular $P_{\lambda\mu\nu}$ so that all we can say a priori about h_0, h_1, h_2 is

$$h_0 \geqq 0,\ h_1 \geqq 0,\ h_2 \geqq 0,\ h_0 + h_1 + h_2 = 1. \tag{3.3}$$

The subscript in h_0, h_1, h_2 exhibits the numbers of the vanishing among λ, μ, ν. If P is $(A+F)/2$, then $h_0 = 0$, $h_1 = h_2 = 1/2$, and the condition (3.2) is fulfilled; this agrees with the existence of the inequality (2.3). If $P = (3+G)/4$, then $h_0 = 1/4$, $h_1 = 0$, $h_2 = 3/4$; the condition (3.2) is again fulfilled, and so we have, by Theorem II,

$$\begin{gathered} \frac{1}{2}A + \frac{1}{2}F < \frac{3}{4}A + \frac{1}{4}G, \\ F < \frac{1}{2}A + \frac{1}{2}G \end{gathered} \tag{3.4}$$

for $a > 0$, $b > 0$, $c > 0$, unless $a = b = c$. If P is given by (1.11) then $h_0=27/75$, $h_1=32/75$, $h_2=16/75$, the condition (3.2) is not fulfilled and so (1.11) is not an upper approximation. With just as little trouble we may see that none of the formulas (1.10) (1.11) (1.12) represents an upper approximation or a lower approximation (see Theorem I).

It follows easily from (3.2) and (3.3) that $h_2 \geqq 1/2$ and so A must be a constituent of all upper approximations. Consider the particular case in which the approximation has but one constituent, being of the form (1.1). We obtain that the only $P_{\lambda\mu\nu}$ that gives an upper approximation of E is $P_{2,0,0}=A$; the inequality (2.2) is the only one of its kind. We can express an essential part of the Theorems I and II by saying that the inequalities (2.1) and (2.3) are the best of their kind.

4. *Nearly spherical ellipsoids.* If P is an approximation that is homogeneous of degree 2 in a, b, c (has the property (II)) its relative error $(P-E)/E$ is a homogeneous function of degree 0 of the semiaxes a, b, c, and so it depends only on the ratios of these semiaxes. That is, the relative error depends only of the shape of the ellipsoid and not on its size; it can be conceived, under condition (2.5), as a function of α^2 and β^2, where

$$(4.1) \qquad \alpha^2 = \frac{a^2-c^2}{a^2} \quad , \quad \beta^2 = \frac{b^2-c^2}{b^2};$$

α and β are the numerical excentricities of two ellipses, intersections of the ellipsoid with two planes, each of which contains the shortest axis and another axis of the ellipsoid.

We assume that the relative error can be expanded into powers of α^2 and β^2 and we write the expansion as follows

$$(4.2) \qquad (P-E)/E = Q_0 + Q_1 + \ldots + Q_n + \ldots,$$

Q_n being a homogeneous polynomial of degree n in α^2 and β^2.

If all coefficients of $Q_0, Q_1, \ldots, Q_{m-1}$ vanish, but Q_m has a non-vanishing coefficient, we shall call Q_m the *initial polynomial, m* the *order of approximation,* and the maximum of $|Q_m|$ for

(4.3) $$0 \leqq \beta \leqq \alpha = 1$$

the modulus of the approxinmation. (It (2.5) holds, $\alpha \geqq \beta$).

If we consider small (infinitesimal) values of α, β, that is, nearly spherical ellipsoids, and we wish to judge the efficiency of the approximation, we must know its order, its modulus, and the initial polynomial. Comparing two approximations whose orders are different, we consider that with the *higher order* as *better*. Comparing two approximations of the same order, we consider that with the *smaller modulus* as *better*.

If we are given the numbers λ, μ, ν; λ', μ', ν'; λ'', μ'', ν''; ... and we have to decide whether there exists or not a semiregular approximation to E of the form (1.7) and of order m, we can solve our problem discussing a system of $1+2+\ldots+m$ linear equations for the unknowns $k, k', k'', \ldots$ Such problems provide an instructive illustration of the chapter of the Calculus that deals with Taylor's series.

For instance, the expansion

(4.4) $$\frac{P_{\lambda\mu\nu}-E}{E} = \frac{5(\lambda^2+\mu^2+\nu^2)-12}{120}(\alpha^4-\alpha^2\beta^2+\beta^4)+\ldots$$

starts as written, the following terms being of degree 3 in α^2 and β^2. Therefore, unless

$$5(\lambda^2+\mu^2+\nu^2)-12=0$$

(which never happens for rational λ, μ, ν) the order of approximation is 2 and the modulus of approximation is the factor of $(\alpha^4-\alpha^2\beta^2+\beta^4)$. To derive (4.4) in some interesting special case, as (1.4), (1,5), or (1.6), may be proposed as a useful exercise in the classroom.

The best approximating linear combination of A, F, G for nearly spherical ellipsoids is

(4.5) $$(-2A+64F-27G)/35.$$

The order of the approximation is 5, the modulus of the approximation 1/20790, and the initial polynomial

$$-(\alpha^2-2\beta^2)\left(\alpha^2-\frac{1}{2}\beta^2\right)(\alpha^6+\beta^6)/20790.$$

The approximation (4.5) gives remarkably good numerical results. For $a=5$, $b=4$, $c=3$ the relative error is in absolute value very nearly 3.10^{-6}. For $a=28$, $b=27$, $c=26$ the tables at disposal were insufficient to find a difference between the approximate value supplied by (4.5) and the true value given by elliptic integrals; the first 7 figures are same.

For the not too easy numerical computation of the intervening elliptic integrals I am indebted to Profesor J. V. Uspensky.

5. *Spheroids.* Let P denote an approximation of the form (1.7). It does not seem quite easy to formulate necessary and sufficient conditions under which P is a unilateral approximation to E, not necessarily for all ellipsoids, but just for spheroids (ellipsoids of revolution) of a certain kind. But I found various sufficient conditions, and I state here a relatively simple condition of this kind.

Assume that $a>c>0$. It is easy to see that there is an identity in α of the form

$$\text{(5.1)} \qquad \frac{d}{d\alpha}\left(\frac{\alpha P(a,a,c)}{4\pi c^2}\right)-\frac{1}{(1-\alpha^2)^2}$$

$$=c_0(1-\alpha^2)^{\gamma_0}+c_1(1-\alpha^2)^{\gamma_1}+\ldots+c_l(1-\alpha^2)^{\gamma_l}$$

where $c_0,\ldots c_l$, $\gamma_0,\ldots\gamma_l$ are independent of α, $\gamma_0,\gamma_1,\ldots\gamma_l$ are different from each other, and c_0, $c_1,\ldots,c_l$ are different from 0. Let us call, as in the preceding section, m the order with which P approximates E for nearly spherical ellipsoids. *If $l=m$, then $P(a,a,c)-E(a,a,c)$ can not vanish for $0<c<a$* and so P is a *unilateral* approximation for *oblate* spheroids. For prolate spheroids, we consider

$$\frac{d}{d\alpha}\left(\frac{\alpha P(a,a,c)}{4\pi a c}\right)-(1-\alpha^2)^{1/2}$$

instead of the left hand side of (5.1) and the condition is similar.

These conditions can be applied, for instance, to the approximation (1.2) due to Peano and to the new approximation (4.5). These two approximations have errors with opposite signs for spheroids, so that the true value is always contained between (1.12) and (4.5)). The approximation (1.12) gives consistently too high values for prolate spheroids and too low values for oblate spheroids, and just the opposite is true of (4.5).