GEOMETRIC PROPERTIES OF GENERALIZED DYNAMICAL TRAJECTORIES

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1. Introduction. A complete characteristic set of the geometric properties of the dynamical trajectories of fields of force has been given by Kasner in his Princeton Colloquium Lectures(1). The fields of force heretofore considered, are those depending only upon the position of the point. In this article, we propose to begin the development of the geometry of the trajectories of fields of force in the plane wich depend not only on the position of the point but also on the direction through the point. That is, the force vector depends upon the lineal element (x, y, y'). We shall also compare our new results concerning these generalized dynamical trajectories with the corresponding ones concerning the trajectories of ordinary positional (depending only upon the point) fields of force.

2. The system of ∞^3 generalized dynamical trajectories. Let a field of force, depending upon the position of the lineal element (x, y, y'), be given by the components ϕ and ψ, not both identically zero, parallel to the x and y axes. We assume that the direction of the force vector does not identically coincide with that of the corresponding lineal element. The equations of motion of a particle of unit mass are

$$(1) \qquad \frac{d^2x}{dt^2} = \phi(x, y, y'), \quad \frac{d^2y}{dt^2} = \psi(x, y, y').$$

Upon shooting a particle from any position in any direction with any speed, a definite trajectory is described. Considering all initial positions, there are ∞^3 trajectories. By eliminating the time t from the equations (1), we find that the differential equation of this system of ∞^3 generalized dynamical trajectories is

$$(2) \qquad (\psi - y'\phi)\, y''' = [\psi_x + y'(\psi_y - \phi_x) - y'^2 \phi_y]\, y'' + [\psi_{y'} - y'\phi_{y'} - 3\phi]\, y''^2.$$

The system of ∞^3 generalized dynamical trajectories is represented by a differential equation of the type (G), namely

$$(G) \qquad y''' = G(x, y, y')\, y'' + H(x, y, y')\, y''^2.$$

Kasner first encountered these differential equations of the type (G) in his study of the geometry of dynamical trajectories of positional field of force (see reference 1). Any system of such type may be characterized by Kasner's geometric Property I which is described in the following way[2].

For each of the ∞^1 trajectories of the system of ∞^3 trajectories which pass trough a given lineal element E, construct the osculating parabola at E. The Property I is then either one of the following three equivalent results.

(I_a). The locus of the foci of the osculating parabolas is a circle passing through the point of E.

(I_b). The ∞^1 directrices of the parabolas form a pencil of straight lines.

(I_c). The envelope of the ∞^1 osculating parabolas is a straight line.

Conversely any system of ∞^3 curves with the Property I may represent the dynamical trajectories of ∞f[2] fields of force which depend upon the position of the lineal element[3].

For the ∞^3 dynamical trajectories of a positional field of force, there are *four additional* independent geometric properties. Moreover any two positional fields of force with the same dynamical trajectories are constant multiples of one another.

3. The angular rate λ. For the development of our theory, it is found necessary to associate a number λ to any lineal element E as follows. As the element E rotates about its point p, the corresponding force vector F also rotates about p. The angular rate λ is the instantaneous rate of change of the inclination of F with respect to the inclination of E. It is given by the formula

$$\lambda = \frac{1+y'^2}{\phi^2+\psi^2}(\phi\psi_{y'} - \psi\phi_{y'}). \tag{3}$$

4. The lines of force and the rest trajectories(4). In this section we wish to compare two special important families of curves — namely the lines of force and the rest trajectories. First we shall consider the lines of force. In general, at a given point p, there is one lineal element E_0 such that the direction of the force vector is identical with that of E_0. Assume the angular rate $\lambda \neq 1$ at all elements E_0 within a given region of the plane. Then these ∞^2 elements E_0 define a differential equation of the first order. The ∞^1 integral curves of this differential equation are called the *lines of force* of the given field field of force.

A *rest trajectory* is defined as the path described in a field of force by a particle which starts from rest. Any rest trajectory must initially contain the lineal element E_0 described above. There are ∞^2 rest trajectories.

Let the rest trajectory have contact of the first order with the lineal element E_0, and at E_0, let the angular rate $\lambda \neq 1$ or 3. Then the line of force has contact of first order with E_0.

The ratio ρ of the curvatures of the rest trajectory and the line of force is

$$\rho = \frac{1-\lambda}{3-\lambda}. \tag{4}$$

For a positional field of force, Kasner proved that this ratio $\rho = 1/3$. From this formula (4), we obtain the following converse of this result.

If $\rho = 1/3$, then the angular rate $\lambda = 0$. This means geometrically that the line of the lineal element E_0 is tangent to the terminal curve C formed by the end points of the force vectors through the given point p.

Next let the rest trajectory have contact of the n-th order with E_0, and at E_0, let the angular rate $\lambda \neq 1$ or $2n+1$. Then the line of force has contact of n-th order with E_0.

The ratio ρ *of the departures of the rest trajectory and the line of force from the line of* E_0 *is*

$$\rho = \frac{1-\lambda}{2n+1-\lambda}. \tag{5}$$

The analogous result for positional field of force is that $\rho = 1/(2n+1)$. Our converse is that $\rho = 1/(2n+1)$ *if and only if* $\lambda = 0$.

Finally we note that if $\lambda = 2n+1$, the line of force has contact with E_0 of order higher than n. Of course, this situation can never arise in positional fields of force. That is, a rest trajectory and a line of force always have the same order of contact with their common tangent line.

In conclusion, we note that all our results belong to the total projective group G_8 consisting of collineations and correlations.

In another paper, we study fields of force, in the plane and space, which depend upon differential elements of higher order.

REFERENCES

1) KASNER, *Differential geometric aspects of dynamics*, Published by the American Mathematical Society, 1913, 1934. Also see the Transactions of the American Mathematical Society 1906-1910.

2) Recently TERRACINI has given an alternate geometric characterization of the Property I by conic sections. See the paper, *Sobre la Ecuación diferencial* $y''' = G(x,y,y')y'' + H(x,y,y')y''^2$, Revista de Matemáticas (Tucumán), Vol. 2, pp. 245-329 (1941).

3) KASNER, *A notation for infinite manifolds*, American Mathematical Monthly, Vol. 49, pp. 243-44, 1942.

4) For generalizations to acceleration fields of higher order, see KASNER and MITTLEMAN, *A general theorem on the initial curvature of dynamical trajectories*, Proceedings of the National Academy of Sciences, Vol. 28, pp. 48-52, 1942; also *Extended theorems in dynamics*, Science, Vol. 95, pp. 249-250, 1942.