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Product of independent random variables involving inverted hypergeometric function type I variables

Produto de variáveis aleatórias independentes envolvendo variáveis com função hipergeométrica invertida tipo I

Producto de variables aleatorias independentes que involucran variables con función hipergeométrica invertida de tipo I

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Se aceptan comentarios y/o discusiones al artículo

Abstract

The inverted hypergeometric function type I distribution has the probability density function proportional to

$$x^{\nu-1}(1+x)^{-(\nu+\gamma)}{}_2F_1(\alpha,\beta;\gamma;(1+x)^{-1}), \quad x>0\,,$$

where ${}_2F_1$ is the Gauss hypergeometric function. In this article, we derive the probability density function of the product of two independent random variables having inverted hypergeometric function type I distribution. We also consider several other products involving inverted hypergeometric function type I, beta type I, beta type II, beta type III, Kummer–beta and hypergeometric function type I variables.

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Key words: Appell's first hypergeometric function, beta distribution, Gauss hypergeometric function, Humbert's confluent hypergeometric function, product, transformation.

Resumo

A distribuição função hipergeométrica invertida tipo I tem a função densidade de probabilidade proporcional a

$$x^{\nu-1}(1+x)^{-(\nu+\gamma)}{}_2F_1(\alpha,\beta;\gamma;(1+x)^{-1}), \quad x>0\,,$$

em que ${}_2F_1$ é a função hipergeométrica de Gauss. Neste artigo, vamos derivar a função densidade de probabilidade do produto de duas variáveis aleatórias independentes tendo distribuição função hipergeométrica invertida tipo I. Também consideramos vários outros produtos envolvendo variáveis com função hipergeométrica invertida tipo I, beta tipo I, beta tipo II, beta tipo III, Kummer–beta e hipergeométrica tipo I.

Palavras chaves: primeira funcão hipergeométrica de Appell, distribução beta, função hipergeométrica de Gauss, função hipergeométrica confluente de Humbert, producto, transformação.

Resumen

La distribución de función hipergeométrica invertida tipo I tiene la función de densidad de probabilidad proporcional a

$$x^{\nu-1}(1+x)^{-(\nu+\gamma)}{}_2F_1(\alpha,\beta;\gamma;(1+x)^{-1}), \quad x>0\,,$$

donde ${}_2F_1$ es la función hipergeométrica de Gauss. En este artículo se deriva la función de densidad de probabilidad del producto de dos variables aleatorias independientes que se distribuyen según la función hipergeométrica inversa tipo I. También se consideran otros productos entre variables aleatorias con distribución beta tipo I, beta tipo II, beta tipo III, función hipergeométrica tipo I, función hipergeométrica inversa tipo I y Kummer–beta.

Palabras claves: primera función hipergeométrica de Appell, distribución beta, función hipergeométrica confluente de Humbert, función hipergeométrica de Gauss, producto, transformación.

1 Introduction

The random variable X is said to have an inverted hypergeometric function type I distribution, denoted by $X \sim IH^I(\nu,\alpha,\beta,\gamma)$, if its probability density function(p.d.f.) is given by (Gupta and Nagar [1], Nagar and Álvarez [2]),

$$\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)}\frac{x^{\nu-1}}{(1+x)^{\nu+\gamma}}\,{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x}\right), \tag{1}$$

where $x>0$, $\nu>0$, $\gamma>0$, $\gamma+\nu>\alpha+\beta$, and ${}_2F_1$ is the Gauss hypergeometric function. For $\alpha=\gamma$, (1) reduces to a beta type II density given by

$$\frac{\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu-\beta)}\frac{x^{\nu-\beta-1}}{(1+x)^{\nu-\beta+\gamma}},\quad x>0\,,$$

and for $\beta=\gamma$, the inverted hypergeometric function type I density slides to

$$\frac{\Gamma(\gamma+\nu-\alpha)}{\Gamma(\gamma)\Gamma(\nu-\alpha)}\frac{x^{\nu-\alpha-1}}{(1+x)^{\nu-\alpha+\gamma}},\quad x>0\,.$$

Further, for $\alpha=0$ or $\beta=0$, the inverted hypergeometric function type I density simplifies to a beta type II density with parameters ν and γ.

Recently, Nagar and Álvarez [2] studied several properties and stochastic representations of the inverted hypergeometric function type I distribution.

In this article, we derive the density function of the product of two independent random variables having inverted hypergeometric function type I distribution. We also derive densities of several other products involving hypergeometric function type I, beta type I, beta type II, beta type III, Kummer–beta and hypergeometric function type I variables. For further results on product of independent random variables, see: Mathai and Saxena [3], Nagar and Zarrazola [4], and Sanchez and Nagar [5].

In section 2, we give definitions of several univariate distributions. Sections 3 and 4 deal with derivations of a number of densities of products of independent random variables. Finally, in appendix, we give definitions and results on Gauss hypergeometric function, Appell's first hypergeometric function F_1 and Humbert's confluent hypergeometric function $\mathbf{\Phi}_1$.

2 Some Univariate Distributions

In this section, we define the beta type I, beta type II, beta type III, hypergeometric function type I and Kummer–beta distributions. These definitions can be found in Gordy [6], Johnson, Kotz and Balakrishnan [7], Nagar and Zarrazola [4], and Sánchez and Nagar [5].

Definition 2.1. *The random variable X is said to have a beta type I distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim B^{I}(a, b)$, if its p.d.f. is given by*

$$\{B(a,b)\}^{-1} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1\,,$$

where $B(a, b)$ is the beta function given by

$$B(a,b) = \Gamma(a)\Gamma(b)\{\Gamma(a+b)\}^{-1}\,.$$

Definition 2.2. *The random variable X is said to have a beta type II distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim B^{II}(a, b)$, if its p.d.f. is given by*

$$\{B(a,b)\}^{-1} x^{a-1}(1+x)^{-(a+b)}, \quad x > 0\,.$$

Definition 2.3. *The random variable X is said to have a beta type III distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim B^{III}(a, b)$, if its p.d.f. is given by*

$$2^{a}\{B(a,b)\}^{-1} x^{a-1}(1-x)^{b-1}(1+x)^{-(a+b)}, \quad 0 < x < 1\,.$$

Definition 2.4. *The random variable X is said to have a Kummer–beta distribution, denoted by $X \sim KB(\alpha, \beta, \lambda)$, if its p.d.f. is given by*

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}\exp\left[\lambda(1-x)\right]}{B(\alpha,\beta)\,{}_1F_1(\beta;\alpha+\beta;\lambda)}, \quad 0 < x < 1\,,$$

where $\alpha > 0$, $\beta > 0$ and $-\infty < \lambda < \infty$.

Note that for $\lambda = 0$ the above density simplifies to a beta type I density with parameters α and β.

Definition 2.5. *The random variable X is said to have a hypergeometric function type I distribution, denoted by $X \sim H^{I}(\nu, \alpha, \beta, \gamma)$, if its p.d.f. is given by*

$$\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)} x^{\nu-1}(1-x)^{\gamma-1}{}_2F_1(\alpha,\beta;\gamma;1-x), \quad 0 < x < 1\,,$$

where $\gamma + \nu - \alpha - \beta > 0$, $\gamma > 0$ and $\nu > 0$.

The following result (Gupta and Nagar [1], Nagar and Álvarez [2]) states that the hypergeometric function type I distribution can be obtained as the distribution of the product of two independent beta type I variables.

Theorem 2.1. *Let* X_1 *and* X_2 *be independent,* $X_i \sim B^I(a_i, b_i)$, $i = 1, 2$. *Then,* $X_1 X_2 \sim H^I(a_1, b_2, a_1 + b_1 - a_2, b_1 + b_2)$.

The matrix variate generalizations of beta type I, beta type II, beta type III, hypergeometric function type I and Kummer–beta distributions have been defined and studied extensively. For example, see Gupta and Nagar [8, 1], and Nagar and Gupta [9].

3 Products of Two Independent Random Variables

In this section we derive distributions of the product of two independent random variables when at least one of them has inverted hypergeometric function type I distribution.

Theorem 3.1. *Let* X_1 *and* X_2 *be independent random variables,* $X_i \sim IH^I(\nu_i, \alpha_i, \beta_i, \gamma_i)$, $i = 1, 2$. *Then, the p.d.f. of* $Z = X_1 X_2$ *is given by*

$$\frac{\Gamma(\nu_1+\gamma_2)\Gamma(\nu_2+\gamma_1)}{\Gamma(\nu_1+\nu_2+\gamma_1+\gamma_2)}\left\{\prod_{i=1}^{2}\frac{\Gamma(\gamma_i+\nu_i-\alpha_i)\Gamma(\gamma_i+\nu_i-\beta_i)}{\Gamma(\gamma_i)\Gamma(\nu_i)\Gamma(\gamma_i+\nu_i-\alpha_i-\beta_i)}\right\}$$
$$\times z^{\nu_1-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_s(\beta_1)_r(\beta_2)_s(\nu_2+\gamma_1)_r(\nu_1+\gamma_2)_s}{(\gamma_1)_r(\gamma_2)_s(\nu_1+\nu_2+\gamma_1+\gamma_2)_{r+s}\, r!\, s!}$$
$$\times {}_2F_1(\nu_1+\gamma_2+s, \nu_1+\gamma_1+r; \nu_1+\nu_2+\gamma_1+\gamma_2+r+s; 1-z)\,, \qquad (2)$$

where $z > 0$.

Proof. Using independence, the joint p.d.f. of X_1 and X_2 is given by

$$\frac{K_1 x_1^{\nu_1-1} x_2^{\nu_2-1}}{(1+x_1)^{\nu_1+\gamma_1}(1+x_2)^{\nu_2+\gamma_2}}\,{}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{1}{1+x_1}\right){}_2F_1\left(\alpha_2, \beta_2; \gamma_2; \frac{1}{1+x_2}\right),$$

where $x_1 > 0$, $x_2 > 0$ and

$$K_1 = \prod_{i=1}^{2}\frac{\Gamma(\gamma_i+\nu_i-\alpha_i)\Gamma(\gamma_i+\nu_i-\beta_i)}{\Gamma(\gamma_i)\Gamma(\nu_i)\Gamma(\gamma_i+\nu_i-\alpha_i-\beta_i)}\,.$$

Transforming $Z = X_1X_2$, $U = 1/(1 + X_2)$ with the Jacobian $J(x_1, x_2 \to z, u) = 1/u(1-u)$ in the joint density of X_1 and X_2 and integrating u, we obtain the p.d.f. of Z as

$$K_1 z^{\nu_1-1}\int_0^1 \frac{u^{\nu_1+\gamma_2-1}(1-u)^{\nu_2+\gamma_1-1}}{[1-(1-z)u]^{\nu_1+\gamma_1}}\,{}_2F_1\left(\alpha_1,\beta_1;\gamma_1;\frac{1-u}{1-(1-z)u}\right)\times{}_2F_1\left(\alpha_2,\beta_2;\gamma_2;u\right)du\,. \tag{3}$$

Now, expanding Gauss hypergeometric functions in the integral (3) in terms of power series we arrive at

$$K_1 z^{\nu_1-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_s(\beta_1)_r(\beta_2)_s}{(\gamma_1)_r(\gamma_2)_s r!s!}\int_0^1\frac{u^{\nu_1+\gamma_2+s-1}(1-u)^{\nu_2+\gamma_1+r-1}}{[1-(1-z)u]^{\nu_1+\gamma_1+r}}\,du\,.$$

Finally, using (A.3) and substituting for K_1 we obtain the desired result. □

Corollary 3.1.1. *Let X_1 and X_2 be independent random variables, $X_1 \sim IH^I(\nu_1,\alpha_1,\beta_1,\gamma_1)$, and $X_2 \sim B^{II}(\nu_2,\gamma_2)$. Then, the p.d.f. of $Z = X_1X_2$ is given by*

$$\frac{\Gamma(\nu_1+\gamma_2)\Gamma(\nu_2+\gamma_1)}{\Gamma(\nu_1+\nu_2+\gamma_1+\gamma_2)}\frac{\Gamma(\gamma_1+\nu_1-\alpha_1)\Gamma(\gamma_1+\nu_1-\beta_1)}{\Gamma(\gamma_1)\Gamma(\nu_1)\Gamma(\gamma_1+\nu_1-\alpha_1-\beta_1)}\frac{\Gamma(\gamma_2+\nu_2)}{\Gamma(\gamma_2)\Gamma(\nu_2)}\times z^{\nu_1-1}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\beta_1)_r(\nu_2+\gamma_1)_r}{(\gamma_1)_r(\nu_1+\nu_2+\gamma_1+\gamma_2)_r\,r!}\times{}_2F_1(\nu_1+\gamma_2,\nu_1+\gamma_1+r;\nu_1+\nu_2+\gamma_1+\gamma_2+r;1-z),\quad z>0\,. \tag{4}$$

Corollary 3.1.2. *Let X_1 and X_2 be independent, $X_i \sim B^{II}(\nu_i,\gamma_i)$, $i = 1, 2$. Then, the p.d.f. of $Z = X_1X_2$ is given by*

$$\frac{\Gamma(\nu_1+\gamma_2)\Gamma(\nu_2+\gamma_1)}{\Gamma(\nu_1+\nu_2+\gamma_1+\gamma_2)}\frac{\Gamma(\gamma_1+\nu_1)}{\Gamma(\gamma_1)\Gamma(\nu_1)}\frac{\Gamma(\gamma_2+\nu_2)}{\Gamma(\gamma_2)\Gamma(\nu_2)}\times z^{\nu_1-1}\,{}_2F_1(\nu_1+\gamma_2,\nu_1+\gamma_1;\nu_1+\nu_2+\gamma_1+\gamma_2;1-z),\quad z>0\,. \tag{5}$$

Note that the Gauss hypergeometric functions in the densities (2), (4) and (5) can be expanded in series form if $0 < z < 1$. However, if $z > 1$, then $1 - 1/z < 1$ and we use (A.4) to rewrite the densities (2), (4) and (5) in series involving Gauss hypergeometric functions having $1 - 1/z$ as argument.

The next theorem gives the density of the product of Kummer–beta and inverted hypergeometric function type I variables.

Theorem 3.2. *Let X_1 and X_2 be independent, $X_1 \sim IH^I(\nu_1, \alpha_1, \beta_1, \gamma_1)$ and $X_2 \sim KB(\nu_2, \gamma_2, \lambda)$. Then, the p.d.f. of $Z = X_1 X_2$ is*

$$\frac{\Gamma(\gamma_1+\nu_1-\alpha_1)\Gamma(\gamma_1+\nu_1-\beta_1)\Gamma(\gamma_2+\nu_2)\Gamma(\gamma_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma_1)\Gamma(\gamma_1+\nu_1-\alpha_1-\beta_1)\Gamma(\gamma_1+\gamma_2+\nu_2)}$$
$$\times\{{}_1F_1(\gamma_2;\nu_2+\gamma_2;\lambda)\}^{-1}\frac{z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\beta_1)_r(\gamma_1+\nu_2)_r}{(\gamma_1+\gamma_2+\nu_2)_r\,(\gamma_1)_r\,r!(1+z)^r}$$
$$\times\mathbf{\Phi}_1\left[\gamma_2,\nu_1+\gamma_1+r;\gamma_1+\gamma_2+\nu_2+r;\frac{1}{1+z},\lambda\right],\quad z>0\,.$$

Proof. The joint p.d.f. of X_1 and X_2 is given by

$$K_2\frac{x_1^{\nu_1-1}x_2^{\nu_2-1}(1-x_2)^{\gamma_2-1}}{(1+x_1)^{\nu_1+\gamma_1}}{}_2F_1\left(\alpha_1,\beta_1;\gamma_1;\frac{1}{1+x_1}\right)\exp[\lambda(1-x_2)]\,, \qquad (6)$$

where $x_1 > 0$, $0 < x_2 < 1$ and

$$K_2=\frac{\Gamma(\gamma_1+\nu_1-\alpha_1)\Gamma(\gamma_1+\nu_1-\beta_1)}{\Gamma(\gamma_1)\Gamma(\nu_1)\Gamma(\gamma_1+\nu_1-\alpha_1-\beta_1)}\{B(\nu_2,\gamma_2){}_1F_1(\gamma_2;\nu_2+\gamma_2;\lambda)\}^{-1}\,.$$

Transforming $Z = X_1X_2$, $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \to z, x_2) = 1/x_2$ in (6) and integrating x_2 we obtain the marginal p.d.f. of Z as

$$K_2\frac{z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}\int_0^1\frac{w^{\gamma_2-1}(1-w)^{\gamma_1+\nu_2-1}}{[1-w/(1+z)]^{\nu_1+\gamma_1}}$$
$$\times\exp(\lambda w){}_2F_1\left(\alpha_1,\beta_1;\gamma_1;\frac{(1+z)^{-1}(1-w)}{1-w/(1+z)}\right)dw\,. \qquad (7)$$

Now, expanding Gauss hypergeometric functions in the integral (7) in terms of power series we arrive at

$$\frac{K_2z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\beta_1)_r(1+z)^{-r}}{(\gamma_1)_r\,r!}\int_0^1\frac{w^{\gamma_2-1}(1-w)^{\gamma_1+\nu_2+r-1}\exp(\lambda w)\,dw}{[1-w/(1+z)]^{\nu_1+\gamma_1+r}}\,.$$

Finally, applying (A.8) and substituting for K_2 we obtain the desired result. □

Corollary 3.2.1. *Let X_1 and X_2 be independent, $X_1 \sim B^{II}(\nu_1, \gamma_1)$, and $X_2 \sim KB(\nu_2, \gamma_2, \lambda)$. Then, the p.d.f. of $Z = X_1 X_2$ is given by*

$$\frac{\Gamma(\gamma_1+\nu_1)\Gamma(\gamma_2+\nu_2)\Gamma(\gamma_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma_1)\Gamma(\gamma_1+\gamma_2+\nu_2)}\{{}_1F_1(\gamma_2;\nu_2+\gamma_2;\lambda)\}^{-1}$$
$$\times\frac{z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}\mathbf{\Phi}_1\left[\gamma_2,\nu_1+\gamma_1;\gamma_1+\gamma_2+\nu_2;\frac{1}{1+z},\lambda\right],\quad z>0\,.$$

Corollary 3.2.2. *Let X_1 and X_2 be independent, $X_1 \sim B^{II}(\nu_1, \gamma_1)$ and $X_2 \sim B^{I}(\nu_2, \gamma_2)$. Then, the p.d.f. of $Z = X_1 X_2$ is given by*

$$\frac{\Gamma(\gamma_1+\nu_1)\Gamma(\gamma_2+\nu_2)\Gamma(\gamma_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma_1)\Gamma(\gamma_1+\gamma_2+\nu_2)}$$
$$\times\frac{z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}{}_2F_1\left(\gamma_2,\nu_1+\gamma_1;\gamma_1+\gamma_2+\nu_2;\frac{1}{1+z}\right),\quad z>0\,.$$

Nagar and Zarrazola [4], while studying the distribution of the product of two independent Kummer–beta variables, have also derived the above result.

Corollary 3.2.3. *Let X_1 and X_2 be independent random variables, $X_1 \sim IH^{I}(\nu_1, \alpha_1, \beta_1, \gamma_1)$ and $X_2 \sim B^{I}(\nu_2, \gamma_2)$. Then, the p.d.f. of $Z = X_1 X_2$ is given by*

$$\frac{\Gamma(\gamma_1+\nu_1-\alpha_1)\Gamma(\gamma_1+\nu_1-\beta_1)\Gamma(\gamma_2+\nu_2)\Gamma(\gamma_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma_1)\Gamma(\gamma_1+\nu_1-\alpha_1-\beta_1)\Gamma(\gamma_1+\gamma_2+\nu_2)}$$
$$\times\frac{z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\beta_1)_r(\gamma_1+\nu_2)_r}{(\gamma_1+\gamma_2+\nu_2)_r(\gamma_1)_r\, r!(1+z)^r}$$
$$\times{}_2F_1\left(\gamma_2,\nu_1+\gamma_1+r;\gamma_1+\gamma_2+\nu_2+r;\frac{1}{1+z}\right),\quad z>0\,.$$

Theorem 3.3. *Let the random variables X_1 and X_2 be independent, $X_1 \sim IH^{I}(\nu_1, \alpha_1, \beta_1, \gamma_1)$ and $X_2 \sim B^{III}(\nu_2, \gamma_2)$. Then, the p.d.f. of $Z = X_1 X_2$ is given by*

$$\frac{\Gamma(\gamma_1+\nu_1-\alpha_1)\Gamma(\gamma_1+\nu_1-\beta_1)\Gamma(\nu_2+\gamma_2)\Gamma(\nu_2+\gamma_1)}{2^{\gamma_2}\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma_1)\Gamma(\gamma_1+\nu_1-\alpha_1-\beta_1)\Gamma(\gamma_1+\gamma_2+\nu_2)}$$
$$\times\frac{z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\beta_1)_r(\gamma_1+\nu_2)_r(1+z)^{-r}}{(\gamma_1)_r(\gamma_1+\gamma_2+\nu_2)_r\, r!}$$

$$\times F_1\left(\gamma_2, \nu_1+\gamma_1+r, \nu_2+\gamma_2; \gamma_1+\gamma_2+\nu_2+r; \frac{1}{1+z}, \frac{1}{2}\right), \quad z>0\,.$$

Proof. The joint p.d.f. of X_1 and X_2 is given by

$$K_3 \frac{x_1^{\nu_1-1} x_2^{\nu_2-1}(1-x_2)^{\gamma_2-1}}{(1+x_1)^{\nu_1+\gamma_1}(1+x_2)^{\nu_2+\gamma_2}}\, {}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{1}{1+x_1}\right), \tag{8}$$

where $x_1>0$, $0<x_2<1$ and

$$K_3 = \frac{\Gamma(\gamma_1+\nu_1-\alpha_1)\Gamma(\gamma_1+\nu_1-\beta_1)}{\Gamma(\gamma_1)\Gamma(\nu_1)\Gamma(\gamma_1+\nu_1-\alpha_1-\beta_1)} 2^{\nu_2}\{B(\nu_2,\gamma_2)\}^{-1}\,.$$

Now, transforming $Z = X_1X_2$, $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \to z, x_2) = 1/x_2$ in (8) and integrating x_2, the marginal p.d.f. of Z is derived as

$$\frac{K_3 z^{\nu_1-1}}{2^{\nu_2+\gamma_2}(1+z)^{\nu_1+\gamma_1}} \int_0^1 \frac{w^{\gamma_2-1}(1-w)^{\nu_2+\gamma_1-1}}{[1-w/(1+z)]^{\nu_1+\gamma_1}\,(1-w/2)^{\nu_2+\gamma_2}} \times {}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{(1+z)^{-1}(1-w)}{1-w/(1+z)}\right) dw\,. \tag{9}$$

Expanding Gauss hypergeometric functions in the integral (9) in series form we arrive at

$$\frac{2^{-(\nu_2+\gamma_2)}K_3 z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}} \sum_{r=0}^{\infty} \frac{(\alpha_1)_r(\beta_1)_r}{(\gamma_1)_r\, r!(1+z)^r} \int_0^1 \frac{w^{\gamma_2-1}(1-w)^{\nu_2+\gamma_1+r-1}\,dw}{[1-w/(1+z)]^{\nu_1+\gamma_1+r}(1-w/2)^{\nu_2+\gamma_2}}\,.$$

Finally, the desired result follows by using (A.7) and substituting for K_3. □

Corollary 3.3.1. *Let the random variables X_1 and X_2 be independent, $X_1 \sim B^{II}(\nu_1,\gamma_1)$ and $X_2 \sim B^{III}(\nu_2,\gamma_2)$. Then, the p.d.f. of $Z = X_1X_2$ is given by*

$$\frac{\Gamma(\gamma_1+\nu_1)\Gamma(\nu_2+\gamma_2)\Gamma(\nu_2+\gamma_1)}{2^{\gamma_2}\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma_1)\Gamma(\gamma_1+\gamma_2+\nu_2)} \frac{z^{\nu_1-1}}{(1+z)^{\gamma_1+\nu_1}} \times F_1\left(\gamma_2, \nu_1+\gamma_1, \nu_2+\gamma_2; \gamma_1+\gamma_2+\nu_2; \frac{1}{1+z}, \frac{1}{2}\right), \quad z>0\,.$$

The above result has also been obtained by Sánchez and Nagar [5].

Theorem 3.4. *Let the random variables* X_1 *and* X_2 *be independent. Further,* $X_1 \sim IH^I(\nu_1, \alpha_1, \beta_1, \gamma_1)$ *and* $X_2 \sim H^I(\nu_2, \alpha_2, \beta_2, \gamma_2)$. *Then, the p.d.f. of* $Z = X_1 X_2$ *is given by*

$$\left\{\prod_{i=1}^{2} \frac{\Gamma(\gamma_i + \nu_i - \alpha_i)\Gamma(\gamma_i + \nu_i - \beta_i)}{\Gamma(\gamma_i)\Gamma(\nu_i)\Gamma(\gamma_i + \nu_i - \alpha_i - \beta_i)}\right\} B(\gamma_2, \nu_2 + \gamma_1)$$
$$\times \frac{z^{\nu_1 - 1}}{(1+z)^{\nu_1+\gamma_1}} \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\alpha_2)_s(\beta_2)_s(\alpha_1)_r(\beta_1)_r(\nu_2+\gamma_1)_r}{(\gamma_1)_r(\nu_2+\gamma_1+\gamma_2)_{s+r}\, s!\, r!}(1+z)^{-r}$$
$$\times {}_2F_1\left(\gamma_2 + s, \nu_1 + \gamma_1 + r; \nu_2 + \gamma_2 + \gamma_1 + s + r; \frac{1}{1+z}\right), \quad z > 0\,.$$

Proof. The joint p.d.f. of X_1 and X_2 is given by

$$\frac{K_4 x_1^{\nu_1-1} x_2^{\nu_2-1}(1-x_2)^{\gamma_2-1}}{(1+x_1)^{\nu_1+\gamma_1}} {}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{1}{1+x_1}\right) {}_2F_1(\alpha_2, \beta_2; \gamma_2; 1-x_2), \tag{10}$$

where $0 < x_1 < \infty$, $0 < x_2 < 1$ and

$$K_4 = \prod_{i=1}^{2} \frac{\Gamma(\gamma_i + \nu_i - \alpha_i)\Gamma(\gamma_i + \nu_i - \beta_i)}{\Gamma(\gamma_i)\Gamma(\nu_i)\Gamma(\gamma_i + \nu_i - \alpha_i - \beta_i)}\,.$$

Now, transforming $Z = X_1 X_2$, $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \to x_1, z) = 1/x_2$ in (10) and integrating x_2, we obtain the p.d.f. of Z as

$$\frac{K_4 z^{\nu_1-1}}{(1+z)^{\nu_1+\gamma_1}} \int_0^1 \frac{w^{\gamma_2-1}(1-w)^{\nu_2+\gamma_1-1}}{[1-w/(1+z)]^{\nu_1+\gamma_1}}\, {}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{(1+z)^{-1}(1-w)}{1-w/(1+z)}\right)$$
$$\times {}_2F_1\left(\alpha_2, \beta_2; \gamma_2; w\right)\, dw, \quad z > 0\,.$$

Now, expanding the Gauss hypergeometric functions in series form, integrating the resulting expression using (A.3), substituting for K_4 and simplifying, we obtain the desired result. □

4 Products of Three Independent Random Variables

In this section we derive distributions of products of three independent random variables involving inverted hypergeometric function type I, beta type I and beta type II variables.

Theorem 4.1. *Let X_1, X_2 and X_3 be independent random variables, $X_i \sim B^I(a_i, b_i)$, $i = 1, 2$ and $X_3 \sim IH^I(\nu, \alpha, \beta, \gamma)$. Then, the p.d.f. of $Z = X_1X_2X_3$ is given by*

$$\frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)\Gamma(a_1+\gamma)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+b_1+b_2+\gamma)}\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\nu)\Gamma(\gamma)\Gamma(\gamma+\nu-\alpha-\beta)}$$
$$\times\frac{z^{\nu-1}}{(1+z)^{\nu+\gamma}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(b_2)_s(a_1+b_1-a_2)_s(\alpha)_r(\beta)_r(a_1+\gamma)_r}{(\gamma)_r(a_1+b_1+b_2+\gamma)_{s+r}\, s!\, r!}(1+z)^{-r}$$
$$\times {}_2F_1\left(b_1+b_2+s, \nu+\gamma+r; a_1+b_1+b_2+\gamma+s+r; \frac{1}{1+z}\right), \quad z > 0\,.$$

Proof. Using Theorem 2.1, $X_1X_2 \sim H^I(a_1, b_2, a_1+b_1-a_2, b_1+b_2)$. Now, using independence of X_1, X_2 and X_3 and Theorem 3.4, we obtain the desired result. □

Corollary 4.1.1. *Let X_1, X_2, and X_3 be independent, $X_i \sim B^I(a_i, b_i)$, $i = 1, 2$ and $X_3 \sim B^{II}(a_3, b_3)$. Then, the p.d.f. of $Z = X_1X_2X_3$ is given by*

$$\frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)\Gamma(a_3+b_3)\Gamma(a_1+b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_1+b_1+b_2+b_3)\Gamma(b_3)}$$
$$\times\frac{z^{a_3-1}}{(1+z)^{a_3+b_3}}\sum_{s=0}^{\infty}\frac{(b_2)_s(a_1+b_1-a_2)_s}{(a_1+b_1+b_2+b_3)_s\, s!}$$
$$\times {}_2F_1\left(b_1+b_2+s, a_3+b_3; a_1+b_1+b_2+b_3+s; \frac{1}{1+z}\right), \ z > 0\,.$$

Appendix: Additional Definitions and Results

Here we give some definitions and additional results which are used throughout this work. We use the Pochammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$. The generalized hypergeometric function of scalar argument is defined by

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}\,, \tag{A.1}$$

where a_i, $i = 1, \ldots, p$; b_j, $j = 1, \ldots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (A.1) are available in the literature, see Luke [10]. From (A.1) it is easy to see that

$$ {}_0F_0(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z), \quad {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, $$

and

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1\,. $$

The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$ {}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} \exp(zt)\, dt\,, \tag{A.2} $$

and

$$ \begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-zt)^{-b}\, dt, \\ &\qquad |\arg(1-z)| < \pi\,, \end{aligned} \tag{A.3} $$

respectively, where $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(c-a) > 0$.

It is easy to check by using (A.3) that

$$ {}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{-z}{1-z}\right). \tag{A.4} $$

The Appell's first hypergeometric function F_1 is defined by

$$ \begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= \sum_{r,s=0}^{\infty} \frac{(a)_{r+s} (b_1)_r (b_2)_s}{(c)_{r+s}} \frac{z_1^r z_2^s}{r!\, s!} \\ &= \sum_{r=0}^{\infty} \frac{(a)_r (b_1)_r}{(c)_r} \frac{z_1^r}{r!} {}_2F_1(a+r, b_2; c+r; z_2) \\ &= \sum_{s=0}^{\infty} \frac{(a)_s (b_2)_s}{(c)_s} \frac{z_2^s}{s!} {}_2F_1(a+s, b_1; c+s; z_1)\,, \end{aligned} \tag{A.5} $$

where $|z_1| < 1$ and $|z_2| < 1$. The Humbert's confluent hypergeometric function $\mathbf{\Phi}_1$ is defined by

$$\begin{aligned}\mathbf{\Phi}_1[a, b_1; c; z_1, z_2] &= \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b_1)_r}{(c)_{r+s}} \frac{z_1^r z_2^s}{r!\, s!}, \\ &= \sum_{r=0}^{\infty} \frac{(a)_r (b_1)_r}{(c)_r} \frac{z_1^r}{r!}\, {}_1F_1(a+r; c+r; z_2) \\ &= \sum_{s=0}^{\infty} \frac{(a)_s}{(c)_s} \frac{z_2^s}{s!}\, {}_2F_1(a+s, b_1; c+s; z_1)\,, \end{aligned} \tag{A.6}$$

where $|z_1| < 1$, $|z_2| < \infty$. The integral representations of F_1 and $\mathbf{\Phi}_1$ are given by

$$F_1(a, b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1}\, dv}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}}, \tag{A.7}$$

where $|z_1| < 1$, $|z_2| < 1$, $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$, and

$$\mathbf{\Phi}_1[a, b_1; c; z_1, z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1} \exp(vz_2)\, dv}{(1-vz_1)^{b_1}}, \tag{A.8}$$

where $|z_1| < 1$, $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$. Note that for $b_1 = 0$, F_1 and $\mathbf{\Phi}_1$ reduce to ${}_2F_1$ and ${}_1F_1$ functions, respectively. For properties and further results on these functions the reader is referred to Luke [10], Srivastava and Karlsson [11], Mathai and Saxena [12], and Mathai [13].

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