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APPLICATIONS OF LEVEL CURVES TO SOME PROBLEMS ON ALGEBRAIC SURFACES

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RESUMEN. En [1] se introdujo un resultado algorítmico para el cálculo de los tipos topológicos de las curvas de nivel de una superficie algebraica. A partir de este resultado, aquí presentamos aplicaciones basadas en las curvas de nivel a la determinación de ciertas características topológicas de superficies (carácter real, compacidad, conexión) y al problema del *plotting*.

ABSTRACT. In [1], a result to algorithmically compute the topology types of the level curves of an algebraic surface, is given. From this result, here we derive applications based on level curves to determine some topological features of surfaces (reality, compactness, connectivity) and to the problem of plotting.

1. Introduction

The study of the level curves of an implicit algebraic surface (i.e. the sections of the surface with real planes parallel to the xy-plane) gives a clue on how the surface is like. Take for example the well-known case of the Whitney Umbrella, whose equation is $x^2 - y^2 z = 0$. It is clear that for $z > 0$ the level curves consist of two intersecting lines; for $z = 0$, the level curve reduces to one line; and for $z < 0$, the level curves consist of just one real isolated point. From this information one may derive a good mental picture of the surface: a non-bounded surface, collapsing onto a line when $z = 0$, and with a handle attached corresponding to $z < 0$.

The problem of computing the different topology types arising in the family of level curves of a given algebraic surface (together with the z-intervals corresponding to each type) has been addressed in [1] and [15]. In our paper, we will see that by processing such information topological features of the surface, which can be useful in order to compute a reliable plotting of the surface, can be algorithmically computed. Some of these results were already announced in [2]; however, here we give a more detailed and rigorous description, and we present preliminary results on an additional question, namely the connectedness of the surface.

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More precisely, the first problem that we consider is to algorithmically decide whether a given implicit algebraic surface is *real*. That is, we provide an algorithm to check whether the intersection of the surface with $\mathbb{R}^3$ is a two-dimensional set (in the Euclidean topology) or not. Furthermore, in the negative case, the algorithm can also be used to analyze whether the real part of the surface is empty (like for example $x^2 + y^2 + z^2 + 1 = 0$), consisting of finitely many points (the case of $x^2 + y^2 + z^2 = 0$, whose real part is the origin) or corresponding to a space curve (the case of $x^2+y^2 = 0$). This kind of surfaces whose real part is not 2-dimensional may get error messages when one tries to draw a plotting.

The second application is concerned with the compactness of the surface. In this case, since one works with implicit algebraic surfaces, which are therefore closed over the usual Euclidean topology, the algorithm essentially checks whether the surface is bounded w.r.t. the variables x, y, z, respectively.

The third application has to do with surface plottings. In order to draw a plotting of a surface, the user has to introduce as an input a "box" $I = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2] \subset \mathbb{R}^3$; so, the output shows the part of the surface lying inside the box. Now, if the user is interested in computing a plotting where the main topological features of the surface are shown (i.e. which makes clear how the surface is like), some previous information must be known in order to properly choose I. Using the information on the topology of the level curves of the surface w.r.t. the variables x, y, z, we provide an algorithm to compute a "good" box I.

Finally, as a fourth application we provide some results on the connectedness of the surface that can be derived form the topology of the level curves, and we sketch a symbolic-numeric algorithm, based on level curves, to compute the number of connected components of an algebraic surface. Some numerical aspects of this algorithm are still under study, and hence part of this section can be considered as a preliminary version of our results on this matter.

The structure of the paper is the following. The second section contains some preliminary notions and related results on level curves. The third section is devoted to the problem of checking whether a given implicit algebraic surface is real, or not. The fourth section analyzes compactness. The problem of computing a suitable box for plotting a surface (so that the output shows the more relevant topological features of the surface) is addressed in the fifth section. The last section is concerned with connectedness.

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2. Preliminaries on Level Curves

In this section we provide some preliminary notions and results concerning level curves, that are taken from [1]. Thus, we refer the interested reader to [1] for further information. Moreover, here we also fix the notation to be used along the paper, together with some hypotheses to be requested on the surface S.

In the sequel, we consider an algebraic surface S defined by an square-free polynomial $F \in \mathbb{R}[x, y, z]$ having no univariate factor only depending on the variable z; so, S has no component which is a plane parallel to the xy-plane. Moreover, we also assume that the leading coefficient of F with respect to the variable y does not depend on x. In this situation, the z-level curves (or z-slices, or simply the *level curves*) of S are the (plane) curves obtained by intersecting S with planes normal to the z-axis. Furthermore, given $b \in \mathbb{R}$ we will denote the level curve corresponding to the plane $z - b = 0$ as S_b. One might similarly define the level curves corresponding to the x-axis and the y-axis, respectively. Thus, when necessary we will speak of ξ-level curves, where $\xi \in \{x, y, z\}$. Notice that the topology type of a ξ-level curve can be described by means of a graph homeomorphic to it; the computation of such a graph is a well-studied problem (see for example [6], [8], [10] and many others). Furthermore, observe that the requirement on $\mathrm{lcoeff}_y(F)$ (namely, that it does not depend on x) can always be fulfilled by applying if necessary a rotation around the z-axis, which does not modify the topology of the level curves of S.

From Hardt's Semi-Algebraic Triviality Theorem (see [4]) it can be derived that the number of topology types of the level curves of S is always finite. In case that S is compact and non-singular the problem of determining these topology types can be solved by using Morse Theory (see [4], [14]). In the more general case of singular surfaces, two approaches can be considered. The first one comes from Differential Topology and uses elements of Whitney Stratification Theory (see [9]). This approach has been used in [15]. The second one comes from Computer Algebra and uses as an essential tool the notion of delineability (see [12]). This second approach has been developed in [1]. So, in the rest of the section we briefly recall some notions and results of[1].

Definition 1. *We say that $a \in \mathbb{R}$ is a* Critical Level Value *if the topology of the level curves of S changes at $z = a$, i.e. $\forall\ \epsilon > 0$ there exists $a_\epsilon \in (a - \epsilon, a + \epsilon)$ such that the level curves corresponding to a and a_ϵ, have different topology types. Moreover, we say that $\mathcal{A} \subset \mathbb{R}$ is a* Critical Level Set *of S, if $\mathcal{A}$ is finite and it contains all the critical level values of S.*

Since the number of topology types of the level curves of S is always finite, the number of critical level values is also finite and therefore a critical level set always exists. Moreover, once that a critical level set $\mathcal{A}$ has been computed, the topology types of the level curves can be obtained. Indeed, writing $\mathcal{A} = \{\alpha_1, \ldots, \alpha_r\}$, we can decompose the z-axis as

$$(-\infty, \alpha_1) \cup \{\alpha_1\} \cup (\alpha_1, \alpha_2) \cup \cdots \cup (\alpha_{r-1}, \alpha_r) \cup \{\alpha_r\} \cup (\alpha_r, \infty).$$

Thus, taking a z-value for each open interval, and applying the existing algorithms for computing the topology type of a plane algebraic curve, one determines the topology type of all the z-level curves with z in some interval. The remaining finitely many level curves, corresponding to the $F(x, y, \alpha_i)$'s, where $i = 1, \dots, r$, are also analyzed with the same strategy.

Therefore, the problem of determining the topology types of the level curves of S reduces to the computation of a critical level set. Now we consider the following notation: $D_w(G)$ denotes the discriminant of a polynomial G w.r.t. the variable w, i.e. $D_w(G) = \mathrm{Res}_w(G, \frac{\partial G}{\partial w})$, $\sqrt{G}$ denotes the square-free part of a polynomial G, and:

$$M(x,z) := \begin{cases} 0 & \text{if } \deg_y(F) = 0 \\ \sqrt{D_y(F)} & \text{otherwise} \end{cases}$$

$$R(z) := \begin{cases} 0 & \text{if } \deg_x(M) = 0 \\ D_x(M(x,z)) & \text{otherwise} \end{cases}$$

Then, the following result holds (see [1]):

Theorem 2. *Let S satisfy the above hypotheses. Then, it holds that:*

(1) *If $R(z)$ is not identically zero, then the set of real roots of $R(z)$ is a critical level set.*
(2) *If $R(z)$ and M are identically zero, then the set of real roots of $D_x(F)$ is a critical level set.*
(3) *If $R(z)$ is identically zero but M is not identically* 0*, then the set of real roots of $M(z)$ is a critical level set.*

Remark 1. *If $R(z)$ is a non-zero constant, then there is just one topology type for all the level curves. Similarly for the case when $M(z)$ is a non-zero constant.*

3. First Application: Reality of Algebraic Surfaces

We say that an algebraic plane curve is *real*, if it has infinitely many real points; so, the real part of a non-real plane curve is either empty or consisting of finitely many real points. Also, one can check whether a given algebraic curve is real by means of well-known algorithms (see [16]). For surfaces, we say that an algebraic surface S is *real*, if $S \cap \mathbb{R}^3$ has dimension 2 over the usual Euclidean topology; hence, if S is not real then its real part is at most contained in a space algebraic curve, and may be even empty or consisting of finitely many points. Now the problem of checking whether S is real can be solved by using C.A.D. techniques. Nevertheless, here we will see that it can also be derived from the information on the level curves. Essentially, we will prove that the question can be reduced to a problem in $\mathbb{R}^2$, i.e. to checking wether certain plane algebraic curves are real or not. Now the following theorem, which can be found in [11] (see Theorem XI.3.6 there), is essential for our purposes. Here, we consider the usual definition of regular point of a surface $F(x, y, z) = 0$, i.e. a point P of the surface is *regular* if some first partial derivative of F P does not vanish at P.

Theorem 3. *S is a real surface if and only if it has at least one real regular point.*

From this theorem, one may derive another result, concerning the level curves of S, which can be used to algorithmically check whether S is real. In order to see this, we need the following previous result. We denote the partial derivatives of F w.r.t. the variables x, y, z, respectively, as F_x, F_y, F_z.

Proposition 4. *Let $F \in \mathbb{R}[x, y, z]$, with $\deg_y(F) > 0$, be the defining polynomial of the surface S, and let $\mathcal{A}$ be the critical level set of S computed by means of Theorem 2. If $a \in \mathbb{R}$ satisfies some of the following conditions:*

(i) *a is a root of the leading coefficient of F w.r.t. y,*
(ii) *a is a root of the leading coefficient of $M(x, z)$ w.r.t. x,*
(iii) *the polynomial $F(x, y, a)$ has multiple factors,*

then $a \in \mathcal{A}$.

Proof. In order to prove the result, we distinguish the cases when the polynomial R, defined in Section 2, is identically 0, or not. Let us see first the case when $R \neq 0$. Here, we observe that if a is a root of the leading coefficient of F w.r.t. y, by using the Sylvester form of the resultant $\mathrm{Res}_y(F, F_y)$ one has that $z - a$ is a factor of $D_y(F) = \mathrm{Res}_y(F, F_y)$, and therefore of M. Thus, it is also a factor of the leading coefficient of M w.r.t. x, and consequently, condition (i) implies condition (ii). Now let us see that condition (ii) implies that $a \in \mathcal{A}$. Indeed, since $R(z) = D_x(M) = \mathrm{Res}_x(M, M_x)$, again from the Sylvester form of the resultant one deduces that any factor of the leading coefficient of M w.r.t. x is also a factor of $R(z)$. Thus, $R(a) = 0$ and $a \in \mathcal{A}$. So, it remains to see that if a verifies (iii) but not (i), then $R(a) = 0$. Let $H(x, z) = D_y(F)$; then $M(x, z) = \sqrt{H(x, z)}$. Since a does not verify condition (i), the discriminant of F w.r.t. y behaves properly under specializations when $z = a$, i.e. $D_y(F_a) = H(x, a)$ (see Lemma 4.3.1 in [17]). However, since F_a has multiple factors, we get that $D_y(F_a) = 0$, and hence $H(x, a) = 0$. Therefore, $z - a$ is a factor of $D_y(F)$, and consequently condition (ii) occurs; but in that case we have already seen that $a \in \mathcal{A}$. Now let us see the case when $R = 0$. Since $\deg_y(F) > 0$, by Theorem 2 it holds that $M \neq 0$ but $M \in \mathbb{R}[z]$. Also by Theorem 2, $\mathcal{A}$ is the set of real roots of M. Thus, if a satisfies condition (ii) then obviously $a \in \mathcal{A}$. Moreover, if a satisfies condition (i), arguing as above condition (ii) is also satisfied, and therefore $a \in \mathcal{A}$. Finally, if (a) satisfies (iii), we argue as in the case $R \neq 0$. □

Now, we can prove the following result concerning the level curves of S.

Theorem 5. *Let $\mathcal{A}$ be a critical set of the surface S determined by applying Theorem 2. Then, S is real if and only if there exists at least one real level curve S_a of S, with $a \in \mathbb{R}$ and $a \notin \mathcal{A}$.*

Proof. If S is real, then by Theorem 3 there exists a regular real point $P \in S$. Thus, the implication ($\Rightarrow$) follows from Implicit Function Theorem. So, let us consider the implication ($\Leftarrow$). For this purpose, we separately analyze the cases when $F \in \mathbb{R}[x, z]$ and when F depends on the variable y, respectively. We start with the case $F \in \mathbb{R}[x, z]$. In this situation, let $\mathcal{C}_{xz}$ be the plane algebraic curve defined by F in the xz-plane. Thus, S is real iff $\mathcal{C}_{xz}$ is real; so, let us see this. By hypothesis there exists $a \in \mathbb{R}$ such that $a \notin \mathcal{A}$, and satisfying that the corresponding level curve S_a is real. Since F does not depend on y, the level curves of S are lines normal to the xz-plane. Therefore, since S_a is a real curve, one has that the intersection point of S_a with the xz-plane, which we denote as P_a, is also real. By Theorem 2, $\mathcal{A}$ is the set of real roots of the discriminant $D_x(F)$. Thus, a is not a root of $D_x(F)$. Therefore, P_a is not a singular point of $\mathcal{C}_{xz}$ and, since it has a real regular point, from the Implicit Function Theorem $\mathcal{C}_{xz}$ is a real curve. Thus, ($\Leftarrow$) holds for the case when $F = F(x,z)$. Finally, let us see that ($\Leftarrow$) also holds when F depends on the variable y, i.e. when $\deg_y(F) > 0$. In order to see this, let S_a be a level curve of S, real, and corresponding to the intersection of S with the real plane $z = a$, where $a \notin \mathcal{A}$. Since $a \notin \mathcal{A}$, by Proposition 4 the polynomial $F_a(x,y) = F(x,y,a)$ is square-free. Thus, since S_a is real and $F_a(x,y)$ is square-free, we have that S_a has at least one real non-singular point $(x_a, y_a) \in S_a$. Then, (x_a, y_a, a) is a real non-singular point of the surface S and therefore by Theorem 3 the surface S is real. □

This theorem can be used to derive an algorithm for checking the reality of an algebraic surface. For this purpose, note that the condition in Theorem 5, i.e. the existence of a real level curve of S corresponding to a non-critical level value, can be tested by checking the reality of the level curves corresponding to intermediate z-values in between two consecutive critical level values. More precisely, one has the following algorithm:

Algorithm: (Reality of an algebraic surface S) Given an algebraic surface S implicitly defined by a real polynomial $F(x,y,z)$, square-free, with no factor only depending on the variable z, and such that $\mathrm{lcoeff}_y(F)$ does not depend on the variable x, the algorithm decides whether S is real.

(1) Compute a critical set of S, $\mathcal{A} = \{a_1, \dots, a_r\}$ by means of Theorem 2. Let $a_0 = -\infty$, $a_{r+1} = \infty$.
(2) Check whether there exists $i \in \{0, \dots, r\}$ such that the plane algebraic curve defined by $F(x,y,\xi_i)$, where ξ_i is taken in the interval (a_i, a_{i+1}), is real. If it is, then **return** $\ll S$ is real$\gg$ **else return** $\ll S$ is not real$\gg$.

Remark 2. *If the surface is not real there are three cases: (i) the real part of the surface reduces to a space curve; (ii) it consists of finitely many points; (iii) it is empty. Then, one can algorithmically identify the case by looking at the level curves. More precisely, in case (iii) all the level curves are empty; in case (ii), there are just finitely many non-empty level curves, all of them corresponding to z-critical level values, and consisting of finitely many real points. Finally, case (i) is identified when (ii) and (iii) do not happen, and all the level curves corresponding to non-critical z-values are either empty or consisting in finitely many real points.*

Example 1. *Let S be the algebraic surface defined by*

$$F(x,y,z) = (x^2-1)^2 + (y^2-1)^2 + (z^2-1)^2 - 3/2.$$

which satisfies the required hypotheses. Let us see whether S is real. For this purpose, by applying Theorem 2 the following z-critical level set of S is computed:

$$\mathcal{A}_z = \{-1.491557867, -1.306562965, -0.5411961001, \\ 0.5411961001, 1.306562965, 1.491557867\}$$

Then we check whether there exists some level curve, corresponding to a z-value not in $\mathcal{A}_z$, which is real. For $z < -1.491557867$ we get that the level curves are empty over $\mathbb{R}^2$, but for $z = -7/5$, which is intermediate between -1.491557867 and -1.306562965, we get the z-slice

$$\{(x^2-1)^2 + (y^2-1)^2 - 723/1250 = 0, z = -7/5\}$$

which is real. Therefore, we conclude that S is real (see Figure 1).

Example 2. *Consider the surface S defined by*

$$F(x,y,z) = x^4 + 2x^2y^2 - 2x^2 + y^4 - 2y^2 + 1 + z^2.$$

In this case, a z-critical set of S is $\mathcal{A}_z = \{0\}$, i.e. the z-slices of S have at most three different topology types. However, the z-slices for $z = -1$ and $z = 1$ are empty curves over $\mathbb{R}$; so, from Theorem 2 we deduce that for $z > 0$ and $z < 0$ the surface is empty over the reals. Therefore, S is not real. In fact, the only real points of the surface are the points of the z-slice corresponding to $z = 0$, which is the circle

$$\{(x^2+y^2-1)^2 = 0, z = 0\}$$

4. Second application: Compactness

Here we show how to use level curves to algorithmically decide whether S is compact. Now since S is implicitly defined by a polynomial $F \in \mathbb{R}[x,y,z]$, then it is obviously closed. Thus, in order to check whether it is compact it suffices to check whether it is bounded, which amounts to deciding whether it is bounded w.r.t. the x, y and z variables, respectively.

Let us see how to check whether S is bounded w.r.t. the variable z. For this purpose, let $\mathcal{A} = \{a_1, \ldots, a_r\}$ be a critical level set of S, where $a_1 < \cdots < a_r$. Then, S is bounded w.r.t. the variable z iff for $z > a_r$ and $z < a_1$ the level curves of S are empty over $\mathbb{R}^2$. Moreover, since by Theorem 2 the topology type of the level curves of S stays invariant for $z > a_r$ and for $z < a_1$, in order to check whether the condition holds it suffices to take $z_0 < a_1$ and $z_{r+1} > a_r$, and then to analyze whether the level curves $S_{z_0}, S_{z_{r+1}}$ are empty or not over $\mathbb{R}^2$. For this purpose one may adapt the strategy for deciding whether a given algebraic curve is real.

Similarly for the x and y variables. However, observe that in order to compute ξ-critical sets, with $\xi \in \{x,y,z\}$, by means of Theorem 2, one needs that the hypotheses of Theorem 2 hold not only for the variable z, but also for x, y. To ensure this, one may always apply if necessary a linear transformation so that the polynomial $F \in \mathbb{R}[x, y, z]$ defining S has no univariate factors, and $\text{lcoeff}_x(F)$, $\text{lcoeff}_y(F)$ and $\text{lcoeff}_z(F)$ are all constant. Observe that such a transformation preserves the topological properties of the surface.

Thus, one may derive the following algorithm:

Algorithm: (Compactness of an algebraic surface S) Given an algebraic surface S implicitly defined by a real polynomial $F(x, y, z)$, square-free, with no univariate factor, and such that $\text{lcoeff}_x(F), \text{lcoeff}_y(F), \text{lcoeff}_z(F)$ are constant, the algorithm decides whether S is compact.

(1) Compute an upper bound k_z of the absolute value of the elements of a critical set of S.
(2) Check if the plane algebraic curves defined by $F(x, y, -k_z-1)$ and $F(x, y, k_z+1)$ are both empty over $\mathbb{R}^2$. If this does not happen, then **return** $\ll S$ is not compact$\gg$.
(3) Proceed in an analogous way for the variables x and y. If all the tested plane curves are empty over $\mathbb{R}^2$, **return** $\ll S$ is compact$\gg$, **else return** $\ll S$ is not compact$\gg$.

Remark 3. *In this case one does not need to compute the real roots of the polynomial provided by Theorem 2, but just upper and lower bounds on them. This can be done by applying existing algorithms (see for example* [13]*).*

Example 3. *Let S be the surface in Example 1, which fulfills all the requirements of the algorithm before, and let us see whether it is compact. Because of the symmetry of the surface, we just need to examine the z-level curves. More precisely, we just have to check whether the z-slices below the least z-critical level value and above the greatest z-critical level value are both empty over $\mathbb{R}^2$. In this sense, for $z = -2$ and $z = 2$ one gets*

$$\{(x^2 - 1)^2 + (y^2 - 1)^2 + 15/2 = 0, z = -2\}$$
$$\{(x^2 - 1)^2 + (y^2 - 1)^2 + 15/2 = 0, z = 2\}$$

which are obviously empty over $\mathbb{R}^2$. Therefore, we deduce that S is bounded, and therefore it is compact (see Figure 1).

An alternative to this approach would be the following: whenever S has no asymptotic plane of the type $z - z_0 = 0$, S is bounded iff: (i) the level curves of S above (resp. below) the highest (resp. the lowest) z-critical level values are empty over the reals; (ii) all the the z-slices of S are bounded curves. Therefore, in this case just the level curves w.r.t. z need to be considered. One may check that if $z - z_0 = 0$ is an asymptotic plane of S, then the homogenization $\hat{F}$ of F, particularized at $z = z_0$, must contain the line of infinity; hence, z_0 must be a real root of $\text{lcoeff}_y(F)$. So, whenever $\text{lcoeff}_y(F)$ is constant (which can be achieved by applying if necessary an affine transformation), S has no asymptotic plane of the considered type. The disadvantage of this alternative approach is that it requires to describe the topology of several plane curves, while in the algorithm that we provided before one just needs to check whether some curves are empty over the reals, or not.

5. Third application: Plotting Boxes

Here we address the problem of computing an interval $I = [-a, a] \times [-b, b] \times [-c, c] \subset \mathbb{R}^3$, so that the plotting of S in I shows the main relevant topological features of S. For this purpose, the information on the ξ-level curves of S, $\xi \in \{x, y, z\}$, is used. More precisely, we consider the following definition, which provides a criterion to compute I.

Definition 6. *We say that the interval $[-m_x, m_x] \times [-m_y, m_y] \times [-m_z, m_z] \subset \mathbb{R}^3$ is* suitable for plotting *S if, for $\xi \in \{x, y, z\}$, $-m_\xi, m_\xi$ are not ξ-critical level values of S and $[-m_\xi, m_\xi]$ contains* all *the ξ-critical level values of S.*

Thus, if I is "suitable for plotting" S, one can be sure that out of I there is no change in the topology type of the ξ-level curves of S. Note that the computation of a suitable I requires to compute critical level sets for the variables x, y, z, respectively, so one requests the same hypotheses as in Section 4. Observe also that Remark 3 also holds for this case. Thus, the following algorithm is derived:

Algorithm: (Suitable interval for plotting an algebraic surface S) Given an algebraic surface S implicitly defined by a real polynomial $F(x, y, z)$, square-free, with no univariate factor, and such that $\text{lcoeff}_x(F)$, $\text{lcoeff}_y(F)$, $\text{lcoeff}_z(F)$ are constant, the algorithm determines a suitable interval $I \subset \mathbb{R}^3$ for plotting S.

(1) For $\xi \in \{x, y, z\}$ compute an upper bound k_ξ of the absolute values of the elements of a ξ-critical set.
(2) **Return** the interval $I = [-k_x, k_x] \times [-k_y, k_y] \times [-k_z, k_z]$.

Example 4. *Consider again the surface in Example 1. Here, we have that*

$$\mathcal{A}_z = \{-1.491557867, -1.306562965, -0.5411961001,\\ 0.5411961001, 1.306562965, 1.491557867\}$$

is a z-critical level set of S. Furthermore, by symmetry,

$$\mathcal{A}_z = \mathcal{A}_x = \mathcal{A}_y$$

Thus, the interval

$$I = [-1.5, 1.5] \times [-1.5, 1.5] \times [-1.5, 1.5]$$

is suitable for plotting S. The picture of the part of S lying in I is shown in Figure 1.

6. Fourth application: Connectedness

In case that $R(z)$ has no real roots, S is homeomorphic to $S_a \times \mathbb{R}$ (where a is any real value), and hence it is connected iff S_a is. Since one can algorithmically decide whether a plane algebraic curve is connected or not (for example, from the graph associated with the curve), this case is easy to address. So, in the sequel we assume that $R(z)$ has real roots. Then the information on the topology types

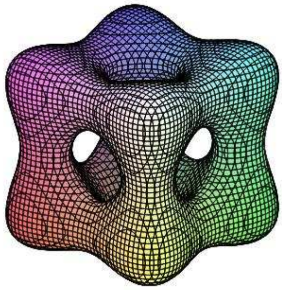

FIGURE 1. $F(x, y, z) = (x^2 - 1)^2 + (y^2 - 1)^2 + (z^2 - 1)^2 - 3/2$.

of the level curves of S provides the following sufficient condition for S to be connected.

Theorem 7. *Assume that S has no asymptotic plane of the type $z - z_0 = 0$, $z_0 \in \mathbb{R}$, and let z_{min}, z_{max} be the least and the greatest real roots of $R(z)$, respectively. Moreover, assume that S fulfills the following two conditions: (i) the level curves corresponding to non-critical z-values in between z_{min}, z_{max} are real; (ii) every level curve corresponding to a critical z-value strictly lying in between z_{min}, z_{max} is connected. Then, S is connected.*

Proof. Since S has no asymptotic plane normal to the z-axis, from the given conditions one may see that S is path-connected. Hence, it is connected. □

Notice that the condition is not necessary (think for example on a surface having several local maxima and minima with respect to z, placed at different heights). Moreover, we have already observed that asymptotic planes normal to the z-axis come from real roots of $\text{lcoeff}_y(F)$; hence, whenever $\text{lcoeff}_y(F)$ is constant (which can be achieved by almost all affine transformations) the surface has no asymptotic planes of the considered type.

Example 5. *Consider the algebraic surface S defined by $F(x, y, z) = x^2 + y^2 + z^2 + 2xyz - 1$. Notice that $\text{lcoeff}_y(F)$ is constant. A z-critical level set of S is $\mathcal{A}_z = \{-1, 1\}$. Because of the symmetry of the surface, we have that $\mathcal{A}_x = \mathcal{A}_y = \mathcal{A}_z$; so, $[-2, 2] \times [-2, 2] \times [-2, 2]$ is suitable for plotting S. A plotting of S in this interval can be seen in Figure 2; this figure was computed with* MAPLE.

Observe that from Figure 2 it is not completely clear whether the surface is connected or not. However, by using the above result we check that S is connected. Indeed, Figure 3 shows the different topology types corresponding to the z-level

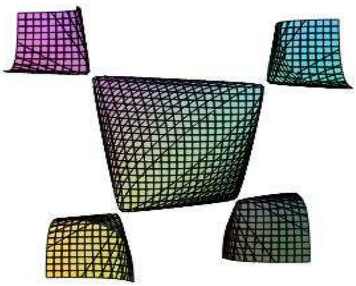

FIGURE 2. The cubic $x^2 + y^2 + z^2 + 2xyz - 1 = 0$, and its level curves.

curves for the cases $z < -1, z = -1, -1 < z < 1, z = 1, z > 1$, *respectively. Here one may check that all the hypotheses of Theorem 7 hold.*

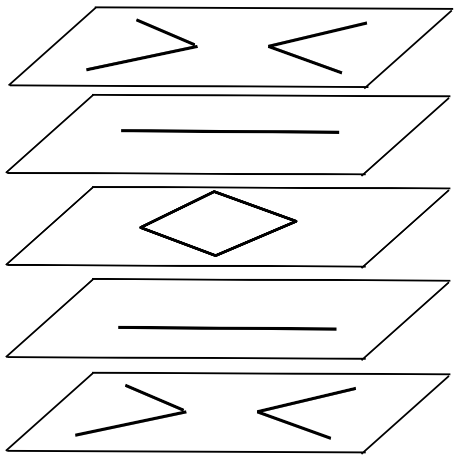

FIGURE 3. Level curves of the cubic $x^2 + y^2 + z^2 + 2xyz - 1 = 0$.

Also, one has the following sufficient condition, derivable from the topological information on the level curves, for S to be connected. The proof of this statement is straightforward.

Theorem 8. *If there exists a non-critical level value $z_a \in \mathbb{R}$ so that the corresponding level curve is empty over the reals, and there also exist $z_1, z_2 \in \mathbb{R}$, $z_1 < z_a < z_2$ so that the level curves corresponding to $z = z_1$, $z = z_2$ are not empty over the reals, then S is not connected.*

However, except in the case when the hypotheses of Theorem 7 are satisfied, the topology of the level curves does not provide enough information to derive the number of connected components of S. In order to this, we need to know how to join the level curves. So, in the sequel we propose a symbolic-numeric algorithm based on level curves to solve this question. We want to remark that numerical aspects and improvements for an efficient implementation of the method are still under study.

Essentially, the idea is to determine how the connected components of a level curve corresponding to a non-critical z-value join to the connected components of the level curves corresponding to the critical level values immediately below and above, respectively; for this purpose, we take a point on each connected component of the non-critical z-value, and we generate a space curve (as the solution of a system of differential equations) which connects it with some connected component of the critical z-level curve immediately below/above (see Figure 2). Thus, once we know how to join the connected components of the level curves corresponding to non-critical and critical z-values, the number of connected components of S can be obtained as the number of "connected chains" (whose elements are connected components of level curves) computed in the process. In Figure 4 we suggest the idea for a surface consisting of the union of two spheres, and an isolated real point; in this case, the output of the algorithm would yield three connected chains (two of them corresponding to the spheres, and one to the isolated point), and therefore we would conclude that the surface has three connected components.

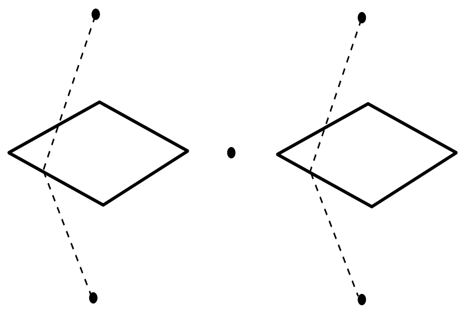

FIGURE 4. Connected chains.

In order to solve this problem, we require some more conditions on the surface S to be analyzed. More precisely, the following hypotheses must be satisfied:

(i) S is defined by an square-free polynomial F, with no factor just depending on the variables y, z. If this holds, then $\gcd(F, F_x) = 1$, and the variety $\mathcal{C}$ defined by $F = F_x = 0$ is a space algebraic curve; recall that F_x denotes the partial derivative of F w.r.t. x.
(ii) There does not exist any plane $z - a = 0$ containing infinitely many points of the curve $\mathcal{C} = V(F, F_x)$, i.e. not containing infinitely many points of S where F_x vanishes.
(iii) S is not asymptotic to any plane of equation $z - b = 0$, where b is a critical level value.

Condition (i) can be trivially checked by factorizing F. To check condition (ii), one computes $\mathrm{Res}_y(F, F_x)$, and checks whether the resultant has some factor only depending on the variable z; if this does not happen, then the condition holds (notice that the zero-set of this resultant contains the projection of $\mathcal{C}$ onto the xz-plane). Finally, condition (iii) can be checked by embedding S in the projective space, and analyzing the form of higher degree of the equation defining the projective closure of S. Furthermore, in case that S does not fulfill some of these conditions, almost all linear transformations lead to a new surface (with the same topological features than the original surface) where the three requirements hold.

Now in the sequel let $\mathcal{A} = \{a_1, \ldots, a_r\}$, where $a_1 < \cdots < a_r$, be a z-critical level set of S; furthermore, we set $a_0 = -\infty$, $a_{r+1} = +\infty$. Moreover, let $b_0 < \cdots < b_r$ verify $a_i < b_i < a_{i+1}$ for all $i \in \{0, \ldots, r\}$. With this notation, our problem is to decide, for each a_i, b_i, a_{i+1}, how to join the connected components of the level curves S_{b_i}, S_{a_i}, and $S_{b_i}, S_{a_{i+1}}$, respectively. Observe that, since S satisfies condition (iii), every connected component of a level curve of S corresponding to a non-critical z-value, joins to some connected component of the level curve corresponding to the z-critical level value immediately below (resp. above). In fact, this is the reason why we request condition (iii).

For this purpose, the strategy is to use a symbolic-numeric algorithm which essentially works as follows:

Algorithm: (Connected components of an algebraic surface S) Given an algebraic surface S implicitly defined by a real polynomial $F(x, y, z)$ satisfying the conditions (i), (ii), (iii), the algorithm determines the number of connected components of S and a description of them in terms of level curves.

(0) Compute the topology graph of S_{b_i} (if $S_{b_i} \cap \mathbb{R}^3 = \emptyset$ take another i), and the singular points of S_{a_i}; the information on the singular points of S_{a_i} will be used at step (2), in some cases (see Remark 4), and also at step (3)).
(1) Take a real point P in each connected component of S_{b_i}. Notice that this information is derived from the computation of the topology graph of S_{b_i}.
(2) Use a path continuation method to connect P with some point Q in S_{a_i}, to be computed by the algorithm; in order to do this, we travel from P to

Q by following a space curve, contained in the surface, which is computed as the solution of a system of differential equations.

(3) Identify the connected component of S_{a_i} where the final point Q belongs to. For this purpose, we compute the topology graph of S_{a_i} by introducing the point Q as a vertex of the graph.
(4) Join by an edge the starting connected component of S_{b_i} and the connected component of S_{a_i} that has been reached.
(5) Proceed in an analogous way to connect P with some point $Q^\star$ in $S_{a_{i+1}}$.
(6) After carrying out the computation for all the connected components of all the S_{b_i}'s, several connected chains are obtained, each one consisting of some connected components of the S_{b_i}'s and the S_{a_i}'s joined by edges. The number of connected chains, is the number of connected components of the surface.

Now let us describe with more detail step (2). We consider the solution to the following system of differential equations:

$$\begin{cases} x' = -F_y + \dfrac{F_z}{F_x} \\ y' = F_x \\ z' = -1 \\ x(0) = x_i; y(0) = y_i; z(0) = b_i \end{cases}$$

where $P := (x_i, y_i, b_i) \in S_{b_i}$. The solution of this differential equation provides a space curve contained in S; moreover, since $z'(t) = -1$ and S fulfills condition (iii), this space curve reaches $z = a_i$. Also, since one may decompose the part of S with $z \in (a_i, b_i)$ into non-intersecting pieces, each one corresponding to a different connected component of S_{b_i} (see [1]), the choice of the initial point for a particular connected component of S_{b_i} does not affect the connected component reached in the end. In other words, the connected component reached at $z = a_i$ is always the same for all the points of a same connected component of S_{b_i} (see Figure 5).

In general the differential system above does not have a symbolic solution, and therefore numerical methods must be applied. In our case, we used the package of MAPLE for numerically integrating differential equations. Here, one may see that as the numerical method goes on, the error considerably grows, so that the point of $z = a_i$ finally reached cannot be recognized as belonging to any connected component of S_{a_i}. For this purpose, at each step the solution provided by numerical integration must be corrected. In order to do this, each solution $\tilde{P}_{i,k} = (\tilde{x}_{i,k}, \tilde{y}_{i,k}, \tilde{z}_{i,k})$ is corrected to $\bar{P}_{i,k} = (\bar{x}_{i,k}, \bar{y}_{i,k}, \tilde{z}_{i,k})$ by computing a point of the level curve $S_{\tilde{z}_{i,k}}$ close to the solution $(\tilde{x}_{i,k}, \tilde{y}_{i,k}, \tilde{z}_{i,k})$ (observe that the z-coordinate is the same in $\tilde{P}_{i,k}$ and in $\bar{P}_{i,k}$). For this purpose, we take the line passing through $\tilde{P}_{i,k}$ in the direction of $\vec{v} = (F_x(\tilde{P}_{i,k}), F_y(\tilde{P}_{i,k}))$, and we compute the intersection points of this line with $S_{\tilde{z}_{i,k}}$. This new point is used to go on

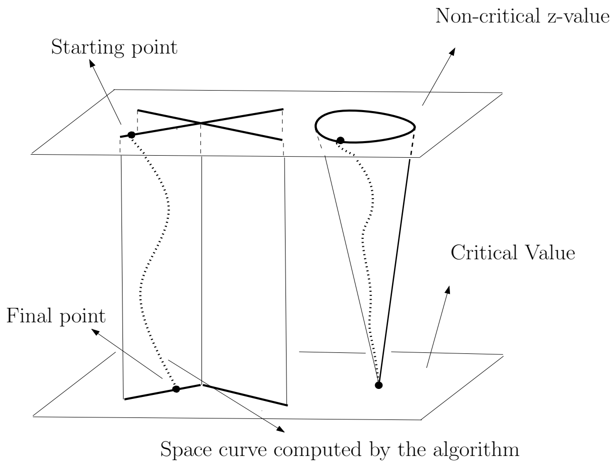

FIGURE 5. Idea of the connectedness algorithm.

with the numerical integration process. In this sense, the following remark must be taken into account.

Remark 4. *If K is a singular point of S_{a_i} (notice that these points are computed in the initial step of the algorithm), then we assume that it is reached whenever the distance between $\bar{P}_{i,k}$ and K is smaller than a sufficiently small ϵ previously fixed. Thus, in this case the computation stops and we assume that the starting point of S_{b_i} is connected with K, i.e. that $Q = K$.*

In addition, there are two more situations which must be examined carefully:

- If, before reaching the level plane $z = a_i$, the numerical integration process hits a point where F_x vanishes, then the method fails. This situation can be prevented by detecting whether $|F_x(\tilde{x}_{i,k}, \tilde{y}_{i,k}, \tilde{z}_{i,k})| < \epsilon$; if this happens, we choose a different point of $S_{\tilde{z}_{i,k}}$ to go on with the numerical process.
- It may happen that S contains a 1-dimensional subset $\mathcal{L}$ of singular points, where $\mathcal{L}$ is "isolated" in the following sense: given any point $P \in \mathcal{L}$, there exists a Euclidean neighborhood E_p of P such that every point of $S \cap E_p \cap \mathbb{R}^3$ is also a point of $\mathcal{L}$. For example, the handle of the Whitney Umbrella $x^2 - y^2 z = 0$, which is obtained for negative values of z, provides an example of this situation. Unless $\mathcal{L}$ is parallel to the xy-plane (in which case no problem arises), this phenomenon can be detected by identifying the presence of isolated points in non-critical level curves. In that situation, the points of $\mathcal{L}$ are singular points of S; so, F_x vanishes at B and therefore the system of differential equations before cannot be used to compute the point A in S_{a_i} which must be connected with B. However, in that case $\mathcal{L}$ carries the information that we wanted to extract from the system of

differential equation, and hence what we must do in this case is to take that information from the topology of the space curve defined by $\mathcal{L}$ (see [3], [5], [7]).

Finally, in order to connect $z = b_i$ and $z = a_{i+1}$, we apply an analogous process to the differential equation system:

$$\begin{cases} x' = -F_y - \dfrac{F_z}{F_x} \\ y' = F_x \\ z' = 1 \\ x(0) = x_i; y(0) = y_i; z(0) = b_i \end{cases}$$

Here, the third equation is different from the system before ($z' = 1$ instead of $z' = -1$) since in this case one has to move "up" from $z = b_i$ to $z = a_{i+1}$. One may see that also the first equation has changed. However, the space curve that one obtains by integrating these equations lies also in the surface S.

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