CONTRIBUCIONES CIENTÍFICAS
EN HONOR DE MIRIAN ANDRÉS GÓMEZ
(Laureano Lambán, Ana Romero y Julio Rubio, editores),
Servicio de Publicaciones, Universidad de La Rioja,
Logroño, Spain, 2010.

OPEN QUESTIONS RELATED TO A_∞-STRUCTURES IN A COMPUTATIONAL FRAMEWORK

A. BERCIANO, M. J. JIMÉNEZ, AND R. UMBLE

Dedicado a la memoria de Mirian Andrés Gómez

Resumen. En este artículo presentamos una breve revisión sobre las A_∞-(co)álgebras, comenzando con su origen, y continuando con algunas líneas actuales de investigación en varias disciplinas académicas y con algunos problemas abiertos relacionados con su cálculo, con una atención especial al papel de la Teoría de Perturbación Homológica.

Abstract. In this paper we give a brief review of A_∞-(co)algebras, beginning with their origin, some actual lines of research in various academic disciplines, and some open questions related to their computation with particular attention to role of Homological Perturbation Theory.

1. Introduction

In the early sixties, J. Stasheff introduced the notion of an A_n-space [29], which is a topological space X whose singular chains $A = C_*(X)$ come equipped with operations $\{m_i : A^{\otimes i} \to A\}_{1\leq i\leq n}$ that relate to one another in a systematic way. The operation m_1 is a degree -1 differential, m_2 is a multiplication, and m_1 is a derivation of m_2; thus, $m_1m_1 = 0$ and $m_1m_2 = m_2(m_1 \otimes 1 - 1 \otimes m\otimes_1)$. If m_2 is *associative up to homotopy*, there is a chain homotopy m_3, called the *associator*, that relates the two associations in three variables, i.e., $m_1m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) = m_2(1 \otimes m_2) - m_2(m_2 \otimes 1)$. In this case, $(A, m_i)_{1\leq i\leq 3}$ is an A_3-*algebra*. If there is a chain homotopy m_4 relating the five chain homotopic associations in four variables, the tuple $(A, m_i)_{1\leq i\leq 4}$ is an A_4-*algebra*, and so on.

Stasheff's definition of an A_n-space was motivated by the space ΩX of base pointed loops on a topological space $(X, *)$, whose points range over all continuous maps $\alpha : ([0, 1], \{0, 1\}) \to (X, *)$. Given $\alpha, \beta \in \Omega X$, the product $\alpha\beta$ is defined by

$$\alpha\beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Thus the product $(\alpha\beta)\gamma$ is not associative, but is *associative up to homotopy*, as indicated by the following linear change of parameter diagram:

Key words and phrases. Bar construction, Cobar construction, A_∞-(co)algebra, A_∞-bialgebra, contraction, Basic Perturbation Lemma.

This work was partially supported by "Computational Topology and Applied Mathematics" PAICYT research project FQM-296, Spanish MEC project MTM2006-03722.

In this case, $C_*(\Omega X)$ comes equipped with operations m_i for all i, and is called an A_∞-*algebra;* the complete family of structure relations is given by equation (1) below (for a more complete exposition see [18]).

In the seventies and eighties, A_∞-algebras were applied extensively in Homotopy Theory [1]; and in the nineties, A_∞-algebras assumed an important role in differential geometry and mathematical physics [30, 11]. Work of Kadeishvili and others developed the computational techniques of Homological Perturbation Theory to transfer an A_∞-algebra structure at the chain level to homology [16, 17, 12, 13]. Today, the structure relations of an A_∞-algebra are encoded by the non-Σ operad $\mathcal{A}_\infty$, and an A_∞-algebra is viewed as an algebra over $\mathcal{A}_\infty$ [21]. And there is the completely dual notion of an A_∞-coalgebra.

A current line of research, pioneered by S. Saneblidze and R. Umble [26], considers the notion of an A_∞-bialgebra, which generalizes the notion of an A_∞-algebra by joining an A_∞-algebra and an A_∞-coalgebra together in a compatible way. Thanks to this joint work, many naturally occurring examples of A_∞-bialgebras, both topological and algebraic, have been found and are now under study (see for example [27, 31, 7, 4]). Indeed, over a field, the homology of a loop space is naturally endowed with an A_∞-bialgebra structure.

In the sections that follow we will amplify and extend these introductory notions sufficiently so that the open problems that interest us can be interpreted clearly. The paper is organized as follows: Section 2 reviews the notion of an A_∞-(co)algebra and the related notions of Homological Perturbation Theory, and Section 3 discusses some open problems related to A_∞-structures from the perspective of HPT.

2. Main concepts

Whereas the aim of this paper is to give an exposition of the ideas behind the open problems itemized in Section 3, we give a somewhat cursory review here. More rigorous treatments can be found in the literature.

To better understand the notion of an A_∞-structure, we begin with a review of some notation and ideas from homological algebra. For additional related material see [19]. Let Λ be a commutative ring with unity. A *differential graded module* (DGM) is a graded module M together with a square zero morphism $d_M : M \to M$ of degree -1. The module M is *connected* whenever $M_0 = \Lambda$, in which case the *reduced module* $\bar{M}$ satisfies $\bar{M}_0 = 0$ and $\bar{M}_n = M_n$ for $n > 1$.

Throughout the paper we will strictly adhere to the *Koszul sign convention*: If $f : M \to M'$ and $g : N \to N'$ are morphisms of DGMs, the map $f \otimes g : M \otimes N \to$

$M' \otimes N'$ satisfies

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y),$$

where $|\cdot|$ denotes the degree.

Given a DGM (M, d_M), the *suspension* of M is the DGM (sM, d_{sM}), where $(sM)_n = M_{n-1}$ and $d_{sM} = -d_M$. Dually, the *desuspension* of M is the DGM $(s^{-1}M, d_{s^{-1}M})$ given by $(s^{-1}M)_n = M_{n+1}$ with differential $-d_M$. The *tensor module* of M is the DGM $TM = \bigoplus_{n\geq 0} M^{\otimes n}$ whose *(tensor) differential* d_t is the linear extension of d. The *tensor degree* of a homogeneous element $a_1 \otimes \cdots \otimes a_n \in TM$ is given by $\sum_{i=1}^{n} |a_i|$. A morphism $f : M \to N$ of DGMs induces a morphism $Tf : TM \to TN$ via $Tf|_{M^{\otimes n}} = f^{\otimes n}$.

A *differential graded algebra* (DGA) is a DGM (A, d_A) endowed with a unital associative product μ_A such that d_A is a derivation of μ_A. In fact, a DGA is an A_∞-algebra with trivial higher order structure. An A_∞*-algebra* is a graded module M together with a family of maps $\left\{\mu_i \in Hom^{i-2}\left(M^{\otimes i}, M\right)\right\}_{i\geq 1}$ such that for all $i \geq 1$

$$\sum_{n=1}^{i} \sum_{k=0}^{i-n} (-1)^{n+k+nk} \mu_{i-n+1}(1^{\otimes k} \otimes \mu_n \otimes 1^{\otimes i-n-k}) = 0. \tag{1}$$

Let A be a connected DGA. The *reduced bar construction* of A, constructed over the *reduced module* $\bar{A} = A/A_0$, is the DGM $\bar{B}(A) = Ts\left(\bar{A}\right)$ with differential $d_{\bar{B}} = d_t + d_s$, where d_t is as above and d_s is the *simplicial differential.* Roughly speaking (up to the appropriate suspensions and desuspensions), the morphism d_s is given by

$$d_s = \sum_{i=1}^{r-1} 1^{\otimes i-1} \otimes \mu_A \otimes 1^{r-i-1}.$$

The bar construction plays an important role in the transfer of an A_∞-algebra structure from chains to homology via Homological Perturbation Theory (HPT).

Let (M, d_M) and (N, d_N) be DGMs. A *contraction* r from N to M [10], denoted by either a tuple $r : \{N, M, f, g, \phi\}$ or an arrow $N \to M$, is an special type of *homotopy equivalence* in which the morphisms $f : N \to M$, $g : M \to N$, and the homotopy operator $\phi : N_* \to N_{*+1}$ satisfy the relations

$$(r1) fg = 1_M\,; \qquad (r2) \phi d_N + d_N \phi + gf = 1_N\,;$$

$$(r3)\, f\phi = 0\,; \quad (r4) \phi g = 0\,; \quad (r5) \phi\phi = 0\,.$$

A contraction $c : \{N,\ M,\ f,\ g,\ \phi\}$ between DGMs induces the *tensor module contraction* [12, 13], between the corresponding tensor modules,

$$T(c) : \{T(N), T(M), T(f), T(g), T(\phi)\},$$

where

$$T(\phi)|_{T^n(N)} = \phi^{[\otimes n]} = \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \phi \otimes (g\, f)^{\otimes n-i-1}.$$

Of course, the homology groups of the modules N and M are isomorphic, but the structure in N does not typically transfer faithfully to M. Indeed, if N is a DGA, a contraction r from N to M induces an A_∞-algebra structure on M, which arises via the following *perturbation process* [13]: A *perturbation* δ of a DGM N is a morphism $\delta : N \to N$ of graded modules such that $(d_N + \delta)$ is a differential. A *perturbation datum* of a contraction $r : \{N, M, f, g, \phi\}$ is a perturbation δ of N such that $\phi\delta$ is pointwise nilpotent, that is, for each $x \neq 0$, there exists an $n \in \mathbb{N}$ such that $(\phi\delta)^n(x) = 0$. The fundamental tool of HPT is the **Basic Perturbation Lemma** (BPL) [28, 9, 11], which is an algorithm whose input is a contraction $r : \{N, M, f, g, \phi\}$ together with a perturbation datum δ of r and whose output is a new contraction r_δ. The pointwise nilpotency of $\phi\delta$ guarantees that all sums in the formulas involved are finite for each $x \in N$.

$$\text{Input:}\quad \overset{\phi}{\curvearrowleft}\; r : (N, d_N) \underset{g}{\overset{f}{\rightleftarrows}} (M, d_M) \quad + \quad \text{perturbation } \delta$$

$$\text{Output:}\quad \overset{\phi_\delta}{\curvearrowleft}\; r_\delta : (N, d_N + \delta) \underset{g_\delta}{\overset{f_\delta}{\rightleftarrows}} (M, d_M + d_\delta)$$

where $f_\delta, g_\delta, \phi_\delta, d_\delta$ are given by

$$d_\delta = f\,\delta\,\Sigma_r^\delta\, g; \qquad f_\delta = f\,(1 - \delta\,\Sigma_r^\delta\,\phi); \qquad g_\delta = \Sigma_r^\delta\, g; \qquad \phi_\delta = \Sigma_r^\delta\,\phi;$$

and $\Sigma_r^\delta = \sum_{i\geq 0}(-1)^i\,(\phi\delta)^i$.

To obtain an A_∞-algebra structure on a small DGM of a contraction, we apply the so-called *tensor trick* given in [13] and expressed by the following algorithm:

- As initial **datum**, let (A, μ) be a DGA and let $r : \{A, M, f, g, \phi\}$ be a contraction.
- Form the tensor module contraction on the suspension
$$Ts(r) : \{Ts(A), Ts(M), Ts(f), Ts(g), Ts(\phi)\}.$$
- Using the simplicial differential as a perturbation datam of $Ts(r)$, apply the BPL and obtain a new contraction
$$\widetilde{r} : \{\bar{B}(A), Ts(M), \widetilde{f}, \widetilde{g}, \widetilde{\phi}\};$$
where $Ts(M)$ is endowed with a differential $\widetilde{d} = d_{Ts} + d_\delta$. Recall that the formula of d_δ is provided by the BPL.

- Finally, the induced A_∞-algebra structure on M can be extracted from the differential $\widetilde{d}$. More precisely, since $\widetilde{d}$ is a differential, and hence $\widetilde{d}\widetilde{d} = 0$, we can construct a family of maps $\{m_i\}_{i\geq 1}$ satisfying the A_∞-algebra structure relations in (1). In fact, this strategy produces the following formulas for the induced A_∞-algebra operations m_i [15]:
$$m_i = (-1)^{i+1} f \mu^{(1)} \phi^{[\otimes 2]} \mu^{(2)} \cdots \phi^{[\otimes i-1]} \mu^{[i-1]} g^{\otimes i},$$

where $\mu^{(k)}$ is some kind of linear extension of the product μ,

$$\mu^{(k)} = \sum_{i=0}^{k-1}(-1)^{i+1}1^{\otimes i} \otimes \mu \otimes 1^{\otimes k-i-1}.$$

It is important to note that if A is a DGA, there are certain conditions under which a contraction r of A preserves the algebra structure (see [24] for details). Consequently, an induced A_∞-structure on M is in some sense "natural" if we systematically control all of the data in the contraction r.

Dually, when the initial DGM N in a contraction is a differential graded coalgebra, there is an induced A_∞-coalgebra structure on M that arises in a completely analogous way.

3. Open questions

At this moment, there are many unanswered questions related to A-structures; we list some of them here. First, we consider lines of research related to A_∞-structures induced by a chain contraction of a DG (co)algebra. Following the pioneering work in [16] and [13], numerous papers considered the problem of computing these strongly homotopy associative structures [14, 15, 2, 6, 3]. An important goal in some of these works is to reduce the computational complexity of the formulas for computing A_∞-(co)algebra operations. Whereas this problem seems to be exponential in nature, the first question that arises is:

- *Is it possible to compute A_∞-(co)algebra operations in polynomial time?*

Numerous questions are related to the computation of these operations in a digital context. In the paper [5], the authors exhibit some experimental results when computing some low-dimensional operations on the homology of simple 3D digital images. These operations are potentially the first higher-order operations in the induced A_∞-coalgebra structure given by a contraction of the underlying simplicial chain complex associated with a given 3D digital image. Thus we ask:

- *Under what conditions is it possible to compute a "complete" A_∞-coalgebra structure on the homology of a finite cell, cubical or simplicial complex? For example, are such structures computable for finite complexes with cells in some range of dimensions?*
- *Are there situations under which the simple-connectivity condition assumed by HPT can be relaxed? Is there a coherent theory that applies the techniques of HPT to compute A_∞-operations in a non-simply-connected setting?*
- *Can A_∞ structures be used to distinguish between non-homotopically equivalent spaces in a digital context?*

Answers to these and related questions will allow us to advance the difficult problem of establishing an algorithm for classifying low dimensional finite cell complexes up to topological invariants such as connected components, holes or tunnels, cavities, and so on.

We conclude with some open questions related to A_∞-bialgebras.

- *In [27], the authors introduce new techniques to compute an induced A_∞-bialgebra structure on homology. Is it possible to apply techniques of classical HPT to induce an A_∞-bialgebra structure on homology and to compute the induced operations?*
- *In [26], the authors define a diagonal on the associahedron K_n and use it to define the tensor product of A_∞-(co)algebras. Is it possible to use a tensor product of contractions to reproduce (up to isomorphism) their tensor product?*
- *In [4] the authors construct algebraic examples of A_∞-bialgebra of type (m,n), which are A_∞-bialgebras with exactly one higher order operation $\omega_m^n : H^{\otimes m} \to H^{\otimes n}$. Do there exit topological spaces $X(m,n)$ whose homology carries an induced A_∞-bialgebra of type (m,n)?*
- *As mentioned in the Introduction, if F is a field, $H = H_*(\Omega X; F)$ is naturally endowed with an A_∞-bialgebra structure, and over $\mathbb{Q}$, this structure is well-understood. At the time of this writing, however, the global A_∞-bialgebra structure of $H_*((\mathbb{Z},n);\mathbb{Z}_p)$ is unknown, although some local results were obtained in [7]. Thus we ask, how are the A_∞-bialgebra operations $\omega_m^n : H^{\otimes m} \to H^{\otimes n}$ on $H_*((\mathbb{Z},n);\mathbb{Z}_p)$ defined?*

References

[1] J. F. Adams. *Infinite loop spaces.* Ann. of Math. Stud. **90**, Princeton Univ. Press, Princeton, N.J., 1978.

[2] A. Berciano-Alcaraz. A computational approach of A_∞-(co)algebras. To appear in *International Journal of Computer Mathematics*, 2009.

[3] A. Berciano, M. J. Jiménez, P. Real. On the computation of A_∞-maps. *Lecture Notes in Computer Science* **4770**, 45–57, 2007.

[4] A. Berciano, S. Evans, R. Umble. A_∞-bialgebras of type (m,n). *Preprint arXiv.908.0674*, 2009.

[5] A. Berciano, M. J. Jiménez, H. Molina-Abril, P. Real. Computing homology generators relations in 3D digital images. In *Proceedings of the Workshop on Computational Topology in Image context 2009*, pp. 11–18. 2009.

[6] A. Berciano, J. Rubio, F. Sergeraert. A case study of A_∞-structure. *Preprint*, 2009.

[7] A. Berciano, R. Umble. Some naturally occurring examples of A_∞-bialgebras. *Preprint arXiv.0706.0703*, 2008.

[8] H. J. Baues. The cobar construction as a Hopf algebra and the Lie differential. *Invent. Math.* **132**, 467–489, 1998.

[9] R. Brown. The twisted Eilenberg–Zilber theorem. In *Celebrazioni Archimedee del secolo XX, Simposio di topologia*, pp. 34–37. 1967.

[10] S. Eilenberg, S. Mac Lane. On the groups $H(\pi, n)$, I, II. *Annals of Math.* **58**, 55–106, 1953 y **60**, 49–139, 1954.

[11] V. K. A. M. Gugenheim. On the chain complex of a fibration. *Illinois J. Math.* **3**, 398–414, 1972.

[12] V. K. A. M. Gugenheim, L. Lambe. Perturbation theory in Differential Homological Algebra. *I. Illinois J. Math.* **33**, 56–82, 1989.

[13] V. K. A. M. Gugenheim, L. Lambe, J. Stasheff. Perturbation theory in Differential Homological Algebra, II. *Illinois J. Math.* **35** (3), 357–373, 1991.

[14] M. J. Jiménez. *A_∞–estructuras y perturbación homológica.* Tesis Doctoral de la Universidad de Sevilla, Spain, 2003.

[15] M. J. Jiménez, P. Real. Rectifications of A_∞–algebras. *Communications in Algebra* **35**(9), 2731–2743, 2007.

[16] T. Kadeishvili. The A_∞-algebra Structure and Cohomology of Hochschild and Harrison. *Proc. of Tbil. Math. Inst.* **91**, 19–27, 1988.
[17] T. Kadeishvili. A_∞-algebra Structure in Cohomology and Rational Homotopy Type. *Forschungsswerpunkt Geometrie*, Universitaet Heidelberg, Math. Inst. Heft 37, 1–64, 1988.
[18] B. Keller. Introduction to A-infinity algebras and modules. *Homology, Homotopy and Applications* **3**(1), 1–35, 2001.
[19] S. Mac Lane. *Homology.* Classics in Mathematics, Springer-Verlag, New York-Berlin-Heidelberg, 1995.
[20] M. Markl. A resolution (minimal model) of the PROP for bialgebras. *J. Pure and Appl. Algebra* **205** 341–374, 2006.
[21] M. Markl, S. Shnider, J. Stasheff. *Operads in algebra, topology and physics.* American Mathematical Society, 2002.
[22] J. P. May. *Simplicial Objects in Algebraic Topology.* D. Van Nostrand Company, 1967.
[23] A. Prouté. *Algèbres différentielles fortement homotopiquement associatives.* Tesis Doctoral Université Paris VII, París, 1984.
[24] P. Real. Homological Perturbation Theory and associativity. *Homology, Homotopy and Applications* **2**, 51–88, 2000.
[25] J. Rubio, F. Sergeraert. Constructive Algebraic Topology. *Bulletin des Sciences Mathematiques* **126**, 389–412, 2002.
[26] S. Saneblidze, R. Umble. Diagonals on the permutahedra, multiplihedra and associahedra. *Homology, Homotopy and Applications* **6**(1), 363–411, 2004.
[27] S. Saneblidze, R. Umble. Morphisms of A_∞-bialgebras and applications. In preparation.
[28] W. Shih. Homologie des espaces fibré. *Inst. Hautes Etudes Sci.* **13**, 293–312, 1962.
[29] J. D. Stasheff. Homotopy associativity of H-spaces I,II. *Transactions of A.M.S.* **108**, 275–292, 293–312, 1963.
[30] J. D. Stasheff. Differential graded Lie algebras, Quasi-Hopf algebras and higher homotopy algebras. *Lecture Notes in Mathematics* **1510**, 120–137, 1992.
[31] R. Umble. Higher homotopy Hopf algebras found: A ten-year retrospective. To appear.

Departamento de Didáctica de la Matemática y de las Ciencias Experimentales, Universidad del País Vasco/Euskal Herriko Unibertsitatea, Spain
E-mail address: ainhoa.berciano@ehu.es

Departamento de Matemática Aplicada I, Universidad de Sevilla, Spain
E-mail address: majiro@us.es

Departament of Mathematics, Millersville University of Pennsylvania, USA
E-mail address: Ron.Umble@millersville.edu