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AN APPROACH TO DYNAMICAL SYSTEMS USING EXTERIOR SPACES

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"Y aunque tú no has de verlas,
para hablar con tu ausencia
estas líneas escribo
únicamente por estar contigo."
(Luis Cernuda)

In memory of Mirian, our friend

Resumen. El objetivo principal del artículo es introducir una forma de usar los espacios exteriores para el estudio de los sistemas dinámicos (flujos). Dado un flujo, para obtener su espacio exterior asociado, consideramos la familia de todos sus "subconjuntos abiertos absorbentes" (subconjuntos abiertos que contienen la "parte futura" de todas las trayectorias). Los límites y espacios finales de espacios exteriores son utilizados para construir límites y espacios finales de sistemas dinámicos. Tomando un punto final a, podemos considerar el subflujo que contiene a todas las trayectorias que finalizan en a. Esto da lugar a una descomposición de un sistema dinámico como unión disjunta de subflujos estables (en el infinito).

Abstract. In this article the main objective proposed by authors is to introduce a way of using exterior spaces to study dynamical systems (flows). Given a flow, we consider the family of all "absorbing open subsets" (open subsets that contain the "future part" of all the trajectories) to obtain an exterior space associated with the flow. The limits and end spaces of exterior spaces are used to construct limits and end spaces of dynamical systems. Taking an end point a, we can consider the subflow containing all trajectories finishing at a. This gives a decomposition of a dynamical system as a disjoint union of stable (at infinity) subflows.

1. Introduction

Many natural phenomena can be modelized by means of an autonomous differential equation which can be put (maybe after some manipulations) in the form $\dot{x} = f(x)$, where x are the coordinates of a point p of a m-dimensional manifold

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M, $\dot{x}$ are the coordinates of a tangent vector at the point $p \in M$ and f is a real valued function whose domain is an open of $\mathbb{R}^m$.

Under the assumption that f be locally lipschitzian, an initial condition $x^p(0) = p$ uniquely determines a maximal solution $x^p(t)$. However, the domain of $x^p(t)$ does not need to be the whole real line $\mathbb{R}$, but only an open interval $(a^p, b^p) \subset \mathbb{R}$, $a^p < 0 < b^p$, which depends on the initial condition. All the solutions give a local flow $\phi\colon W \to M$, $\phi(t,p) = x^p(t)$, where W is an open subset of $\mathbb{R} \times M$ containing $\{0\} \times M$ and if we denote $\phi_s(p) = \phi(s,p)$, $(s,p) \in W$, ϕ satisfies $\phi_0 = \mathrm{id}_M$, $\phi_t\phi_s = \phi_{t+s}$, wherever it makes sense. The space M is called the phase space and ϕ is also called the phase map. The trajectory of a point $p \in M$ is the subset $\gamma(p) = \{\phi(t,p) | t \in (a^p, b^p)\}$. It is easy to check that M is a disjoint union of trajectories. We note that the if a trajectory has more than one point, a natural orientation is induced by increasing times. Then, we can consider M as a disjoint union of critical trajectories and oriented trajectories to obtain a phase portrait of the dynamical system ϕ . It is well known the following result (see [3]): If φ be a local flow on M, then there exists a global ($W = \mathbb{R} \times M$) flow ϕ in M such that the oriented trajectories of φ and ϕ coincide. Consequently their phase portraits are the same. Therefore we can reduce our study to the case of global flows as a consequence of this fact.

On the other side, for every topological space X, a continuous map $\varphi\colon \mathbb{R} \times X \to X$ induces a group homomorphism $\bar{\varphi}\colon \mathbb{R} \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of homeomorphisms of X provided with the compact-open topology. This fact permits to study a flow as a particular case of a transformation group.

The pioneering work of H. Poincaré [25, 26] in the late XIX century studied the topological properties of solutions of autonomous ordinary differential equations. We can also mention the work of A. M. Liapunov [15] which developed his theory of stability of a motion (solution) of a system of n first order ordinary differential equations. While many Poincaré work studied the global properties of the system, Liapunov work looks at the local stability of a dynamical system. The theory of dynamical systems reached a great development with the work of G.D. Birkhoff [2], who may be considered as the founder of this theory. He establised two main lines in the study of dynamical systems: the topological theory and the ergodic theory.

In this paper we describe some basic ideas that permit a new approach to the study of dynamical system using exterior spaces. An exterior space is a topological space provided with a distinguished subfamily of open subsets which is called an externology [9, 10] . The exterior homotopy theory can be considered as an extension of the proper homotopy theory [24] that provides many tools which can be used to classify non-compact manifolds and to study the shape of compact metric spaces.

In our approach, the main key to establish a connection between exterior spaces and dynamical sytems is the notion of absorbing open region with respect to a flow (i.e., an open subset that contains the "future part" of all the trajectories). The nice property is that the family of absorbing open regions has the structure of an externology.

This paper presents some initial results of an ongoing project which main objective is to apply the properties of exterior spaces to study and classify dynamical systems.

2. Preliminaires on exterior spaces and dynamical systems

2.1. The proper and exterior categories. There are families of spaces, for example non compact manifolds and some classes of pathological spaces, whose study requires an adaptation of the standard techniques of Algebraic Topology developed for the category of topological spaces and continuous maps **Top**.

The origins of the Proper Homotopy Theory go back to the classification of non compact surfaces given by Kerérjártó [14] in 1923 using "ideal points". This last concept was extended for more general spaces by Freudenthal [8] in 1931 to the notion of "end points" (points at infinity), which is the first invariant in proper homotopy. A great impetus to this theory came from the work of L. Siebenmann [21] in 1965 when in his thesis he proposed to use proper maps (continuous at infinity) instead of continuous maps in order to study non compact manifolds.

Definition 2.1. *A continuous map $f : X \to Y$ is said to be a* **proper map** *if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X.*

The category **P** of topological spaces and proper maps and its corresponding proper homotopy category $\pi\mathbf{P}$ are very useful for the study of non compact spaces and taking into account its connection with Shape Theory [17](Chapman's theorem) is also an interesting tool for the study of compact spaces (surveys of proper homotopy can be seen at [6], [24]).

Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera. An answer to this problem is given by the notion of exterior space:

Definition 2.2. *Let $(X, \mathbf{t})$ be a topological space, where X is the subjacent set and $\mathbf{t}$ its topology. An externology on $(X, \mathbf{t})$ is a non empty collection ε of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \mathbf{t}$ and $E \subset U$ then $U \in \varepsilon$. If an open subset is a member of ε is said to be an exterior open subset.*

An **exterior space** *$(X, \varepsilon, \mathbf{t})$ consists of a space $(X, \mathbf{t})$ together with an externology ε.*

A map $f : (X, \varepsilon, \mathbf{t}) \to (X', \varepsilon', \mathbf{t}')$ is said to be an **exterior map** *if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

For instance, given a topological space $(X, \mathbf{t})$, we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ and the total exterior space if one takes $\varepsilon = \mathbf{t}$. Another important externology is the family $\varepsilon^{\mathbf{c}}(X)$ of the complements of closed-compact subsets of X , that will be called the cocompact externology.

The new category of exterior spaces and exterior maps, **E**, is complete and cocomplete and contains **P** as a full subcategory via the full embedding

$$(\cdot)^{\mathbf{c}} : \mathbf{P} \hookrightarrow \mathbf{E} \ .$$

The functor $(\cdot)^{\mathbf{c}}$ carries a topological space X to the exterior space $X^{\mathbf{c}}$ which is provided with the topology of X and the externology $\varepsilon^{\mathbf{c}}(X)$. A map $f\colon X \to Y$ is carried to the exterior map $f^{\mathbf{c}}\colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ given by $f^{\mathbf{c}} = f$. It is easy to check that a continuous map $f : X \to Y$ is proper if and only if $f = f^{\mathbf{c}}\colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ is exterior.

An important role in this paper will be played by the following construction $(\cdot)\bar{\times}(\cdot)$:

Let $(X, \varepsilon^X, \mathbf{t}_X)$ be an exterior space, $(Y, \mathbf{t}_Y)$ a topological space and for $y \in Y$ we denote by $(\mathbf{t}_Y)_y$ the family of open neighborhoods of Y at y. We consider on $X \times Y$ the product topology $\mathbf{t}_{X\times Y}$ and the externology $\varepsilon^{X\bar{\times}Y}$ given by those $E \in \mathbf{t}_{X\times Y}$ such that for each $y \in Y$ there exists $U_y \in (\mathbf{t}_Y)_y$ and $T^y \in \varepsilon^X$ such that $T^y \times U_y \subset E$. This exterior space will be denoted by $X\bar{\times}Y$ in order to avoid a possible confusion with the product externology. This construction gives a functor

$$(\cdot)\bar{\times}(\cdot)\colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}.$$

When Y is a compact space, we have that E is an exterior open subset if and only if it is an open subset and there exists $G \in \varepsilon^X$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon^X = \varepsilon^{\mathbf{c}}(X)$ then $\varepsilon^{X\bar{\times}Y}$ coincides with the externology of complements of closed-compact subsets of $X \times Y$.

We note that if Y is a discrete space, then E is an exterior open subset if and only if it is open and for each $y \in Y$ there is $T^y \in \varepsilon^X$ such that $T^y \times \{y\} \subset E$.

For more properties and applications of exterior homotopy categories we refer the reader to [13, 9, 10, 11, 7, 4, 12] .

2.2. Dynamical Systems and Ω-Limits. Next we recall some basic notions about dynamical systems.

Definition 2.3. *A* **flow** *on a topological space X is a continuous map $\varphi\colon \mathbb{R}\times X \to X$ such that*

(i) $\varphi(0, p) = p, \quad \forall p \in X$

(ii) $\varphi(t, \varphi(s, p)) = \varphi(t+s, p), \quad \forall p \in X, \forall t, s \in \mathbb{R}$. A flow on X will be denoted by (X, φ) and when no confusion be possible, we use X and $t\cdot x = \varphi(t, x)$ for short.

Given two flows $\phi\colon \mathbb{R}\times X \to X$, $\psi\colon \mathbb{R}\times Y \to Y$, a **flow morphism** *$f\colon (X, \phi) \to (Y, \psi)$ is a continuous map $f\colon X \to Y$ such that $f(r \cdot p) = r \cdot f(p)$ for every $r \in \mathbb{R}$ and for every $p \in X$.*

A subset S of a flow X is said to be **invariant** *if for every $p \in S$ and every $t \in \mathbb{R}$, $t \cdot p \in S$.*

We denote by $\mathbf{F}$ the category of flows and flows morphisms.

Given a flow $\varphi\colon \mathbb{R}\times X \to X$, it is interesting to note that one has a subgroup $\{\varphi_t\colon X \to X | t \in \mathbb{R}\}$, $\varphi_t(x) = \varphi(t, x)$, of homeomorphisms and a family of motions $\{\varphi^p\colon \mathbb{R} \to X | p \in X\}$, $\varphi^p(t) = \varphi(t, p)$. One has also a family of maps $\{\varphi_t^{-1}\colon \mathbf{t}_X \to \mathbf{t}_X | t \in \mathbb{R}\}$ where $\varphi_t^{-1}(U)$ is the inverse image of the open subset $U \in \mathbf{t}_X$ and $\mathbf{t}_X$ denotes the topology of X .

Definition 2.4. *Let X be a flow. The* $\omega^{\mathbf{r}}$**-limit** *set of a point $p \in X$ is given by*

$$\omega^{\mathbf{r}}(p) = \{q \in X | \exists t_n \to +\infty \text{ such that } t_n \cdot p \to q\}$$

and the Ω^r**-limit** *of X by*

$$\Omega^{\mathbf{r}}(X) = \bigcup_{p \in X} \omega^{\mathbf{r}}(p).$$

If $\overline{A}$ denotes the clousure of a subset A of a topological space, we note that the subset $\omega^{\mathbf{r}}(p)$ admits the alternative definition

$$\omega^{\mathbf{r}}(p) = \bigcap_{t \geq 0} \overline{[t, +\infty) \cdot p}$$

which has the advantage of showing that $\omega^{\mathbf{r}}(p)$ is closed.

The notions above can be dualized to obtain the notion of the $\omega^{\mathbf{l}}$-limit set of a point p and the $\Omega^{\mathbf{l}}$- limit of X .

Now we introduce the basic notions of critical, periodic and **r**-Poisson stable points.

Definition 2.5. *Let X be a flow. A point $x \in X$ is said to be a* **critical** *point (or a rest point, or an equilibrium point) if for every $r \in \mathbb{R}$, $r \cdot x = x$.*

A point $x \in X$ is said to be **periodic** *if there is $r \in \mathbb{R}$, $r \neq 0$, such that $r \cdot x = x$.*

We denote by $C(X), P(X)$ the invariant subsets of critical and periodic points of X, respectively.

It is clear that a critical point is a periodic point . Then

$$C(X) \subset P(X) .$$

If $x \in X$ is a periodic point but not critical, then there exists $r > 0$ such that $r \cdot x = x$ and r is the smallest positive such that $r \cdot x = x$. Futher it $r' \in \mathbb{R}$ is such that $r' \cdot x = x$ then there is $z \in \mathbb{Z}$ such that $r' = zr$. The smallest positive period of x is called the fundamental period of x .

Definition 2.6. *Let (X, φ) be a flow . A point $x \in X$ is said to be* **r-Poison stable** *if there is a divergent sequence $t_n \to +\infty$ such that $t_n \cdot x \to x$.*

We denote by $P^{\mathbf{r}}(X)$ the invariant subset of **r**-Poison stable points of X. The reader can check easily that

$$P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X) .$$

3. Ends and Limits of an exterior space

Given an exterior space $X = (X, \varepsilon(X))$, its externology $\varepsilon(X)$ can be seen as an inverse system of spaces, then we define the limit of X as the topological space:

$$L(X) = \lim \varepsilon(X).$$

Note that for each $E' \in \varepsilon(X)$ the canonical map $\lim \varepsilon(X) \to E'$ is continuous and factorizes as $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E \to E'$. Therefore the canonical map $\lim \varepsilon(X) \to \cap_{E \in \varepsilon(X)} E$ is continuous. On the other side, by the universal property of the inverse system the family of maps $\cap_{E\in\varepsilon(X)}E \to E', E' \in \varepsilon(X)$ induces a continuous map $\cap_{E\in\varepsilon(X)}E \to \lim \varepsilon(X)$. This implies that the canonical map $\lim \varepsilon(X) \to \cap_{E\in\varepsilon(X)}E$ is a homeomorphism.

We recall that for a topological space Y, $\pi_0(Y)$ denotes the set of path-components of Y and we have a continuous canonical map $Y \to \pi_0(Y)$ which induces a quotient topology on $\pi_0(Y)$. We remark that if Y is locally path-connected, then $\pi_0(Y)$ is a discrete space.

Definition 3.1. *Given an exterior space $X = (X, \varepsilon(X))$, the topological subspace:*

$$L(X) = \lim \varepsilon(X) = \cap_{E\in\varepsilon(X)}E$$

will be called the **limit space** *of X .*

The **end space** *of X is the inverse limit:*

$$\check{\pi}_0(X) = \lim \pi_0\varepsilon(X) = \lim_{E\in\varepsilon(X)} \pi_0(E)$$

provided with the inverse limit topology of the spaces $\pi_0(E)$.

Note that an end point $a \in \check{\pi}_0(X)$ is represented by the filter base:

$$\{U_a^E | U_a^E \text{ is a path-component of } E, E \in \varepsilon(X)\} .$$

It is interesting to observe that a locally path-connected exterior space $X = (X, \varepsilon(X))$ induces the following family of exterior spaces

$$\{(X, \varepsilon(X, a)) | a \in \check{\pi}_0(X)\}$$

where $\varepsilon(X, a)$ is the externology generated by the filter base of open subsets:

$$\{U_a^E | U_a^E \text{ is a path-component of } E, E \in \varepsilon(X\} .$$

On the other hand, when X is locally path-connected, then we have that $\check{\pi}_0(X)$ is a prodiscrete space.

We have mention in 2.1 the Freudenthal set of end points of a topological space. Using the cocompact externology, the set of Freudenthal end of a topological space Y is given by $\check{\pi}_0(Y^{\mathbf{c}})$ of the exterior space $Y^{\mathbf{c}} = (Y, \varepsilon^{\mathbf{c}}(Y))$. For an exterior space $X = (X, \varepsilon(X))$ we denote by $X_{\mathbf{t}}$ the underlying topological space. If we consider the cocompact externology $\varepsilon^{\mathbf{c}}(X_{\mathbf{t}})$, in general neither $\varepsilon(X) \subset \varepsilon^{\mathbf{c}}(X_{\mathbf{t}})$ nor $\varepsilon^{\mathbf{c}}(X_{\mathbf{t}}) \subset \varepsilon(X)$. For instance, in the case we have the inclusion $\varepsilon^{\mathbf{c}}(X_{\mathbf{t}}) \subset \varepsilon(X)$ we will have the corresponding canonical set maps $\check{\pi}_0(X) \to \check{\pi}_0((X_{\mathbf{t}})^{\mathbf{c}})$, $X \sqcup \check{\pi}_0(X) \to X \sqcup \check{\pi}_0((X_{\mathbf{t}})^{\mathbf{c}})$. If the space $X_{\mathbf{t}}$ verifies some additional conditions, with the topology induced by $\mathbf{t}$ and the filter base above $X \sqcup \check{\pi}_0((X_{\mathbf{t}})^{\mathbf{c}})$ is the Freudenthal compactification of $X_{\mathbf{t}}$. In a similar way, one question that arises at this point is to provide to $X \sqcup \check{\pi}_0(X)$ the structure of a topological space and to analyse the compactness of this new space.

Given an exterior space $X = (X, \varepsilon(X))$ one has a canonical continuous map

$$e \colon L(X) \to \check{\pi}_0(X).$$

This permits to decompose the limit of an exterior space:

Definition 3.2. *Given an exterior space X, an end point $a \in \check{\pi}_0(X)$ is said to be* **representable** *by $b \in L(X)$ if $e(b) = a$. Notice that the map $e\colon L(X) \to \check{\pi}_0(X)$ induce an e-decomposition*

$$L(X) = \bigsqcup_{a \in \check{\pi}_0(X)} L_a(X)$$

where $L_a(X) = e^{-1}(a)$ will be called the a**-component** *of the limit $L(X)$.*

Concerning this e-decomposition of the limit there are some interesting questions that have to be studied; for instance, under which conditions one has that $L(X)$ or $L_a(X)$ are compact spaces. It will also be interesting to analyze the exterior spaces whose limit components $L_a(X)$ are continua (recall that the inverse limit of continua is continuum, see [5]).

Suppose that X, Y are exterior spaces and $f\colon X \to Y$ is an exterior map, then f induces continuous maps $L(f)\colon L(X) \to L(Y)$, $\check{\pi}_0(f)\colon \check{\pi}_0(X) \to \check{\pi}_0(Y)$ and we have the functors:

$$L, \check{\pi}_0\colon \mathbf{E} \to \mathbf{Top}.$$

It is not difficult to check that L preserves exterior homotopies and $\check{\pi}_0$ is invariant by exterior homotopy:

Proposition 3.1. *Suppose that X, Y be exterior spaces and $f, g\colon X \to Y$ are exterior maps.*

(i) *If $H\colon X\bar{\times}I \to Y$ is an exterior homotopy from f to g, then $L(H) = H|_{L(X)\times I}\colon L(X\bar{\times}I) = L(X)\times I \to L(Y)$ is a homotopy from $L(f)$ to $L(g)$.*
(ii) *If $H\colon X\bar{\times}I \to Y$ is an exterior homotopy from f to g, then $\check{\pi}_0(f) = \check{\pi}_0(g)$.*

Then, if $\pi\mathbf{E}$ and $\pi\mathbf{Top}$ are the exterior homotopy category and the usual homotopy category corresponding to $\mathbf{E}$ and $\mathbf{Top}$ respectively, one has the following result.

Proposition 3.2. *The functors $L\colon \mathbf{E} \to \mathbf{Top}$, $\check{\pi}_0\colon \mathbf{E} \to \mathbf{Top}$ induce functors $L\colon \pi\mathbf{E} \to \pi\mathbf{Top}$, $\check{\pi}_0\colon \pi\mathbf{E} \to \mathbf{Top}$.*

4. End and Limit space of a flow via exterior flows

We have the following basic fundamental example: Suppose that $X = \mathbb{R}$ with the usual topology and consider the flow $\varphi\colon \mathbb{R} \times X \to X$ given by $\phi(r, s) = r + s$. In this case the motions $\varphi^r, r \in \mathbb{R}$ are injective and there is only one trajectory.

We consider the following externology:

$$\mathbf{r} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } [n, +\infty) \subset U\}$$

and we denote the corresponding exterior space by $\mathbb{R}^{\mathbf{r}}$. It is interesting to note that a base for $\mathbf{r}$ is given by

$$\mathcal{B}(\mathbf{r}) = \{[n, +\infty) | n \in \mathbb{N}\} .$$

In this paper we propose the following notion that mixes the structures of dynamical system and exterior space:

Definition 4.1. *Let M be an exterior space, $M_{\mathbf{t}}$ the subjacent topological space and $M_{\mathbf{d}}$ the set M provided with the discrete topology. An* **r-exterior flow** *is a continuous flow $\phi\colon \mathbb{R}\times M_{\mathbf{t}} \to M_{\mathbf{t}}$ such that $\phi\colon \mathbb{R}^{\mathbf{r}}\bar{\times} M_{\mathbf{d}} \to M$ is exterior and for any $t\in\mathbb{R}$, $F_t\colon M\bar{\times} I\to M$, $F_t(x,s)=\varphi(ts,x)$, $s\in I,\ x\in M$, is also exterior. An* **r-exterior flow morphism** *of* **r**-*exterior flows $f\colon M\to N$ is a flow morphism such that f is exterior.*

Denote by $\mathbf{E^rF}$ the category of **r**-exterior flows and **r**-exterior flow morphisms.

In section 3 we have defined the end and limit space of an exterior space. In particular, since an **r**-exterior flow X is a exterior space, we can consider the end space $\check{\pi}_0(X)$ and the limit space $L(X)$. Notice that one has the following properties:

Proposition 4.1. *Suppose that (X,ϕ) is an* **r**-*exterior flow. Then,*

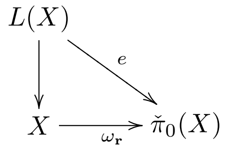
(i) *the space $L(X)$ is invariant,*
(ii) *there is a trivial induced flow on $\check{\pi}_0(X)$.*

Proof. (i): We have that $L(X)=\cap_{E\in\varepsilon(X)}E$. Note that for any $s\in\mathbb{R}$, $\phi_s(E)\in\varepsilon(X)$ if and only if $E\in\varepsilon(X)$. Then $\phi_s(L(X))=\phi_s(\cap_{E\in\varepsilon(X)}E)=\cap_{E\in\varepsilon(X)}\phi_s(E)=\cap_{E\in\varepsilon(X)}E=L(X)$.

(ii): For any $s\in\mathbb{R}$, consider the exterior homotopy $H\colon X\bar{\times} I\to X$, $H(x,t)=\phi(ts,x)$, from id_X to ϕ_s. By Proposition 3.2, it follows that $\mathrm{id}=\check{\pi}_0(\phi_s)$. Therefore the induced action is trivial. □

We note that for an **r**-exterior flow X, each trajectory has an end point given as follows:

If $p\in X$ and $E\in\varepsilon(X)$, there is $T^p\in\mathbf{r}$ such that $T^p\cdot p\subset E$. We can suppose that T^p is path-connected, then $T^p\cdot p$ is path-connected and there is a unique $\omega_{\mathbf{r}}(p,E)$ path-component of E such that $T^p\cdot p\subset\omega_{\mathbf{r}}(p,E)\subset E$. This gives maps $\omega_{\mathbf{r}}(\cdot,E)\colon X\to\pi_0(E)$ and $\omega_{\mathbf{r}}\colon X\to\check{\pi}_0(X)$ such that the following diagram commutes:



The map $\omega_{\mathbf{r}}$ permits to divide an **r**-exterior flow in simpler **r**-exterior flows:

Definition 4.2. *Let X be an* **r**-*exterior flow. The invariant space denoted by*

$$X_{(\mathbf{r},a)}=\omega_{\mathbf{r}}^{-1}(a),\quad a\in\check{\pi}_0(X)$$

will be called the **r-basin** *at a.*

The induced partition of X in simpler **r**-*exterior flows:*

$$X=\bigsqcup_{a\in\check{\pi}_0(X)}X_{(\mathbf{r},a)}$$

will be called the $\omega_{\mathbf{r}}$**-decomposition** *of the* $\mathbf{r}$*-exterior flow* X*.*

Given an $\mathbf{r}$-exterior flow $(M,\phi)\in \mathbf{E^rF}$, one also a flow $(M_{\mathbf{t}},\phi)\in \mathbf{F}$. This gives a forgetful functor

$$(\cdot)_{\mathbf{t}}\colon \mathbf{E^rF}\to \mathbf{F}\ .$$

Now given a flow (X,φ), an open $N\in \mathbf{t}_X$ is said to be **r-exterior** if for any $x\in X$ there is $T^x\in \mathbf{r}$ such that $\varphi(T^x\times\{x\})\subset N$. It is easy to check that the family of $\mathbf{r}$-exterior subsets of X is an externology that will be denoted by $\varepsilon^{\mathbf{r}}(X)$ which gives an exterior space $X^{\mathbf{r}}$, the map $\varphi\colon \mathbb{R}^{\mathbf{r}}\bar{\times} X_{\mathbf{d}}\to X^{\mathbf{r}}$ is exterior and $F_t\colon X^{\mathbf{r}}\bar{\times} I\to X^{\mathbf{r}}$ is also exterior for every $s\in\mathbb{R}$. Therefore we obtain an $\mathbf{r}$-exterior flow. The pair $(X^{\mathbf{r}},\varphi)$ is said to be $\mathbf{r}$-exterior flow associated to X . When there is no possibility of confusion, $(X^{\mathbf{r}},\varphi)$ will be briefly denoted by $X^{\mathbf{r}}$. Then we have a functor

$$(\cdot)^{\mathbf{r}}\colon \mathbf{F}\to \mathbf{E^rF}.$$

Note that for a flow (X,φ), if E is an open subset such that $\overline{E}$ is compact, then E is an $\mathbf{r}$-exterior subset if and only if $\overline{E}$ is an "absorbing region" in the sense of Definition 1.4.2 in [1].

The forgetful functor and the given constructions of exterior flows are related as follows:

Proposition 4.2. *The functor* $(\cdot)^{\mathbf{r}}\colon \mathbf{F}\to\mathbf{E^rF}$ *is left adjoint to the functor* $(\cdot)_{\mathbf{t}}\colon \mathbf{E^rF}\to\mathbf{F}$ *. Moreover* $(\cdot)_{\mathbf{t}}\,(\cdot)^{\mathbf{r}}=\mathrm{id}$ *and* $\mathbf{F}$ *can be considered as a full subcategory of* $\mathbf{E^rF}$ *via* $(\cdot)^{\mathbf{r}}$*.*

Proof. Let X be in $\mathbf{F}$ and M be in $\mathbf{E^rF}$. If $f\colon X^{\mathbf{r}}\to M$ is a morphism in $\mathbf{E^rF}$, then it is clear that $f\colon X=(X^{\mathbf{r}})_{\mathbf{t}}\to M_{\mathbf{t}}$ is a morphism in $\mathbf{F}$. Now if $g\colon X\to M_{\mathbf{t}}$ is a morphism in $\mathbf{F}$, suppose that $E\in\varepsilon(M)$, given $x\in X$, one has $g(x)\in M$, since M is an $\mathbf{r}$-exterior flow, there is $T^{g(x)}$ such that $T^{g(x)}\cdot g(x)\subset E$. This implies that $T^{g(x)}\cdot x\subset g^{-1}(E)$. Therefore $g^{-1}(E)\in\varepsilon(X^{\mathbf{r}})=\varepsilon^{\mathbf{r}}(X)$. □

Definition 4.3. *Given a flow* X*, the space* $\check{\pi}_0^{\mathbf{r}}(X)=\check{\pi}_0(X^{\mathbf{r}})$ *is said to be the* **end space of the flow** X *and the space* $L^{\mathbf{r}}(X)=L(X^{\mathbf{r}})$ *is said to be the* **limit space of the flow** X *.*

We remark that if X is a flow their associated $\mathbf{r}$-exterior structure permits to decompose the flow X using the decomposition of $X^{\mathbf{r}}$. The $\omega_{\mathbf{r}}$-decomposition

$$X^{\mathbf{r}}=\bigsqcup_{a\in\check{\pi}_0^{\mathbf{r}}(X)} X^{\mathbf{r}}_{(\mathbf{r},a)}$$

can be considered as generalization for a continuous flow of a disjoint union of "stable" submanifolds of a differentiable flow (see [22]). On the other side, the above decomposition generalizes Morse-Smale's decompositions of dynamical system associated to Morse functions (see [19],[20]).

It is interesting to note that the $\omega_{\mathbf{r}}$-decomposition of X is compatible with the e-decomposition of the limit subspace $L(X)$.

The relation of the limit space of a flow or an $\mathbf{r}$-exterior flow and the subflow of periodic points is analysed in the following results:

Lemma 4.1. *Let X be an* **r***-exterior flow and suppose that $x \in X$. If x is a periodic point, then for every $E \in \varepsilon(X)$, $x \in E$.*

Proof. Suppose that x is a periodic point. If $E \in \varepsilon(X)$, where X is an **r**-exterior flow, there is $T \in \mathbf{r}$, such that $T \cdot x \subset E$. Since x is periodic, $T \cdot x = \mathbb{R} \cdot x$. Taking into account that $x \in \mathbb{R} \cdot x$, we have that that $x \in E$. □

Lemma 4.2. *Let X be a flow and suppose that X is a T_1-space. Then, for every $x \in X$ the following statements are equivalent:*

(i) *x is a non-periodic point,*
(ii) *$X \setminus \{x\}$ is an* **r***-exterior subset of X.*

Proof. (i) implies (ii): Take $y \in X$. If the trajectory of y is different of the trajectory of x, then for every $T \in \mathbf{r}$, $T \cdot y \subset X \setminus \{x\}$. If y is in the trajectory of x, taking into account that x is not periodic, there is $T \in \mathbf{r}$ such that $T \cdot y \subset X \setminus \{x\}$. Then, one has that $X \setminus \{x\} \in \varepsilon^{\mathbf{r}}(X)$.

Conversely, suppose that x is a periodic point, by Lemma above $X \setminus \{x\}$ is not **r**-exterior. □

Proposition 4.3. *Let X be an* **r***-exterior flow. Then, $P(X) \subset L(X)$.*

Proof. It follows as an easy consequence of Lemma 4.1 □

Theorem 4.1. *Let X be a flow and suppose that X is a T_1-space. Then, $L^{\mathbf{r}}(X) = P(X)$ the set of periodic points of X .*

Proof. Let $x \in X \setminus P(X)$ and take $y \in X$. If the trajectory of y is different of the trajectory of x, then for every $T \in \mathbf{r}$, $T \cdot y \subset X \setminus \{x\}$. If y is in the trajectory of x, taking into account that x is not periodic, there is $T \in \mathbf{r}$ such that $T \cdot y \subset X \setminus \{x\}$. Then one has that $X \setminus \{x\} \in \varepsilon^{\mathbf{r}}(X)$ and $\bigcap_{x \notin P(X)} X \setminus \{x\} = X \setminus (\bigcup_{x \notin P(X)} \{x\}) = P(X) \subset \bigcap_{E \in \varepsilon(X)} E = L(X)$. □

Taking into account the result above, if X is flow and X is T_1 we have that

$$L^{\mathbf{r}}(X) = P(X) \subset P^{\mathbf{r}}(X) \subset \Omega^{\mathbf{r}}(X) \subset X \ .$$

With respect to decompositions, it will be interesting to find topological and dynamical conditions to ensure that the $\omega_{\mathbf{r}}$-decomposition of a flow X divides $\Omega^{\mathbf{r}}(X)$ without dividing $\omega^{\mathbf{r}}(x)$ for each $x \in X$.

We note that if we take on $\mathbb{R}$ the externology

$$\mathbf{l} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (-\infty, -n] \subset U\}$$

or we take the reversed flow, we have the notion of **l**-exterior flow and we obtain the corresponding dual results.

In this paper we have presented some initial applications of exterior spaces to the study of dynamical systems. The authors together other collaborators want to develop a research project to study more applications of these new techniques.

In particular, we are interested in the following subjects:

(i) using the externology of the flow to study the shape of the limit space,

(ii) to find some relations between global stability and the exterior homotopy type of the limit space,

(iii) to find some connections between end points and end spaces of a flow and attractors, local stability, etc.,

(iv) to apply the properties of algebraic invariants developed in shape theory and exterior homotopy theory to classification problems of some families of dynamical systems.

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