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RESONANCES AND STABILITY FOR A TWO DEGREES OF FREEDOM HAMILTONIAN SYSTEM FROM A GEOMETRIC POINT OF VIEW

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Dedicado a la memoria de Mirian Andrés Gómez

Resumen. La estabilidad de sistemas Hamiltonianos es una parte esencial en el estudio de un amplio número de problemas en varias ramas de la Ciencia, tales como Mecánica Clásica, Mecánica Celeste, Física Atómica, etc. Además, es un tema de alto interés para las Matemáticas. No obstante, el problema es difícil de abordar, incluso para sistemas con dos grados de libertad, donde existen algunas situaciones especiales que deben ser resueltas.

En este trabajo se presentan algunos de los resultados clásicos sobre estabilidad en sistemas Hamiltonianos con dos grados de libertad y cómo éstos se pueden unificar en un enunciado sencillo. Esto se consigue atendiendo a la geometría del problema cuando se ha reducido éste a uno más simple, manteniendo las propiedades esenciales del sistema original.

Abstract. Stability of Hamiltonian systems is an essential piece in the study of a number of problems in various scientific branches, such as Classical Mechanics, Celestial Mechanics, Atomic Physics, etc. Furthermore, it is a subject of high mathematical interest. Nevertheless, the problem is difficult to tackle even for systems with two degrees of freedom where there are some special situations to be solved.

In this work we present some of the classical results on the stability question for two degrees of freedom Hamiltonian systems and how they can be unified in a very simple statement. This is done paying attention to the geometry of the problem when it is reduced to a simpler one, retaining the essential properties of the original system.

1. Introduction

The stability of equilibrium positions of mechanical systems is an old question and can be traced back to the 17th century, just after the publication of Newton's Principia. One of the problems in touch was the stability of the solar system, which astronomers and mathematicians studied under the name of the n-body problem. This problem showed to be very difficult to handle and even a precise concept of stability was not clear and different definitions were used.

Key words and phrases. Nonlinear stability, normal forms, resonances.

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The usual concept of stability is due to Lyapunov [15] and, for an equilibrium solution, reads as

Definition 1. *The equilibrium solution* $\mathbf{x}_0$ *of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ *is said to be stable in the sense of Lyapunov (or Lyapunov stable) if for each* $\epsilon > 0$ *there exists* $\delta 0$ *such that if* $\mathbf{x}(t)$ *is any solution satisfying* $\|\mathbf{x}(t_0) - \mathbf{x}_0\| < \delta$, *then* $\|\mathbf{x}(t) - \mathbf{x}_0\| < \epsilon$ *for all* $t \geq t_0$. *If the equilibrium is not stable it is said to be unstable.*

For a linear differential equation, $\dot{\mathbf{x}} = A\mathbf{x}$, the stability of the trivial solution $\mathbf{x} = 0$ follows from the eigenvalues of the matrix A. Indeed, if all the eigenvalues of A have zero or negative real part and the corresponding Jordan blocks of the zero real part eigenvalues are diagonal, the equilibrium is stable. Otherwise the equilibrium is unstable.

Thanks to this result, it is possible to establish the stability of an equilibrium point in the first approximation, that is to say of its linear approximation. The next step is whether linear stability implies Lyapunov stability. In this way, let us consider a system of differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}) = A\mathbf{x} + \dots, \tag{1}$$

where f is a sufficient differentiable function satisfying $f(0) = 0$ and A the Jacobian matrix at $\mathbf{x} = 0$. Then, if all the eigenvalues of A have negative real part, the equilibrium $\mathbf{x} = 0$ is Lyapunov stable. On the contrary, if at least one of the eigenvalues has positive real part, the equilibrium is unstable.

The case when all the eigenvalues have zero real part is a critical one and more sophisticated results are needed. One of them is due to Dirichlet [7], although the idea was original of Lagrange, who proved that if the system (1) has a first integral and $\mathbf{x} = 0$ is an extremum of that integral, then the equilibrium is stable. This result is specially interesting in the case of Hamiltonian systems.

Let us consider a canonical system of differential equations

$$\frac{dq_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j} \quad (j = 1, 2) \tag{2}$$

System (2) is called a two degrees of freedom Hamiltonian system and the function $\mathcal{H}$ is called the Hamiltonian function.

We suppose that $\mathcal{H}$ is analytic in a neighborhood of the equilibrium $q_j = p_j = 0$. Thus, it can be written as

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \dots + \mathcal{H}_k + \dots, \tag{3}$$

where $\mathcal{H}_k$ $(k \geq 2)$ are homogeneous polynomials of order k with respect to q_j, p_j with constant coefficients.

By virtue of Liouville theorem, if λ is an eigenvalue of the linear part, also $-\lambda$ and $\bar{\lambda}$ are eigenvalues. Thus, in order to have Lyapunov stability, a necessary condition is that all eigenvalues have zero real part. In this way, the stability of the origin in the sense of Lyapunov depends on properties of the Hamiltonian function $\mathcal{H}$. As (2) is autonomous, then $\mathcal{H}$ is a first integral and if $\mathcal{H}$ is definite then, by

Dirichlet's theorem, the origin is stable. In fact it is enough that the quadratic form $\mathcal{H}_2$ be definite.

When $\mathcal{H}_2$ is not definite, Arnold [3] and Moser [19, 23] proved that the equilibrium position is stable if certain non-degeneracy conditions are fulfilled. This criterion works in the so called *general elliptic case*, when the eigenvalues are purely imaginary, $\pm\omega_1 i$, $\pm\omega_2 i$. However, it is necessary that $\omega_1/\omega_2 \notin \mathbb{Q}$, which means that ω_1 and ω_2 do not satisfy a resonant condition.

Definition 2. *ω_1 and ω_2 are said to be resonant if there exist a pair of integers n and m such that $n\omega_1 - m\omega_2 = 0$. The number $|n| + |m|$ is called the order of the resonance.*

There are well known examples for which the origin is unstable in the presence of resonances despite of its stability in the linear approximation [6, 14]. The first one who obtained rigorous results of stability in the presence of resonances was Markeev [16, 17], but for concrete resonance values. A general result for resonant cases is due to Cabral and Meyer [5], which also accounts for the classical result of Arnold and Moser.

All these stability and instability criteria are established with the help of technical theorems, like Moser's invariant curve theorem, KAM theory and Chetaev's instability theorem. However, a geometric approach to these results is also possible as it was shown by Alfriend [1, 2] when he studied the stability of Lagrangian points in the circular restricted three-body problem, for certain resonant cases. Later on, Elipe et al. [10] gave a different geometric approach that was generalized in [11, 21]. The study of stability under a geometric point of view has been one of our research lines along the last years, while Mirian was a staff member at our Department. In what follows, we will give a summary of the classical results and some geometric counterparts we have derived.

2. Classical results

The key idea in the derivation of stability criteria lies on the reduction of system (2) to another one expressed in a simpler way; the new system retaining the essential properties of the original one in a small neighborhood of the equilibrium point. This was one of the tasks in Poincaré's thesis [22], who tried to reduce the system to a linear one. Unfortunately, this is not possible for a Hamiltonian system and we have to satisfy ourselves with a new system derived from a *normalized* Hamiltonian function. This new Hamiltonian is obtained step by step by constructing a new *formal* first integral which, in the general elliptic case, is the term $\mathcal{H}_2$ in (3). In this way, we arrive to the so called Birkhoff-Gustavson normal form, after Birkhoff [4] and Gustavson [12].

Theorem 1 (Birkhoff)**.** *Let $\mathcal{H}$ be given by (3), where*

$$\mathcal{H}_2 = \frac{1}{2}\omega_1(p_1 2 + q_1 2) \pm \frac{1}{2}\omega_2(p_2 2 + q_2 2),$$

and $\omega_1/\omega_2 \notin \mathbb{Q}$. *Then, there exists a canonical change of variables* $(q_k, p_k) = w(\xi_k, \eta_k)$ *in such a way that*

$$\mathcal{H}(w(\xi_k, \eta_k)) = \Gamma(\xi_1 2 + \eta_1 2, \xi_2 2 + \eta_2 2)$$

is a formal power series of $\xi_k 2 + \eta_k 2$, $k = 1, 2$.

Based on this theorem, we have the first stability result, due to Arnold, we reproduce in a more readable version given by Meyer and Schmidt [18].

Theorem 2 (Arnold)**.** *Let us consider a two degrees of freedom Hamiltonian system* $\mathcal{H}$ *expressed in the Poincare's variables* $(\Psi_1, \Psi_2, \psi_1, \psi_2)$, *as*

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \cdots + \mathcal{H}_{2n} + \tilde{\mathcal{H}},$$

where

1. $\mathcal{H}$ *is a real analytic function in a neighborhood of the origin.*
2. $\mathcal{H}_{2k}$, $1 \leq k \leq n$, *is a homogeneous polynomial of degree* k *in* Ψ_1, Ψ_2, *with real coefficients and independent of* ψ_1, ψ_2. *In particular,*

$$\mathcal{H}_2 = \omega_1 \Psi_1 - \omega_2 \Psi_2, \quad \omega_1, \omega_2 > 0;$$

$$\mathcal{H}_4 = \tfrac{1}{2} \left(A \Psi_1 2 - 2B \Psi_1 \Psi_2 + C \Psi_2 2 \right).$$

3. $\tilde{\mathcal{H}}$ *has a power expansion in* Ψ_1, Ψ_2 *which starts with terms at least of order* $2n + 1$ *and whose coefficients are real trigonometric sums in* ψ_1, ψ_2.

Under these assumptions, the origin is Lyapunov stable if there exists some k, $2 \leq k \leq n$ *such as* $\mathcal{H}_2$ *does not divide* $\mathcal{H}_{2k}$, *likewise if* $\mathcal{D}_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1) \neq 0$.

Several things must to be noted about this theorem. The first one is the use of a sort of polar coordinates, the Poincaré variables, accounting for the toroid structure of phase space. Ψ_k are radial distances defining the two radius of the torus, whereas ψ_k are the two angular coordinates on the torus. The second thing is that $\mathcal{H}$ is supposed to be in Birkhoff normal form up to order $2n$. As a consequence, the two fundamental frequencies ω_1 and ω_2 do not satisfy any resonant condition, that is $\omega_1/\omega_2 \notin \mathbb{Q}$.

To handle resonant cases, Markeev provided some specialized theorems [16] for third and fourth order resonances, that is, when $\omega_1 = 2\,\omega_2$ and $\omega_1 = 3\,\omega_2$. The most important difference is that in resonant cases the normal form is no longer like the Birkhoff normal form. It retains some terms containing the angular variables, in such a way that the normal form looks like different depending on the order of the resonance. These theorems are presented below.

Theorem 3 (Markeev, 1:2 resonance)**.** *Let us consider a Hamiltonian system under a 1:2 resonance whose normal form is written in terms of the Poincaré variables as*

$$\mathcal{H} = 2\omega_2 \Psi_1 - \omega_2 \Psi_2 + \delta \Psi_1^{1/2} \Psi_2 \cos(\psi_1 + 2\psi_2) + \overline{\mathcal{H}},$$

with $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\Psi_1, \Psi_2, \psi_1, \psi_2) = O((\Psi_1 + \Psi_2)2)$. *If* $\delta \neq 0$, *the equilibrium is unstable.*

Theorem 4 (Markeev, 1:3 resonance)**.** *Let us consider a Hamiltonian system under a 1:3 resonance whose normal form is written in terms of the Poincaré variables as*

$$\mathcal{H} = 3\omega_2\Psi_1 - \omega_2\Psi_2 + \delta\Psi_1^{1/2}\Psi_2^{3/2}\cos(\psi_1 + 3\psi_2) + \frac{1}{2}(A\Psi_1 2 + 2B\Psi_1\Psi_2 + C\Psi_2 2) + \overline{\mathcal{H}},$$

where $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\Psi_1, \Psi_2, \psi_1, \psi_2) = O((\Psi_1 + \Psi_2)^{5/2})$*. We denote* $D = A + 6B + 9C$*. If* $6\sqrt{3}|\delta||D|$*, the equilibrium is unstable and if* $6\sqrt{3}|\delta| < |D|$*, the equilibrium is stable.*

In addition to these results, there are stability criteria for low order resonances [13, 24, 25], when the normal form does not fit with the Birkhoff-Gustavson case. Now, the formal integral to be constructed is related with the decomposition of the linear part in its semisimple and nilpotent components [20]. Excluding low order resonances, Cabral and Meyer formulated a very general theorem to give stability conditions for both resonant and nonresonant cases. Its statement is very similar to that of Arnold's theorem, and it reads as follows:

Theorem 5 (Cabral & Meyer)**.** *Let us consider the Hamiltonian* $\mathcal{H}$ *whose normal form up to order* r *is expressed as*

$$\mathcal{H} = \mathcal{H}_2(\Psi_1, \Psi_2) + \mathcal{H}_4(\Psi_1, \Psi_2) + \cdots + \mathcal{H}_{2l}(\Psi_1, \Psi_2) + \mathcal{H}_r(\Psi_1, \Psi_2, n\psi_1 + m\psi_2) + \cdots$$

where $r = 2l + 1$ *or* $r = 2l + 2$*, and*

1. $\mathcal{H}$ *is a real analytic function in a neighborhood of the origin and* 2π*–periodic in* $n\psi_1 + m\psi_2$*.*
2. $\mathcal{H}_{2k}$ $(1 \le k \le l)$ *is a homogeneous polynomial of degree* k *in* Ψ_1*,* Ψ_2 *with real coefficients independent of* ψ_1, ψ_2*. In particular,*
$$\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2.$$
3. ω_1 *and* ω_2 *satisfy a resonance condition* $n\omega_1 - m\omega_2 = 0$*. If* $n = m = 1$*, we assume that the corresponding linear system is diagonalizable.*
4. $\mathcal{H}_r(\Psi_1, \Psi_2, n\psi_1 + m\psi_2)$ *is a homogeneous polynomial of degree* r *in* $\sqrt{\Psi_1}$*,* $\sqrt{\Psi_2}$*, with coefficients which are finite Fourier series in the angle* $n\psi_1 + m\psi_2$*.*
5. *The dots denote terms of degree bigger than* r *in the variables* $\sqrt{\Psi_1}, \sqrt{\Psi_2}$*.*

Let us define $\mathcal{D}_{2k} = \mathcal{H}_{2j}(\omega_2, \omega_1)$ $(1 \le k \le l)$ *and*

$$\Psi(\psi) = \mathcal{H}_r(\omega_2, \omega_1, \psi),$$

where

$$\psi = n\psi_1 + m\psi_2.$$

Under these assumptions,

- *If* $\mathcal{D}_{2k} \neq 0$ *for some* $k = 2, \ldots, l$*, then the origin is Lyapunov stable.*
- *If* $\mathcal{D}_{2k} = 0$ *for all* $2 \le k \le l$*, and*
  - $\Psi(\psi) \neq 0$ *for all* ψ*, the origin is Lyapunov stable.*
  - Ψ *has a simple zero, that is, if there exists* ψ^* *such that* $\Psi(\psi^*) = 0$ *and* $\Psi'(\psi^*) \neq 0$*, the origin is unstable.*

As in Arnold's theorem there is an implicit assumption in the hypothesis of this theorem. Indeed, it is supposed that r is the first term in the normal form containing angular variables. Thus, r is the order of the resonance. However, this assumption can be weakened as we will see in the geometric approach.

The proof of theorem 5 strongly lies on a basic lemma, due to Sokolsky [25], that reduces the problem to the computation of real roots of an appropriate function.

Lemma 1. *Let $K(s,\psi,t) = \Psi(\psi)s^n + O(s^{n+\frac{1}{2}})$, where $n = m/2$ with $m \geq 3$, an integer. Let us assume that K is an analytic function of $\sqrt{s},\psi,t$, τ-periodic in ψ and T-periodic in t. If $\Psi(\psi) \neq 0$, for all ψ, then the origin $s = 0$ is a stable equilibrium for the Hamiltonian system*

$$\dot{s} = \frac{\partial K}{\partial \psi}, \qquad \dot{\psi} = -\frac{\partial K}{\partial s},$$

in the sense that given $\epsilon > 0$, there exists $\delta > 0$ such that if $s(0) < \delta$, then the solution is defined for all t and $s(t) < \epsilon$. If $\Psi(\psi)$ has a simple zero, i.e., if there exists ψ^* such that $\Psi(\psi^*) = 0$ and $\Psi'(\psi^*) \neq 0$, then the equilibrium $s = 0$ is unstable.

Now we are in position to give a geometric counterpart of the stability criteria, focusing on resonant cases. To this end, we proceed to identify the structure of the phase space after normalization and study the shape of the orbits on the reduced phase space. Indeed, after normalization, a new formal integral is introduced (namely $\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2$) and the phase space can be considered as a foliation of two dimensional surfaces in terms of the value of $\mathcal{H}_2$. Taking this into account, we do not follow the standard procedure of the isoenergetic reduction and consider the motion at the energy level $\mathcal{H} = 0$ (this is the basis of the proof given by Cabral & Meyer). On the contrary, we consider the motion at the surface $\mathcal{H}_2 = 0$ where the origin lies.

3. The reduced phase space and the geometric criterion

A normalization is a step by step procedure where in each step a canonical change of variables is made to kill, order by order, those monomials in the Hamiltonian function which are incompatible with the formal integral $\mathcal{H}_2$. Thus, the new Hamiltonian is generated by a set of *invariant* monomials that act as new variables of the system.

Thanks to the new formal integral these invariants are not independent. They are linked by a functional relation which is different depending on the order of the resonance. There are several ways to construct the invariants. We will use that given in [11], based on an alternative set of polar variables, specially designed to handle resonant oscillators [8, 9]. Using these variables, we denote by M_1, M_2, C, S, and assuming $n\omega_1 - m\omega_2 = 0$ ($\omega_1/m = \omega_2/n = \omega$), the normalized Hamiltonian, up to order N, can be written as

$$\mathcal{H} = \mathcal{H}_2 + \sum_{j=3}^{N} \mathcal{H}_j$$

The formal integral is $\mathcal{H}_2 = 2\omega M_2$ and

$$\mathcal{H}_j = \sum_{2(\gamma_1+\gamma_2)+(n+m)(\gamma_3+\gamma_4)=j} a_{\gamma_1\gamma_2\gamma_3\gamma_4} M_1^{\gamma_1} M_2^{\gamma_2} C^{\gamma_3} S^{\gamma_4}, \qquad 3 \le j \le N.$$

The invariants are not independent and they satisfy the equation

$$C2 + S2 = (M_1 + M_2)^n (M_1 - M_2)^m, \tag{4}$$

together with the relation

$$M_1 \ge |M_2|. \tag{5}$$

Equations (4) and (5) define the reduced phase space as a revolution surface for each constant value of M_2, with a vertex at the point $M_1 = |M_2|$, $C = S = 0$. In particular, the origin is the vertex of the surface corresponding to $M_2 = 0$. Figure 1 shows different surfaces of revolution for a 1:3 resonance depending on the value of M_2.

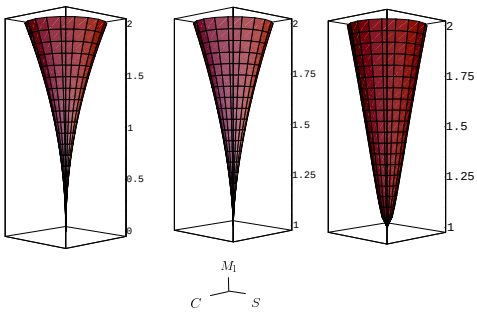

FIGURE 1. The reduced phase space for a 1:3 resonance and different values of M_2. From left to right $M_2 = 0$, $M_2 > 0$ and $M_2 < 0$.

Once the reduced phase space is determined, it is possible to know the flow of the normalized system, when it is truncated to a prescribed order. Indeed, the flow results as the intersection of the normalized Hamiltonian function with the surface defined by the reduced phase space (see figure 2).

Now, a geometric criterion based on how the phase portrait looks like in a neighborhood of the origin can be stated, on the basis we are only interested on the flow in the manifold $M_2 = 0$. In this sense, if the orbits are closed trajectories, then the origin will be stable, whereas if there are asymptotic orbits to the origin, it will be unstable. In fact, this comes down to determining the zero level energy curves on the surface (4) for $M_2 = 0$, that is, the intersection of the surfaces (4) and $\mathcal{H} = 0$ (for $M_2 = 0$).

Let us assume that the Hamiltonian is normalized up to a certain order $N \ge r$, with $\mathcal{H}_N$ the first term in the normal form does not vanish for $M_2 = 0$. Under these conditions, we get the following result.

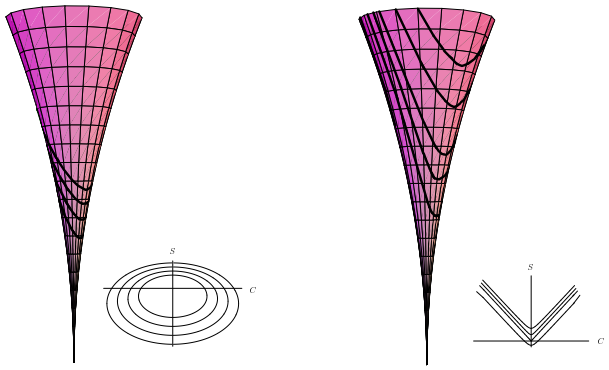

FIGURE 2. Orbits and projections, when $M_2 = 0$, in stable and unstable cases respectively.

Theorem 6 (Geometric criterion)**.** *Let us consider the two surfaces* $\mathcal{G}_1 \equiv C2 + S2 = M_1^r$ *and* $\mathcal{G}_2 \equiv \mathcal{H}(C, S, M_1; M_2 = 0) = 0$*. If their only common point is the origin, then it is stable. If they intersect each other transversally, then the origin is unstable.*

This result is a bit more general than previous results, as it is not implicit that the normal form is given up to the order of the resonance. Even more, it can be generalized to include all kind of resonances, not only those of order greater than 3, but also low order resonances. In this way, we must take into account the structure of the phase space, after normalization, and the new formal integral. Thus, we have a very general result

Theorem 7 (Geometric criterion II)**.** *Let be* I *the formal integral introduced in the normalization procedure and let be* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *the following two surfaces*

- $\mathcal{G}_1$*, defined by the first term in the normal form does not vanish for* $I = 0$*.*
- $\mathcal{G}_2$*, the manifold of the reduced phase space for* $I = 0$*.*

Under these assumptions, if the two surfaces have the origin as an isolated intersection point, the origin is stable. Otherwise, if they are not tangent, the origin is unstable.

A remarkable thing is the fact that $\mathcal{G}_1$ is a definite function in the neighborhood of the origin in the case of stability, whereas it is not sign defined if the equilibrium is unstable. In the last case, the non trivial level set $\mathcal{H} = 0$ acts as a separatrix and, since the origin is an equilibrium point, asymptotic orbits must appear.

4. The geometry of the 1:3 resonance

In order to exemplify the geometric criterion, we consider the case of a 1:3 resonance. Now, the normal form, up to fourth order can be written as

$$\mathcal{H} = 2\omega M_2 + a_4 M_1 2 + b_4 M_1 M_2 + c_4 M_2 2 + \alpha S + \beta C,$$

where a_4, b_4, c_4, α and β are known parameters for a specific problem, such that $a_4 2 + \alpha 2 + \beta 2 \neq 0$. On the other hand, the reduced phase space is given by

$$C2 + S2 = (M_1 + M_2)(M_1 - M_2)3. \tag{6}$$

It can be proved that a rotation about the M_1 axis leaves invariant the structure of the phase flow. In this way, it is possible to reduce the number of parameters in the normal form and it can be expressed as

$$\mathcal{H} = 2\omega M_2 + a_4 M_1 2 + b_4 M_1 M_2 + c_4 M_2 2 + \gamma S. \tag{7}$$

Taking into account (6) and (7), the two surfaces appearing in the geometric criterion are, after doing $M_2 = 0$,

$$\mathcal{G}_1 = \{(C, S, M_1) \in \mathbb{R}3; \quad a_4 M_1 2 + \gamma S = 0\},$$

and

$$\mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}3; \quad C2 + S2 = M_1 4\}.$$

The intersection of these two surfaces is given by

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}3; \quad S = -\frac{a_4 M_1 2}{\gamma}, C2 = (1 - \frac{a_4 2}{\gamma 2}) M_1 4\} \tag{8}$$

if $\gamma \neq 0$. If $\gamma = 0$, the origin is the only intersection point. As the variables C, S and M_1 are real values, from the intersection given by (8) two cases must be distinguished. On the one hand, if $a_4 2 > \gamma 2$ it must be $M_1 = 0$ and so, S and C, then the only common point of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the origin and it is stable. On the other hand, if $a_4 2 < \gamma 2$ there is an intersection curve, defining an asymptotic orbit, and therefore the origin is unstable.

It is easy to visualize the two cases by means of a projection of the two surfaces onto the plane $C = 0$. If the projection of $\mathcal{G}_1$ lies in the region defined by $S2 - M_1 4 > 0$, the origin is unstable. Otherwise, if the projection is not one of the two sheets of $S2 - M_1 4 = 0$, the origin is stable. Figure 3 shows the two cases, depending if the projection of $\mathcal{G}_1$ lies inside or outside the projection of the conic surface $\mathcal{G}_2$.

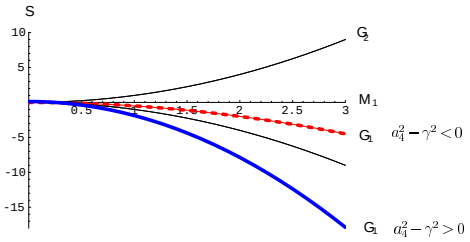

FIGURE 3. Surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ projected onto the plane $C = 0$ in two cases: stable (solid line) and unstable (dashed line).

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