Geometric illustrations of the conjugacy principle

Fernando Martins
Escola Superior de Educação - Instituto Politécnico de Coimbra

M. A. Facas Vicente
Department of Mathematics - University of Coimbra

Abstract

In this paper, we illustrate the use of the conjugacy principle in several geometric transformations: translations, reflections and rotations. Our context are the real spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. The main role is played by matrices, as it is to be expected. The referred to transformations are fundamental when studying motion questions in the Mechanics, Computer Graphics, Robotics, Computer Games, etc.. Our didactical goals are focused on the acquisition and development of spatial abilities, in order to get a better perception, interpretation and forecasting of the geometrical transformations in our physical world.

Key-words

Didactics of mathematics, Geometry, Conjugacy principle, Translation, Reflection, Rotation

Resumo

Neste artigo, mostramos a utilização do princípio da conjugação na resolução de problemas usando as transformações geométricas: translações, reflexões e rotações. O nosso contexto são os espaços reais $\mathbb{R}^2$ e $\mathbb{R}^3$. Nesta abordagem, a teoria de matrizes, tal como esperado, desempenha um papel preponderante. As transformações geométricas referidas anteriormente são fundamentais no estudo de problemas relativos a movimento, presentes, por exemplo, em Mecânica, Computação Gráfica, Robótica e Jogos. Temos como objectivos de ordem didáctica a aquisição e desenvolvimento de capacidades espaciais, proporcionando desta forma meios de percepcionar o mundo físico e de interpretar, modificar e antecipar transformações relativamente aos objectos.

Palavras-chave

Didáctica da matemática, Geometria, Princípio da conjugação, Translação, Reflexão, Rotação

1. Introduction

The world is essentially geometric and, therefore, the study of geometry increases its understanding.

The notion of transformation – which adds a dynamical perspective to geometry – is a tool to study and organize geometrical concepts. The ability to understand and solve problems in geometry is greater when geometric transformations are used. In this sense, we can say that geometric transformations foster the increase of the ability of spatial perception.

In geometry, there are some problems that either can not be solved by direct application of geometric transformations or the achievement of their solutions may become a very difficult task. The conjugacy principle helps us to find a solution to some of these hard problems. This principle [5, vol II, p.374] can be summarized by the following: *in order to solve a difficult problem A, we obtain and solve an easier one T, by using a transformation S and its inverse* S^{-1}. Furthermore, the relation $A=STS^{-1}$ linking *A* and *T* is called conjugacy; it is an equivalence relation.

The plan of this paper is as follows: in Section 2 we materialize the conjugacy principle using translations; in Section 3, to deal with reflections and rotations, the conjugacy principle is again used.

Some abuse of notation is patent in this paper: we use the sign ":=" to identify these situations. We consider an orthonormal referential $\{O;(\vec{e_1},\vec{e_2},\vec{e_3})\}$ and – under the umbrella of adequate isomorphisms – we write points and vectors in several ways, according to our needs in each moment.

2. Translation

A **translation** consists in moving every point in a constant distance in a specified direction. It is one of the rigid motions (other rigid motions include rotation and reflection). A translation can also be interpreted as the addition of a constant vector to each point, or as the shifting of the origin of the coordinate system.

In a more physical approach, the translation means the motion of an object or figure from a point to another point, along a given direction and always parallel to itself. In the mathematics literature it is mentioned that the translation is associated to a vector, this being a mathematical object defined by a direction, orientation and length.

In a great deal of situations we observe translation motions, for instance the earth motion, the motion of a rolling stair, a chair lift or a lift.

Undergraduate Science and Engineering students at Universities (classic and technical) are expected to deal with distance problems in space geometry – which we include in the category of best approximation problems – using cross, $\times$, and dot, $\bullet$, products of vectors. Usually textbooks just present formulas for distances. Not much insistence is put on exhibiting the foot of the perpendicular. When using the conjugacy principle, we consider a translation *t* (which is a non linear application) defined by

$$t_{\vec{v}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\vec{x} \longrightarrow \vec{x} + \vec{v}$$ where, of course, we have

$$\left(t_{\vec{v}}\right)^{-1} = t_{(-\vec{v})} .$$

In this section, we state and prove the results we are going to use, by constructing the solutions, instead of simply enunciating them, and then particularizing the results known in Approximation Theory. In this constructive approach we present the foot of the perpendicular in terms of dot and cross products of vectors. For computing the dot product, •, and the cross product, ×, we use matrix computations.

Given two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$,

we have [6, p.155] $\vec{u} \bullet \vec{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{u} \times \vec{v} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

In this section we are going to show the role of the translation, associated to the conjugacy principle, by considering two situations:

(i) the best approximation on the line *l* to an exterior point *P;*
(ii) the best approximation pair of two skew lines l_1 and l_2.

2.1. The best approximation from a point to a line

The best approximation on the line *l* to an external point *P* is the foot *S,* onto the line *l*, of the perpendicular that passes through the point *P*.

In the next result, we establish a formula for the foot of the perpendicular, in terms of the cross product.

Proposition 1 *Let l be a line, which passes through the point* $M := \vec{m}$ *and is parallel to the vector* $\vec{u}$ *, given by* $\vec{u} \times \left(\vec{x} - \vec{m}\right) = \vec{0}$ *. Let* $P := \vec{p}$ *be a point such that* $P \notin l$ *. Then the foot of the perpendicular S is given by the formula:* $S = \vec{p} - \dfrac{\vec{u} \times \vec{r}}{\left\|\vec{u}\right\|^2}$, *with* $\vec{r} = \vec{u} \times \left(\vec{m} - \vec{p}\right)$.

Proof: We have a pair (P,l) formed by a point $P := \vec{p}$ and a

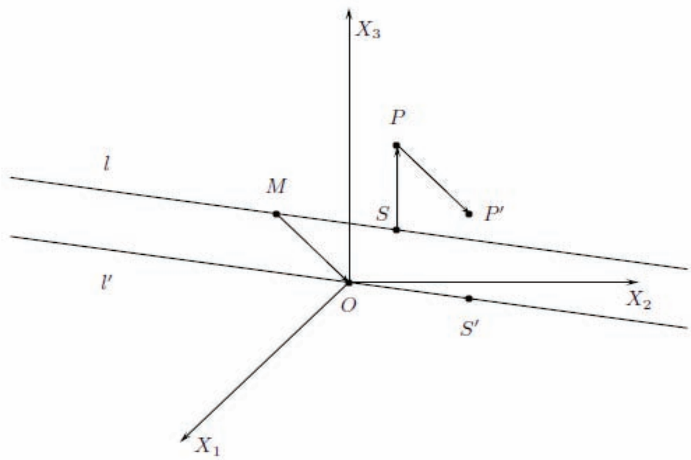

Figure 1: Foot of the perpendicular, S, drawn from the point P

line l with equation $\vec{u} \times (\vec{x} - \vec{m}) = \vec{0}$ such that $P \notin l$. We have to displace the line l to the origin and, for that purpose, we perform the translation $\overrightarrow{MO} = O - M := -M := -\vec{m}$. So, the pair (P',l') enters into consideration, where $P' = P - M := \vec{p}' = \vec{p} - \vec{m}$, and $l' := \vec{u} \times \vec{x}' = \vec{0}$.

We have $d(P,l) = d(P',l')$, as distance is invariant under translations [3].

We look for the foot $S' := \vec{s}'$ of the perpendicular drawn from the point P' onto the line l'.

We build this proof in two steps:

consider a plane π through the point P' and perpendicular to the line l': π is given by $\overrightarrow{P'X'} \bullet \vec{u} = 0$;

intersect the constructed plane π and the line l', thus obtaining the foot S', $S' = l' \cap \pi$; we have, successively:

$$\begin{cases} \vec{u} \times \vec{x}' = \vec{0} \\ \overrightarrow{P'X'} \bullet \vec{u} = 0 \end{cases} \text{ or } \begin{cases} X' = \alpha \vec{u} \\ (X' - P') \bullet \vec{u} = 0 \end{cases} (\alpha \in \mathbb{R}) \text{ or } (\alpha \vec{u} - \vec{p}') \bullet \vec{u} = 0$$

, thus getting

$a = \frac{\vec{p'} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}$.

So, we get the foot of the perpendicular

$$S' := \vec{s'} = \frac{\vec{p'} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u} = \vec{p'} + \frac{\vec{p'} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\vec{u} - \vec{p'} = \vec{p'} + \frac{\vec{u}\left(\vec{p'} \bullet \vec{u}\right)}{\vec{u} \bullet \vec{u}} - \frac{\vec{p'}\left(\vec{u} \bullet \vec{u}\right)}{\vec{u} \bullet \vec{u}}$$

$$= \vec{p'} - \frac{-\vec{u}\left(\vec{p'} \bullet \vec{u}\right) + \vec{p'}\left(\vec{u} \bullet \vec{u}\right)}{\vec{u} \bullet \vec{u}} = \vec{p'} - \frac{-\vec{u}\left(\vec{p'} \bullet \vec{u}\right) - \vec{p'}\left(\vec{u} \bullet \left(-\vec{u}\right)\right)}{\vec{u} \bullet \vec{u}}$$

$$= \vec{p'} - \frac{\vec{u} \times \left(-\vec{u} \times \vec{p'}\right)}{\vec{u} \bullet \vec{u}} = \vec{p} - \vec{m} - \frac{\vec{u} \times \left(-\vec{u} \times \left(\vec{p} - \vec{m}\right)\right)}{\vec{u} \bullet \vec{u}} = \vec{p} - \vec{m} - \frac{\vec{u} \times \left(\vec{u} \times \left(\vec{m} - \vec{p}\right)\right)}{\left\|\vec{u}\right\|^2}.$$

After being performed the reverse translation $\overrightarrow{OM} = M - O = \vec{m}$, we obtain the foot of the perpendicular

$$S = S' + M := \vec{s'} + \vec{m} = \vec{p} - \frac{\vec{u} \times \left(\vec{u} \times \left(\vec{m} - \vec{p}\right)\right)}{\left\|\vec{u}\right\|^2}.$$

In the case where the line *l* passes through the origin of the coordinates we have the following result.

Corollary 1 *Let l_0 be a line [containing the origin O and which is parallel to the vector $\vec{u}$] given by $\vec{u} \times \vec{x} = \vec{0}$ and $P := \vec{p}$ be a point such that $P \notin l_0$. Then the foot of the perpendicular S is given by the formula $S = \vec{p} - \frac{\vec{u} \times \vec{r}}{\left\|\vec{u}\right\|^2}$, with $\vec{r} = \vec{p} \times \vec{u}$.*

2.2. The best approximation of two skew lines

In this section we present a process to determine the pair (S_1, S_2) of points $S_1 \in l_1$ and $S_2 \in l_2$ that are closest to each one. The key idea is that we invoke twice the distance from a point to a line through the origin. We have two translation movements: each line has, once, to be displaced to the origin. Afterwards, two corresponding reverse translations have to be done, as well. The feet of the perpendiculars (one foot on each line) depend on parameters. The vector whose extremities are the feet of the perpendiculars also depends on these parameters.

Let us consider the skew lines l_1 and l_2 given, respectively, by $\vec{u} \times \left(\vec{x} - \vec{p}\right) = \vec{0}$,

$\vec{v} \times (\vec{x} - \vec{q}) = \vec{0}$, with $P := \vec{p}$ and $Q := \vec{q}$.

We look for the foot of the perpendicular S_1, onto l_1, and the foot of the perpendicular S_2, onto l_2.

Then we form a vector which achieves the distance between the two lines, for example $\overrightarrow{S_1S_2}$. The distance is given by $d(l_1, l_2) = \left\| \overrightarrow{S_1S_2} \right\|$.

We may follow the next steps.

(a) Translation of the line l_1 to the origin.

We translate the pair (l_1, l_2), so obtaining the pair (l_1', l_2'), where

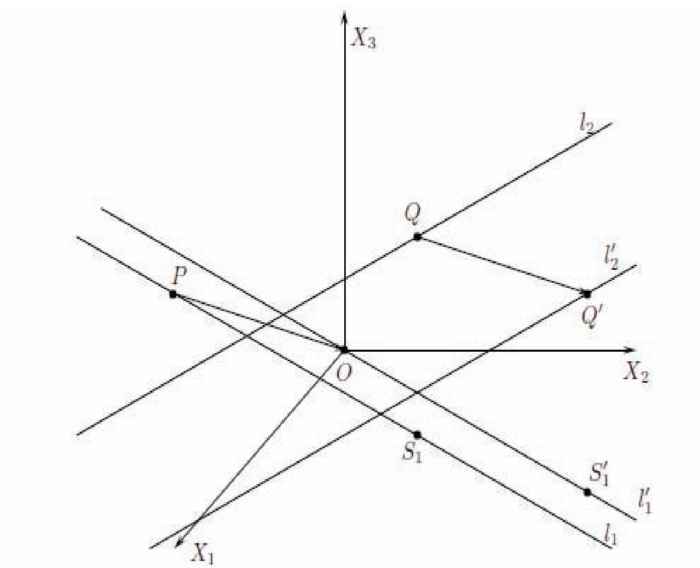

Figure 2: The pair (l_1, l_2) turns into the pair (l_1', l_2').

$l_1' := \vec{u} \times \overrightarrow{x'} = \vec{0}$, $l_2' := \vec{v} \times (\overrightarrow{x'} - \overrightarrow{q'}) = \vec{0}$, with $\overrightarrow{q'} = \vec{q} - \vec{p}$. We take the current point, $Q_2'(b), b \in \mathbb{R}$, on the line l_2', which is $Q_2'(b) = (q_1 - p_1 + bv_1, q_2 - p_2 + bv_2, q_3 - p_3 + bv_3) := \overrightarrow{q_2'}(b)$. Our problem, now, is to determine the distance, $d(Q_2'(b), l_1')$, between the point $Q_2'(b)$ and the line l_1'.

Applying Corollary 1, we have $S_1'(b)$, the foot of the perpendicular onto the line l_1', $S_1'(b) = \overrightarrow{q_2'}(b) - \dfrac{\vec{u} \times \vec{r}}{\|\vec{u}\|^2}$, *with* $\vec{r} = \overrightarrow{q_2'}(b) \times \vec{u}$.

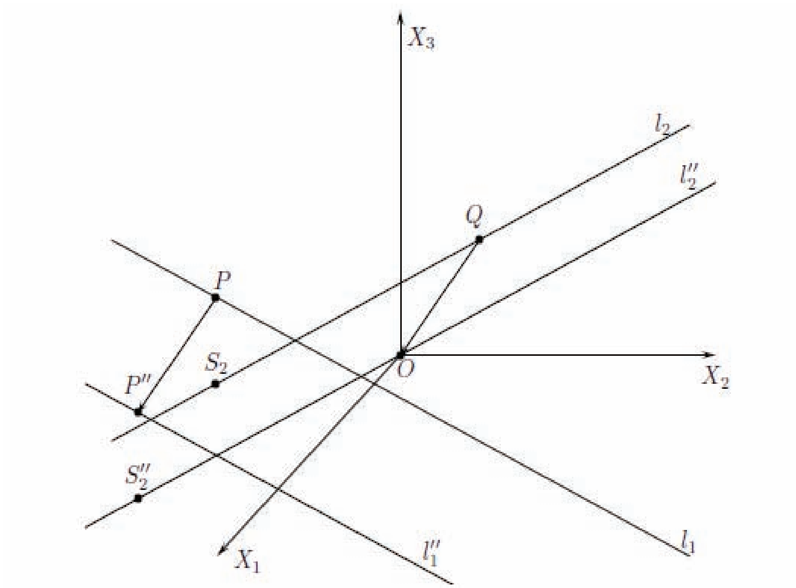

Figure 3: The pair (l_1, l_2) turns into the pair (l''_1, l''_2).

(b) Translation of line l_2 to the origin.

We do another translation of the pair (l_1, l_2), so obtaining the pair (l''_1, l''_2) where $l''_2 := \vec{v} \times \vec{x'} = \vec{0}$, $l''_1 := \vec{u} \times \left(\overrightarrow{x''} - \overrightarrow{q''}\right) = \vec{0}$, with $\overrightarrow{p''} = \vec{p} - \vec{q}$.

The current point of line l''_1 is

$P''_1(a) = (p_1 - q_1 + a\,u_1, p_2 - q_2 + a\,u_2, p_3 - q_3 + a\,u_3) := \overrightarrow{p''_1}(a), a \in □$, and, applying again Corollary 1, we obtain the foot, $S''_2(b)$, of the perpendicular onto the line l''_2 , $S''_2(a) = \overrightarrow{p''_1}(a) - \frac{\vec{v} \times \vec{r}}{\|\vec{v}\|^2}$, *with* $\vec{r} = \overrightarrow{p''_1}(a) \times \vec{v}$.

(c) Doing the reverse translations.

Turning back to the original pair (l_1, l_2), we have

1 - $S_1(b) = S'_1(b) + P$, onto the line l_1;

2 - $S_2(a) = S''_2(a) + Q$, onto the line l_2;

3 - $P(a) = P''_2(a) + Q$, onto the line l_1;

4 - $Q(b) = Q_2'(b) + P$, onto the line l_2.

The relations (1), (2), (3) and (4) form a system with six equations and two unknowns a, b. This system is consistent, by geometrical reasons. Solving this system, we get $a = a^*$ and $b = b^*$.

(d) Final step.

Once we have the needed concretization, a^* and b^*, of the parameters

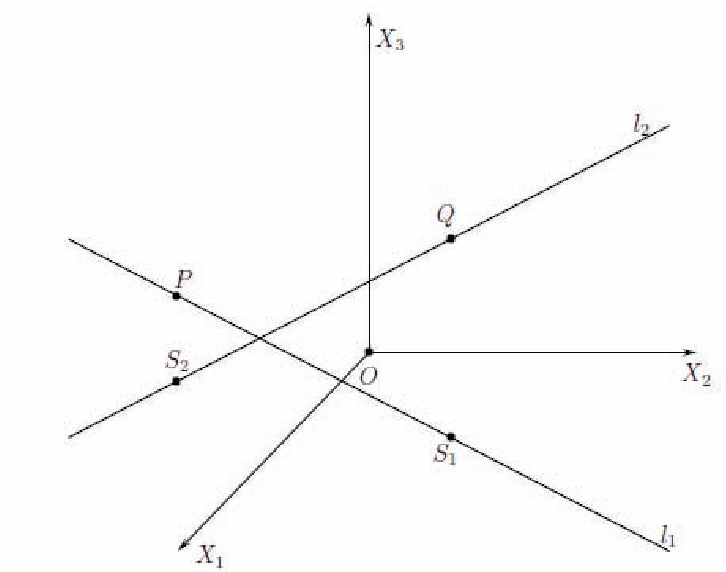

Figure 4: The best approximation pair (S_1, S_2).

a and b, we obtain the feet of the perpendicular $S_1 = S_1(b^*)$ and $S_2 = S_2(a^*)$.

Just for verification of the numerical results, we have, due to geometrical considerations, $S_1 = P(a^*)$ and $S_2 = Q(b^*)$.

2.2.1. Algorithm

Now, let us present the four steps procedure.

1) Find the foot $S_1(\mathrm{b})$ of the perpendicular drawn from the line l_2 onto the line l_1:

(i) Perform the translation associated to the vector $\overrightarrow{PO}$ of the lines l_1 and l_2, so obtaining, respectively, the lines l_1' and l_2' ;

(ii) Find the current point, $Q_2'(\mathrm{b})$, of the line l_2' ;

(iii) Obtain the foot $S_1'(\mathrm{b})$ of the perpendicular drawn from the point $Q_2'(\mathrm{b}) := \vec{q}_2'(\mathrm{b})$ onto the line l_1' , using the formula

$$S_1'(\mathrm{b}) = \vec{q}_2'(\mathrm{b}) - \frac{\vec{u}\times\vec{r}}{\|\vec{u}\|^2}, \quad \text{with } \vec{r} = \vec{q}_2'(\mathrm{b})\times\vec{u} \text{ ;}$$

(iv) Find $S_1(\mathrm{b})$ and $Q(\mathrm{b})$ doing the inverse of the translation associated to the vector $\overrightarrow{PO}$, i. e., $S_1(\mathrm{b}) = S_1'(\mathrm{b}) + P$ and $Q(\mathrm{b}) = Q_2'(\mathrm{b}) + P$.

2) Find the foot $S_2(\mathrm{a})$ of the perpendicular drawn from the line l_1 onto the line l_2:

(i) Perform the translation associated to the vector $\overrightarrow{QO}$ of the lines l_1 and l_2, so obtaining, respectively, the lines l_1'' and l_2'' ;

(ii) Find the current point, $P_1''(\mathrm{a})$, of the line l_1'';

(iii) Obtain the foot $S_2''(\mathrm{a})$ of the perpendicular drawn from the point $P_1''(\mathrm{a}) := \vec{p}_1''(\mathrm{a})$ onto the line l_2'' , using the formula $S_2''(\mathrm{a}) = \vec{p}_1''(\mathrm{a}) - \frac{\vec{v}\times\vec{r}}{\|\vec{v}\|^2}$, with $\vec{r} = \vec{p}_1''(\mathrm{a})\times\vec{v}$;

(iv)Find $S_2(\mathrm{a})$ and $P(\mathrm{a})$ doing the inverse of the translation associated to the vector $\overrightarrow{QO}$, i. e., $S_2(\mathrm{a}) = S_2''(\mathrm{a}) + Q$ and $P(\mathrm{a}) = P_1''(\mathrm{a}) + Q$.

3) Solve, for a and b , the system $\begin{cases} S_1(\mathrm{b}) = P(\mathrm{a}) \\ S_2(\mathrm{a}) = Q(\mathrm{b}) \end{cases}$.

4) Finally, find the feet S_1 and S_2 of the perpendiculars by entering the values of a and b , obtained in 3), into $S_1(\mathrm{b})$ and $S_2(\mathrm{a})$.

Next we prove an interesting geometrical criterium [4, p. 151] for the skewness of two lines in $\square^{3}$.

Proposition 2 *The lines* $l_1 := \vec{a}_1\times\vec{x} = \vec{b}_1$ *and* $l_2 := \vec{a}_2\times\vec{x} = \vec{b}_2$ *are skew if and only if* $\vec{a}_1\times\vec{a}_2 \neq \vec{0}$ *and* $\vec{a}_1\bullet\vec{b}_2 + \vec{a}_2\bullet\vec{b}_1 \neq 0$.

Proof. **(⇒)**

By hypothesis, the lines l_1 and l_2 *are* skew. So, $\vec{a_1}\times\vec{a_2}\neq\vec{0}$ and $d(l_1,l_2)\neq 0$.

Let us write the lines in the following form:

$$l_1 := \vec{x} = -\frac{\vec{a_1}\times\vec{b_1}}{\left\|\vec{a_1}\right\|^2}+\alpha_1\vec{a_1} := \vec{x} = M_1+\alpha_1\vec{a_1},\ \alpha_1\in\mathbb{R}, \qquad (1)$$

$$l_2 := \vec{x} = -\frac{\vec{a_2}\times\vec{b_2}}{\left\|\vec{a_2}\right\|^2}+\alpha_2\vec{a_2} := \vec{x} = M_2+\alpha_2\vec{a_2},\ \alpha_2\in\mathbb{R}. \qquad (2)$$

We have $d(l_1,l_2)=\frac{\left|\left(\vec{m_2}-\vec{m_1}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)\right|}{\left\|\vec{a_1}\times\vec{a_2}\right\|^2}$ [6, p. 177], where $\vec{m_1}:=M_1$ and $\vec{m_2}:=M_2$.
From $d(l_1,l_2)\neq 0$ and $\vec{a_1}\times\vec{a_2}\neq\vec{0}$, follows $\left(\vec{m_2}-\vec{m_1}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)\neq 0$. Hence,

$$\left(\vec{m_2}-\vec{m_1}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)\neq 0 \Leftrightarrow -\frac{1}{\left\|\vec{a_2}\right\|^2}\left(\vec{a_2}\times\vec{b_2}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)+\frac{1}{\left\|\vec{a_1}\right\|^2}\left(\vec{a_1}\times\vec{b_1}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)\neq 0$$

$$\Leftrightarrow \frac{1}{\left\|\vec{a_2}\right\|^2}\left(\vec{a_2}\times\vec{b_2}\right)\bullet\left(\vec{a_2}\times\vec{a_1}\right)+\frac{1}{\left\|\vec{a_1}\right\|^2}\left(\vec{a_1}\times\vec{b_1}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)\neq 0$$

$$\Leftrightarrow -\frac{1}{\left\|\vec{a_2}\right\|^2}\left(\left\|\vec{a_2}\right\|^2\left(\vec{b_2}\bullet\vec{a_1}\right)\right)+\frac{1}{\left\|\vec{a_1}\right\|^2}\left(\left\|\vec{a_1}\right\|^2\left(\vec{b_1}\bullet\vec{a_2}\right)\right)\neq 0$$

$$\Leftrightarrow \vec{a_1}\bullet\vec{b_2}+\vec{a_2}\bullet\vec{b_1}\neq 0.$$

(⇐)

By hypothesis, we have $\vec{a_1}\times\vec{a_2}\neq\vec{0}$ and $\vec{a_1}\bullet\vec{b_2}+\vec{a_2}\bullet\vec{b_1}\neq 0$. The lines l_1 and l_2 are not parallel, since to $\vec{a_1}\times\vec{a_2}\neq\vec{0}$. From the relations (1) and (2), we get $\vec{b_1}=\vec{a_1}\times\vec{m_1}$ and $\vec{b_2}=\vec{a_2}\times\vec{m_2}$.

Hence, $\vec{a_1}\bullet\vec{b_2}+\vec{a_2}\bullet\vec{b_1}\neq 0 \Leftrightarrow \vec{a_1}\bullet\left(\vec{a_2}\times\vec{m_2}\right)+\vec{a_2}\bullet\left(\vec{a_1}\times\vec{m_1}\right)\neq 0$. As

$$\vec{a_1}\bullet\left(\vec{a_2}\times\vec{m_2}\right)+\vec{a_2}\bullet\left(\vec{a_1}\times\vec{m_1}\right)\neq 0 \Leftrightarrow \vec{m_2}\bullet\left(\vec{a_1}\times\vec{a_2}\right)+\vec{m_1}\bullet\left(\vec{a_2}\times\vec{a_1}\right)\neq 0$$

$$\Leftrightarrow \vec{m_2}\bullet\left(\vec{a_1}\times\vec{a_2}\right)-\vec{m_1}\bullet\left(\vec{a_1}\times\vec{a_2}\right)\neq 0 \Leftrightarrow \left(\vec{m_2}-\vec{m_1}\right)\bullet\left(\vec{a_1}\times\vec{a_2}\right)\neq 0,$$

we have $d(l_1,l_2)\neq 0$. As a consequence, the lines l_1 and l_2 are skew.

Remark 1 In the proof of the Proposition 2 we used two tools: vector divisions [1, pp.34-35] and the identity $(\vec{r} \times \vec{u}) \bullet (\vec{v} \times \vec{w}) = \begin{vmatrix} \vec{r} \bullet \vec{v} & \vec{u} \bullet \vec{v} \\ \vec{r} \bullet \vec{w} & \vec{u} \bullet \vec{w} \end{vmatrix}$ [1, p.41].

3. Rotations and reflections

Mathematically, a **rotation** is a rigid body movement which, unlike a translation, keeps a point fixed. This definition applies to rotations within both two and three dimensions (in a plane and in space, respectively). A 2-dimensional object rotates around a center (or point) of rotation. A rotation in 3-dimensional space keeps an entire line fixed, i.e. a rotation in 3-dimensional space is a rotation around an axis. A **reflection** is a map that transforms an object into its mirror image.

3.1. Rotation and reflection in 2-dimensional space

First of all, we will treat the problem of a 2-dimensional rotation and reflection.

Usually, the effect of a rotation can be obtained using a rotation matrix. For an angle 2-dimensional positive rotation (the positive side of the axis moves towards the positive side of the axis), the generic point is transformed into P'=(x',y'), such that [6, p.277]

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

A reflection relatively to the line of equation y=x tan θ, transforms the point P=(x,y) into P'=(x',y'), such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The above reflection matrix can be obtained using the conjugacy principle. In this case, we consider a transformation that is given by a rotation in 2-dimensional space R_α, defined by

$$R_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$P \longrightarrow R_\alpha P$$

where $R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, and, of course, we have (R_α)^(-1)=R_((-α)).

So, first we do a rotation R_((-θ)) with angle –θ, then reflecting about the OX axis and, finally, rotating R_θ with angle θ. So, the above reflection matrix can be given by the following product of matrices [6, p.293]

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

3.2. Rotation in 3-dimensional space

A rotation in the 3-dimensional space (around an axis) can be described by a (real and orthogonal) rotation matrix

$$A = \begin{bmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{bmatrix},$$

where the unit vectors $\hat{u}$, $\hat{v}$ and $\hat{w}$ form the basis of the new (rotated) system of axis. In particular, when the axis of rotation are the coordinate axis, we have the three basic rotation matrices (respectively around the OX, OY and OZ axis):

$$\begin{matrix} 0 & 0 \\ \text{os}\,\alpha & -\sin\alpha \\ \text{in}\,\alpha & \cos\alpha \end{bmatrix}, R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$, and

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The direction of the rotation is determined by the right-hand rule: R_X rotates the *y*-axis towards the *z*-axis, R_Y rotates the *z*-axis towards the *x*-axis, and R_Z rotates the *x*-axis towards the *y*-axis.

It can be proven that any general rotation [2, pp. 64-66] around any axis can be obtained by three consecutive elementary rotations around the three coordinate axis.

3.3. Rotation of an object around a line in 3-dimensional space using the conjugacy principle

Let us consider a 3-dimensional orthogonal and direct referential OXYZ with origin O and basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$. We pretend to rotate, by an angle *Y*, an object *A* around a line, $l := S = Q + \alpha\vec{u}$, $\alpha \in \mathbb{R}$, which is defined by a point Q=(x_Q,y_Q,z_Q) and a direction given by the unit vector $\vec{u}$=(u_1,u_2,u_3), $(\|\vec{u}\| = 1)$.

The process we use does appeal the conjugacy principle in the following way. First we construct a new appropriated referential associated with the line and write the line and the object in the new referential, then we perform an elementary rotation of the object and, finally, reverse the previous transformation, writing the rotated object in the initial system of axis.

In this case, for using the conjugacy principle, we consider the following transformation, consisting in a change of basis of the 3-dimensional coordinate system (described by a rotation matrix) and a change of origin of coordinates (defined by a translation vector):

$$C_{M,Q}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$P \longrightarrow M^T(P - Q)$$

,

where $\left(C_{M,Q}\right)^{-1}(P) = MP + Q$.

Without the loss of generality, we consider that the origin of the new (orthogonal and also direct) referential is located at the point *Q* of the line *L* and the new z-axis has the direction of the vector $\vec{u}$. So, if *(X,Y,Z)* are the coordinates of any point of the space in the initial referential, the correspondent ones *(X´,Y´,Z´)* in the new referential *QX´Y´Z´* are given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - M^{-1} \begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix}, \quad (3)$$

with the orthogonal matrix (because is the matrix of transformation between two direct and orthogonal systems of axis: $M^{-1} = M^T$) defined, for example, by

$$M = \begin{bmatrix} v_1 & w_1 & u_1 \\ v_2 & w_2 & u_2 \\ v_3 & w_3 & u_3 \end{bmatrix},$$

with, without the loss of generality(1), $\vec{v} = (v_1, v_2, v_3) = \frac{\vec{u}\times\vec{e_1}}{\|\vec{u}\times\vec{e_1}\|}$, where X stands for the usual cross product in $\mathbb{R}^3$, and $\vec{w} = (w_1, w_2, w_3) = \frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|} = \frac{\vec{u}\times(\vec{u}\times\vec{e_1})}{\|\vec{u}\times(\vec{u}\times\vec{e_1})\|}$.

Obviously, in QX´Y´Z´, the line has equation $l' := S' = Q' + \alpha\overrightarrow{u'} = \alpha\overrightarrow{u'} = \alpha\overrightarrow{e_3}$, $\alpha \in \mathbb{R}$, with $\overrightarrow{u'} = M^T\vec{u}$.

Now, let $P = (x_P, y_P, z_P)$ be a generic point of the object A with coordinates X_P, Y_P, Z_P, with respect to $OXYZ$ and $P' = (x'_P, y'_P, z'_P)$ with coordinates x'_P, y'_P, z'_P with respect to $QX'Y'Z'$, given by equation (3), [P and P' represent the same point in two different referentials]. The result of the rotation of P around the line L is a point $P_\gamma = (x_{P\gamma}, y_{P\gamma}, z_{P\gamma})$ whose coordinates in $OXYZ$ we are looking for. In $QX'Y'Z'$, the same rotated point $P'_\gamma = (x'_{P\gamma}, y'_{P\gamma}, z'_{P\gamma})$ is given by the elementary rotation about the QZ' axis:

$$\begin{bmatrix} x'_{P\gamma} \\ y'_{P\gamma} \\ z'_{P\gamma} \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'_P \\ y'_P \\ z'_P \end{bmatrix}.$$

The legitimacy of the above rotation is due to the fact that is an invariant (its sign and amplitude do not depend on the chosen referential).

Finally, we perform the reverse transformation, in order to obtain the coordinates of P_γ in $OXYZ$. Those are given by applying the inverse relation of the initially used:

$$\begin{bmatrix} x_{P\gamma} \\ y_{P\gamma} \\ z_{P\gamma} \end{bmatrix} = M \begin{bmatrix} x'_{P\gamma} \\ y'_{P\gamma} \\ z'_{P\gamma} \end{bmatrix} + \begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix}.$$

Naturally, the previous process can be applied to all the defining points of the object A, in order to obtain its transformation A_γ by the pretended rotation.

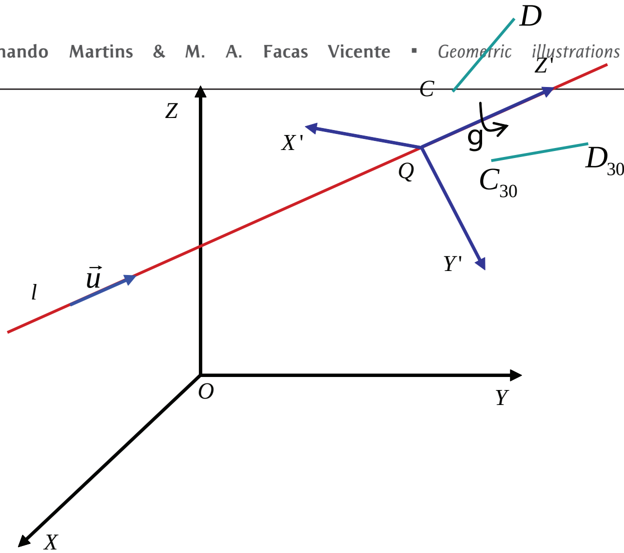

Figure 5: The effect of a rotation on a line segment.

More details of the change of coordinate system can be found in [2, pp. 59-64].

3.4. Example

We illustrate the described method by the following example. Let us consider a line $l := S = (1,2,3) + \alpha\left(\frac{2}{3}\overrightarrow{e_1} - \frac{2}{3}\overrightarrow{e_2} + \frac{1}{3}\overrightarrow{e_3}\right), \alpha \in \mathbb{R}$, and a line segment *[CD]* defined by $C = (5,-4,3)$ and $D = (-3,4,5)$ of length $\overline{CD} = \sqrt{132}$. We want to obtain the image $[C_{30}D_{30}]$ of $[CD]$ by a positive rotation of amplitude *30* degrees around the line l. The new system of axis is obtained by a translation of the origin defined by the vector $Q = \overrightarrow{e_1} + 2\overrightarrow{e_2} + 3\overrightarrow{e_3}$ and by a rotation matrix *M* which third column is the unit vector $\vec{u} = \frac{2}{3}\overrightarrow{e_1} - \frac{2}{3}\overrightarrow{e_2} + \frac{1}{3}\overrightarrow{e_3}$, the first column is $\frac{\vec{u}\times\overrightarrow{e_1}}{\|\vec{u}\times\overrightarrow{e_1}\|} = \frac{1}{\sqrt{5}}\overrightarrow{e_2} + \frac{2}{\sqrt{5}}\overrightarrow{e_3}$ and the second column is $\frac{\vec{u}\times(\vec{u}\times\overrightarrow{e_1})}{\|\vec{u}\times(\vec{u}\times\overrightarrow{e_1})\|} = -\frac{5}{3\sqrt{5}}\overrightarrow{e_1} - \frac{4}{3\sqrt{5}}\overrightarrow{e_2} + \frac{2}{3\sqrt{5}}\overrightarrow{e_3}$. In $QX'Y'Z'$, the points *C* and *D* have coordinates C' and D', given by

$$C' = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{5}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} -\frac{6}{\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{20}{3} \end{bmatrix}$$

and, by replacing the vector $\begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}$ with $\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$, $D' = \begin{bmatrix} \frac{6}{\sqrt{5}} \\ \frac{16}{3\sqrt{5}} \\ -\frac{10}{3} \end{bmatrix}$.

In the new referential, the line l is naturally defined by $l' := S' = \alpha\overrightarrow{e_3},\ \alpha \in \mathbb{R}$. Now, we perform the positive rotation of amplitude *30* degrees around the line $l \equiv l' \equiv QZ'$, defined by the matrix

$$R_{z'}(30) = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and we obtain the points $C'_{30} = \left(-\frac{3\sqrt{3}}{\sqrt{5}} - \frac{2}{3\sqrt{5}}, \frac{2\sqrt{3}}{3\sqrt{5}} - \frac{3}{\sqrt{5}}, \frac{20}{3}\right)$ and $D'_{30} = \left(\frac{3\sqrt{3}}{\sqrt{5}} - \frac{8}{3\sqrt{5}}, \frac{8\sqrt{3}}{3\sqrt{5}} + \frac{3}{\sqrt{5}}, -\frac{10}{3}\right)$. Finally, we perform the reverse change of system of coordinates, transforming C'_{30} and D'_{30} in C_{30} and D_{30}, respectively. So, we obtain

$$C_{30} = \begin{bmatrix} 0 & -\frac{5}{3\sqrt{5}} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{3\sqrt{3}}{\sqrt{5}} - \frac{2}{3\sqrt{5}} \\ \frac{2\sqrt{3}}{3\sqrt{5}} - \frac{3}{\sqrt{5}} \\ \frac{20}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{-2\sqrt{3}+58}{9} \\ \frac{-7\sqrt{3}-16}{9} \\ \frac{-10\sqrt{3}+41}{9} \end{bmatrix}$$

and, substituting, in the above relation, $\begin{bmatrix} -\frac{3\sqrt{3}}{\sqrt{5}} - \frac{2}{3\sqrt{5}} \\ \frac{2\sqrt{3}}{3\sqrt{5}} - \frac{3}{\sqrt{5}} \\ \frac{20}{3} \end{bmatrix}$ by $\begin{bmatrix} \frac{3\sqrt{3}}{\sqrt{5}} - \frac{8}{3\sqrt{5}} \\ \frac{8\sqrt{3}}{3\sqrt{5}} + \frac{3}{\sqrt{5}} \\ -\frac{10}{3} \end{bmatrix}$, we get

$$D_{30} = \begin{bmatrix} \frac{-8\sqrt{3}-20}{9} \\ \frac{-\sqrt{3}+26}{9} \\ \frac{14\sqrt{3}+11}{9} \end{bmatrix}.$$

The image by the rotation around the line of the line segment $[CD]$ is $[C_{30}D_{30}]$ with, just for control, length $\overline{C_{30}D_{30}} = \sqrt{132}$, as expected.

4. Final remarks and conclusions

By invoking the conjugacy principle when dealing with geometric transformations, we are able to solve problems of some complexity. Solving problems in the way presented in this paper, spatial abilities are acquired and developed. Such abilities are very important and fundamental towards the perception and understanding of day to day phenomena. In this manner, we strengthen both the mathematical reasoning and the geometrical thinking, so fundamental to the art of problem solving.

In this paper, we showed the application of the conjugacy principle in several problems in geometry. There are also some interesting cases which were not tackled, such as the reflection in a plane in the ordinary space. Another interesting approach would be to establish some relation between the elementary rotations and the rotations through the Euler angles using the conjugacy principle. We hope that these problems might constitute a challenging problem for the reader.

Bibliography

Alves, A. S. (1988). *Mecânica geral*. Lisboa: Instituto Nacional de Investigação Científica.

Alves, A. S. (1996). *Metrologia geométrica*. Lisboa: Fundação Calouste Gulbenkian.

Deutsch, F. (2001). *Best approximation in inner product spaces*. New York: Springer.

Fenn, R. (2001). *Geometry*. London: Springer.

Melzak, Z. A. (2007). *Companion to concrete mathematics – two volumes bound as one*. New York: Dover Publications [Vol. I published in 1973 and Vol. II published in 1976 by J. Wiley].

Vitória, J. & Lima, T. P. (1998). *Álgebra linear*. Lisboa: Universidade Aberta.

Notas

1) If is collinear with , i.e. or , then change for .

Correspondence

Fernando Martins
Escola Superior de Educação de Coimbra
Praça Heróis do Ultramar, Solum,
3030-329 Coimbra, Portugal.
Project supported by Instituto de Telecomunicações, Pólo de Coimbra, Delegação da Covilhã, Portugal.
fmlmartins@esec.pt

M. A. Facas Vicente
Department of Mathematics - University of Coimbra
Apartado 3008, 3001-454
Coimbra, Portugal.
Project supported by INESC-C---Instituto de Engenharia de Sistemas e Computadores–Coimbra, Rua Antero de Quental, 199, 3000-033 Coimbra, Portugal.
vicente@mat.uc.pt