# A differential redistributive analysis of bilinear tax reforms

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**Abstract**

We analyze differential redistributive effects, voting preferences and revenue elasticities of bilinear tax reforms that are applied to dual taxes or, more generally, to two different one-dimensional taxes. We prove that a partial order -which induces a lattice- based on the Lorenz dominance criterion can be established if certain conditions on the tax reform policy and the income distribution hold. We illustrate empirically our theoretical results in the case of the Spanish dual Personal Income Tax.

*Keywords:* Dual taxes, bilinear tax reforms, Lorenz domination, lattice

*JEL Classification:* E60, E62

## 1 Introduction

In the current situation of global economic crisis, there is agreement that fiscal policies have a prominent role to play. Strategies related to public spending, targeted transfers and tax reforms have been considered or adopted by several governments. Some countries, such as US and Japan have committed their policies

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to reduce taxation to stimulate the aggregated demand. Other countries have increased taxes to avoid increases in budget deficits. Examples are VAT and taxation on capital income in the cases of Germany, UK and Spain, respectively. Changes in fiscal policies (and in what direction) are giving rise to heated discussion not only in the political agenda but also in the society at large.

If policy makers primary concern is to reduce (or increase) tax liability, Pfähler's (1984) analysis of 'linear' tax reforms [3] would be very useful for its simplicity and normative results. He considered three neutral-revenue tax cuts defined respectively as a fraction of tax liability, as a fraction of post-tax income or as a fraction of pre-tax income. According to his results, tax cuts (resp. tax hikes) defined as a fraction of post-tax income (resp. tax liability) are the most redistributive policy, in the sense that they shift the tax burden from low to high income recipients more than the other two reforms. Whenever the income distribution is positively skewed, this latter reform is the most welfare improving. Furthermore, this option would be chosen in a majority vote process for income distributions with more 'poor' than 'rich' tax-payers. Formby et al. (1992) stress the importance of analyzing the distributional effects on income distribution of the three above linear tax reforms and the political implications they had in the US during the eighties.

Pfähler's analysis deals with the case of only one tax function, specifically, a comprehensive income tax. Nevertheless, this approach is restrictive insofar as tax cuts or hikes are applied to dual taxation schemes [4] or, more generally, to different taxes simultaneously.

In the case of the dual income tax, changes in tax burden could be split between labor and capital, not only because policy makers decide to distribute the effects of the reforms between the different income sources but also to prevent tax payers to discriminate between tax bases (tax-base shifting). The possibility of tax-base shifting or altering the status of the corporate business is also the reason for which changes in personal income and corporate taxation should rather be treated together.

In general, there is little research, if any, to be found concerning tax linear reforms applied either to dual taxes or to different taxes with independent tax bases. Ebert and Lambert (1999) analyzed progressivity and income redistribution issues of a combined tax (personal income tax and social security contributions), but in this case each tax function relied on the same tax base.

The main objective of this paper is to examine differential distributional effects of bilinear tax reforms applied to dual income taxes or, more generally, to two

[3] A tax reform is named 'linear' if the post-reform average tax rate is obtained as a linear transformation of the initial average tax rate.

[4] Dual income taxation has become relevant in Europe. It was implemented in the Nordic countries during the 1990 decade (see Sørensen, 1998). Other European countries have been partially or totally adopted dual income taxes, such as The Netherlands, Germany, Austria, Belgium, Italy or Spain.

different one-dimensional taxes. We do not investigate what conditions on the tax schedules are needed to reduce income inequality but we compare different tax reforms on the basis of their redistributive effects [5]. Our analysis can be easily extended to linear reforms of more than two one-dimensional taxes.

We show that a lattice based on the Lorenz dominance criterion can be established for dual taxes if certain conditions on the tax reform policy and income distribution hold. We also prove that the lattice structure is preserved when the criterion to compare tax reforms is based on revenue elasticity instead of income redistribution. It is also shown that, under an additional mild assumption, the set of tax-payers can be partitioned into two connected subsets ('rich' and 'poor') according to their voting preferences over the set of bilinear tax reforms.

Despite the lattice framework focusing on distributional issues, it by no means neglects efficiency aspects of the tax design. Whatever the policy adopted, the lattice structure offers a range of bilinear tax reforms which could have different consequences, not only on income redistribution but also on the economic incentives of tax-payers. These factors should be considered by policymakers when determining the tax rate structures of each policy. Therefore, the lattice setting can be considered as a benchmark to guide reforms.

The paper is organized as follows. Section 2 describes Pfähler's specific linear tax cuts. We identify a whole class of linear tax reforms based on progressive taxes. Section 3 introduces bilinear tax reforms applied to dual income taxes. The analysis of linear tax reforms combining two different one-dimensional taxes is, mathematically speaking, equivalent to the dual tax framework. In Section 4 we analyze distributional effects of bilinear tax reforms and how they can be partially ordered in a lattice. Section 5 discusses the robustness of the lattice setting by relaxing some initial assumptions. In Section 6 we use a micro-simulation model to illustrate the differential incidence analysis of bilinear tax cuts. Section 7 contains a summary and a concluding discussion.

## 2 The one-dimensional framework

As we already mentioned, Pfähler (1984) proposes three different (one-dimensional) tax reforms (either cuts or hikes) of a given progressive tax schedule $T(x)$, to be defined respectively either as fraction $a$ of the tax liability $T(x)$, as a fraction $b$ of the post-tax income $V(x) = x - T(x)$ or as a fraction $c$ of the pre-tax income $x$. Each one of these reforms is neutral with respect to a local measure of tax progres-

[5] A number of papers have studied the relationship between progressive taxes and income inequality. Except for the case of a one-dimensional tax and homogeneous tax payers (Jakobsson 1976, Kakwani, 1976), what we have learned is the impossibility in general of establishing conditions (on tax schedules or/and income) that guarantee income-inequality reduction, whatever the pre-tax income distribution is (see for example Moyes and Shorrock (1998)).

sivity: *liability progression* $\alpha(x) = \frac{dT(x)}{dx}\frac{x}{T(x)}$, *residual progression* $\psi(x) = \frac{dV(x)}{dx}\frac{x}{V(x)}$ and *average rate* $\beta(x) = \frac{d}{dx}\left(\frac{T(x)}{x}\right)$ *progression.* Let $T_1(x)$, $T_2(x)$ and $T_3(x)$ be the three reforms:

$$
\begin{aligned}
T_1(x) &= T(x) - aT(x) \iff && t_1(x) = (1-a)t(x)\\
T_2(x) &= T(x) - bV(x) \iff && t_2(x) = (1+b)t(x) - b\\
T_3(x) &= T(x) - cx \iff && t_3(x) = t(x) - c,
\end{aligned}
$$

where $t(x) = T(x)/x$ denotes the average tax rate. The three above transformations of the original tax are yield-equivalent if and only if $b = ag/(1-g)$ and $c = ag$, where $g = \frac{\overline{T}}{\overline{x}}$ is the quotient between the average tax liability $\overline{T}$ and the average pre-tax income $\overline{x}$. According to Pfähler, post-tax income distributions of $T_1(\cdot)$, $T_2(\cdot)$ and $T_3(\cdot)$ can always be ordered using the Lorenz Dominance (LD) criterion, being $T_2(\cdot)$ the most progressive tax cut and $T_1(\cdot)$ the least progressive tax cut. Formally, it is shown that $L_{V_1} \prec_L L_{V_3} \prec_L L_{V_2}$ and $L_{T_1} \succ_L L_{T_3} \succ_L L_{T_2}$, where $\prec_L$denotes the Lorenz Dominance criterion and $L_V$ (resp. $L_T$) denote the Lorenz curve of post-tax income (tax liability) distribution. For tax hikes, the order is reversed. Pfähler's result relies, in the way he proves it, on the well-known relationship between LD and progressivity of a progressive tax schedule[6] .

We next show that Pfähler's differential income redistribution analysis of $T_1(\cdot)$, $T_2(\cdot)$ and $T_3(\cdot)$ can be easily extended to the set of all yield-equivalent linear reforms of a given one-dimensional tax schedule $T(x)$ without making use of the former relationship between LD and progressivity. This can be done since in fact each yield-equivalent linear tax cut depends on a single parameter, either the slope or the constant term that define it (see Proposition 1 below).

A *linear tax reform* of $T(x)$ is defined by $\rho T(x) + \sigma x$ or, equivalently, by $\rho t(x) + \sigma$. Note that parameters $(\rho, \sigma)$ are equal to $(1-a, 0)$, $(1+b, -b)$ and $(1, -c)$ for the tax reforms $T_1(\cdot)$, $T_2(\cdot)$ and $T_3(\cdot)$ respectively.

It is worth noting the following remark. When the derivative of the tax schedule $T(x)$ is a sum of step functions defined by marginal tax rates $t_k$ and income thresholds $m_k$, it can be shown (see for instance Calonge and Tejada, 2009) that applying a linear tax reform $\rho t(x) + \sigma$ on the average tax rate $t(x)$ is equivalent to transforming the marginal tax rates $t_k$ according to the linear function that defines the tax reform, that is $\rho t_k + \sigma$. This result is used throughout the paper.

Let $\Delta R_{\widehat{T}}$ denote the aggregate (either negative or positive) amount of income that tax-payers get due to reform $\widehat{T}(x)$, i.e. $\Delta R_{\widehat{T}} > 0$ if $\widehat{T}(x)$ is a tax cut and $\Delta R_{\widehat{T}} < 0$ if $\widehat{T}(x)$ is a tax increase. We say that two tax schedules $\widehat{T}(x)$ and $\widetilde{T}(x)$ are *yield-equivalent* if $\Delta R_{\widehat{T}} = \Delta R_{\widetilde{T}} = \Delta R_T$.

[6] If the residual progression of a particular tax schedule $A$ is pointwise weakly larger than the residual progression of a tax schedule $B$, then the post-tax income distribution of $B$ is Lorenz dominated by post-tax income distribution of $A$ (Jakobbson, 1976). An analogous result is obtained considering liability progression and tax liability distributions.

The following proposition extends Pfähler's redistributional analysis of the above three specific linear reforms to the set of all linear reforms.

**Proposition 1** *Let $\{x_1 \leq ... \leq x_n\}$ be a discrete pre-tax income distribution and $T(\cdot)$ a progressive*[7] *tax schedule. Let $\widehat{T}(x) = \widehat{\rho}T(x) + \widehat{\sigma}x$ and $\widetilde{T}(x) = \widetilde{\rho}T(x) + \widetilde{\sigma}x$ be two different yield-equivalent linear tax reforms of $T(\cdot)$. Then, the following **statements** are equivalent:*

*(1) $\widehat{\rho} > \widetilde{\rho}$.*
*(2) $\widehat{\sigma} < \widetilde{\sigma}$.*
*(3) $L_{\widehat{V}} \succ_L L_{\widetilde{V}}$.*
*(4) $L_{\widetilde{T}} \succ_L L_{\widehat{T}}$.*
*(5) $\psi_{\widehat{T}}(x_i) \geq \psi_{\widetilde{T}}(x_i)$ for all tax-payer $i$, with at least one strict inequality.*
*(6) $\alpha_{\widehat{T}}(x_i) \leq \alpha_{\widetilde{T}}(x_i)$ for all tax-payer $i$, with at least one strict inequality.*

**Proof.** By means of simple algebra, it can be checked that the yield-equivalent hypothesis is equivalent to

$$g - \frac{\Delta R_T}{n\overline{x}} = \widehat{\rho}g + \widehat{\sigma} = \widetilde{\rho}g + \widetilde{\sigma}, \tag{1}$$

where $g$ is the total tax ratio before reform and $g - \frac{\Delta R_T}{n\overline{x}}$ is the total tax ratio after reform. This equation proves the equivalence between Statements 1 and 2.

Next, we prove the equivalence between Statements 1 and 4. Since $T(x)$ is progressive, i.e. $t(x)$ is a non-decreasing function,

$$T(x_i) - gx_i \lesseqgtr 0 \text{ if } x \lesseqgtr x_g := t^{-1}(g). \tag{2}$$

By definition of $g$ we have $\sum_{i=1}^{n}(T(x_i) - gx_i) = 0$, which, together with (2), implies that

$$\sum_{i=1}^{p}(T(x_i) - gx_i) \leq 0 \text{ for all } p \in \{0, ..., n\}. \tag{3}$$

By the yield-equivalent condition, Statement 4 holds if and only if $\sum_{i=1}^{p}\widetilde{T}(x_i) \geq \sum_{i=1}^{p}\widehat{T}(x_i)$ for all $p = 0, ..., n$ with at least one strict inequality. This latter expression is equivalent, using (1), to $(\widetilde{\rho} - \widehat{\rho})\sum_{i=1}^{p}(T(x_i) - gx_i) \geq 0$. Hence, by (3), we have that $L_{\widetilde{T}} \succ_L L_{\widehat{T}}$ if and only if $\widetilde{\rho} - \widehat{\rho} < 0$.

The equivalence between Statements 3 and 4 is straightforward thanks to the yield-equivalent condition.

Finally, the equivalence between Statements 3 and 5, as well as Statements 4 and 6, respectively follows from Proposition 1 in Jakobbson (1976), where the discrete definitions of $\alpha(x)$ and $\psi(x)$ are used[8]. ■

[7] That is, $\frac{d}{dx}(\frac{T(x)}{x}) \geq 0$.
[8] one-dimensional measures of progressivity were introduced by Musgrave and Thin (1948) as the limit of discrete tax elasticities. Given a discrete income distribution $\widetilde{x} = \{x_0, x_1, ..., x_i, x_{i+1}, ..., x_m\}$, when $x_i \approx x_{i+1}$ both discrete and continuous defini-

Observe that a 'linear' transformation $\widehat{T}(x) = \rho T(x) + \sigma x$ is in fact an 'average' between $T(x)$ and $x$. Since $T(x)$ is more 'progressive' than $x$, then raising $\rho$, which under the yield-equivalent condition induces a decrease in $\sigma$, implies simply that $\widehat{T}(x)$ is more 'progressive' than before, and hence it is by no means surprising that Proposition 1 holds.

We want to stress that progressivity of $T(x)$ is crucial to Proposition 1 and that equivalence among Statements 1, 2, 3 and 4 is proved independently to their equivalence to Statements 5 and 6. Finally, if, after the tax cut, we do not want any tax-payer to pay a negative amount or to pay more than before, we need to impose bounds on the possible slopes that define the linear reforms. Indeed, by means of simple algebra it can be checked that, considering the conditions of the above proposition, no tax-payer pays more in $\widehat{T}(x) = \rho T(x) + \sigma x$ than in $T(x)$ if and only if $\rho \geq 1 - \Delta R / \left[ n\overline{x} \left( t(x_n) - g \right) \right]$ and no tax-payer pays a negative amount in $\widehat{T}(x)$ if and only if $\rho \leq \left[ g + \Delta R / n\overline{x} \right] / \left[ g - t(x_1) \right]$, where $\Delta R > 0$ is the aggregate tax cut. Analogous expressions can be found in the case of tax increases.

## 3 The dual tax function

In what follows let $x \geq 0$ be the taxable labor income before tax and $y \geq 0$ the taxable capital income before tax[9]. Then, a *dual tax schedule* is defined by

$$T(x, y) = L(x) + K(y), \tag{4}$$

where $L(\cdot), K(\cdot)$ are one-dimensional tax schedules [10]. When both $L(\cdot), K(\cdot)$ are progressive, we say that $T(x, y)$ is *pairwise-progressive.* It is important to point out that a pairwise-progressive dual tax schedule need not be progressive on the total income nor vice versa[11].

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tions coincide. This could be the case for real income distributions, except for the upper tail of the income distribution. Further, for these specific tax-payers, the tax schedule is almost flat and both discrete and continuous measures also coincide.

[9] In our theoretical analysis we shall not consider the role of allowances. This approach is the same that Pfähler implicitly followed. When it comes to real tax functions, allowances can be roughly approximated by linear transformations of the pre-tax income (see for instance Fries et al, 1982).

[10] Ebert and Lambert (1999) consider an specific case of the tax function $T(x, y)$ where $x = y$, i.e. the combined tax components are based on the same income ($x$ is the personal income tax and $y$ is the national insurance contribution).

[11] In the case of dual taxes a fundamental problem arises when we try to link progressivity with redistribution since there is not a notion of tax-progressivity analogous to the one-dimensional case. The drawbacks of all possible approaches are the same as those arising when we try to extend the complete order structure of $\mathbb{R}$ to $\mathbb{R}^2$.

A *bilinear tax reform* $\widehat{T}(x,y)$ applied to a given dual tax schedule $T(x,y)$ is

$$\widehat{T}(x,y)=\overbrace{\rho_L L(x)+\sigma_L x}^{\widehat{L}(x)}+\overbrace{\rho_K K(y)+\sigma_K y}^{\widehat{K}(y)}\,. \tag{5}$$

The specific bilinear tax reforms where $\widehat{L}(x)$ is a Pfähler-type reform of type $i$ applied to $L(x)$ and $\widehat{K}(y)$ is a Pfähler-type reform of type $j$ applied to $K(y)$ shall be denoted respectively by $T_{i,j}(x,y)$. As an example, the parameters of $T_{1,2}(x,y)$ are $\rho_L=1-a_L$, $\sigma_L=0$, $\rho_K=1+b_K$ and $\sigma_K=-b_K$.

We want to stress that $\widehat{L}(x)$ and $\widehat{K}(y)$ may be respectively either a tax cut or a tax hike of $L(x)$ and a tax cut or a tax hike of $K(y)$, thus giving rise to four different scenarios: *labor tax cut/capital tax cut*, *labor tax hike/capital tax cut*, *labor tax cut/capital tax hike* and *labor tax hike/capital tax hike*. Let $\Delta R_{\widehat{T}}:=\Delta R_{\widehat{L}}+\Delta R_{\widehat{K}}$, which may be either negative or positive, where $\Delta R_{\widehat{L}}$ is the aggregate (either negative or positive) amount of labor income that tax-payers get due to reform $\widehat{T}(x,y)$, and $\Delta R_{\widehat{K}}$ is the aggregate (either negative or positive) amount of capital income that tax-payers get due to reform $\widehat{T}(x,y)$.

We consider a finite set of tax-payers with pre-tax labor incomes and pre-tax capital incomes given by distributions $\widetilde{x}=(x_1,...,x_n)$ and $\widetilde{y}=(y_1,...,y_n)$ respectively, where $(x_i,y_i)$ are the incomes of the tax-payer $i$. Let $\overline{x}$ be the *average pre-tax labor income*, $\overline{y}$ the *average pre-tax capital income*, $\overline{L}$ be the *average labor-tax rate* and $\overline{K}$ the *average capital-tax rate*. We also introduce the following initial rates, $g_L=\frac{\overline{L}}{\overline{x}}$ and $g_K=\frac{\overline{K}}{\overline{y}}$. Notice that all the above are ex-ante parameters.

By means of simple algebra it can be proved that two different bilinear tax reforms $\widehat{T}(x,y)=\widehat{\rho}_L L(x)+\widehat{\sigma}_L x+\widehat{\rho}_K K(y)+\widehat{\sigma}_K y$ and $\widetilde{T}(x,y)=\widetilde{\rho}_L L(x)+\widetilde{\sigma}_L x+\widetilde{\rho}_K K(y)+\widetilde{\sigma}_K y$ are yield-equivalent, i.e. $\Delta R_{\widehat{T}}=\Delta R_{\widetilde{T}}=\Delta R_T$, if and only if

$$\begin{aligned}&\delta g_L+(1-\delta)\,g_K-\frac{\Delta R_T}{n\left(\overline{x}+\overline{y}\right)}\\ =&\,\delta\cdot\left[\widehat{\rho}_L g_L+\widehat{\sigma}_L\right]+(1-\delta)\cdot\left[\widehat{\rho}_K g_K+\widehat{\sigma}_K\right]\\ =&\,\delta\cdot\left[\widetilde{\rho}_L g_L+\widetilde{\sigma}_L\right]+(1-\delta)\cdot\left[\widetilde{\rho}_K g_K+\widetilde{\sigma}_K\right],\end{aligned} \tag{6}$$

where $\delta=\overline{x}/\left(\overline{x}+\overline{y}\right)$. The above expression is the bidimensional counterpart of expression (1) and tells that the total tax ratio after a bilinear reform $\widehat{T}(x,y)$ is obtained transforming the corresponding labor and capital total tax ratios according to $\widehat{T}(x,y)$.

## 4 Differential redistributional analysis of bilinear tax reforms

### *4.1 Global effects*

In the first part of this section we carry out, using the Lorenz Domination criterion, a differential distributional analysis to the set of bilinear reforms applied to a pairwise-progressive dual tax schedule.

We say that labor and capital income distributions are *perfectly aligned* when labor and capital income distributions satisfy that, for any two tax-payers $i \neq j$, $x_i \geq x_j$ if and only if $y_i \geq y_j$. If we want to establish comparisons between two different bilinear tax reforms we need to impose the following two conditions:

- **(Condition 1)** Both bilinear tax reforms are pairwise yield-equivalent.
- **(Condition 2)** Labor and capital income distributions are perfectly aligned.

Observe that Condition 1 applies to the policy design $\widehat{T}(x,y) = \rho_L L(x) + \sigma_L x + \rho_L K(y) + \sigma_K y$, whereas Condition 2 applies to income distributions and it is an ex-ante condition. By means of an example, below in the paper we discuss the consequences of the failure of the above conditions. Since it shall be used in the upcoming sections, we explicitly write Condition 1, which is composed of the following two equations:

$$\rho_L g_L + \sigma_L = g_L - \frac{\Delta R_L}{n\overline{x}}, \tag{7}$$

and

$$\rho_K g_K + \sigma_K = g_K - \frac{\Delta R_K}{n\overline{y}}. \tag{8}$$

Before proving the main results of this section, consider the following fundamental technical lemma, which is proved in the Appendix.

**Lemma 1** *Let $\widetilde{x^1} = (x_1^1 \leq ... \leq x_n^1)$, $\widetilde{x^2} = (x_1^2 \leq ... \leq x_n^2)$, $\widetilde{y^1} = (y_1^1 \leq ... \leq y_n^1)$ and $\widetilde{y^2} = (y_1^2 \leq ... \leq y_n^2)$ be income distributions so that $\widetilde{x^1} \succ_L \widetilde{x^2}$ and $\widetilde{y^1} \succ_L \widetilde{y^2}$. If $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^1$ and $\sum_{i=1}^n y_i^2 = \sum_{i=1}^n y_i^1$, then $\widetilde{x^1 + y^1} \succ_L \widetilde{x^2 + y^2}$.*

Let $\overrightarrow{\alpha_T}(x,y) := (\alpha^L(x), \alpha^K(y))$, $\overrightarrow{\psi_T}(x,y) := (\psi_L(x), \psi_K(y))$ and $\overrightarrow{\beta_T}(x,y) := (\beta_L(x), \beta_K(y))$. Let also $\geq$ be the ordinary partial order on $\mathbb{R}^2$: $(x_1, y_1) \geq (x_2, y_2)$ if $x_1 \geq x_2$ and $y_1 \geq y_2$. Next we prove the bidimensional counterpart of Proposition 1.

**Proposition 2** *Let $\widetilde{x} = (x_1 \leq ... \leq x_n)$ and $\widetilde{y} = (y_1 \leq ... \leq y_n)$ be income distributions that satisfy Condition 2. Let also $\widehat{T}(x,y) = \widehat{\rho}_L L(x) + \widehat{\sigma}_L x + \widehat{\rho}_K K(y) + \widehat{\sigma}_K y$ and $\widetilde{T}(x,y) = \widetilde{\rho}_L L(x) + \widetilde{\sigma}_L x + \widetilde{\rho}_K K(y) + \widetilde{\sigma}_K y$ be two bilinear tax reforms of an arbitrary pairwise-progressive dual tax schedule $T(x,y) = L(x) + K(y)$ that satisfy Condition 1. Then,*

*(1) $(\widehat{\rho}_L, \widehat{\rho}_K) \gneqq (\widetilde{\rho}_L, \widetilde{\rho}_K)$ if and only if $(\widehat{\sigma}_L, \widehat{\sigma}_K) \lneqq (\widetilde{\sigma}_L, \widetilde{\sigma}_K)$.*
*(2) $(\widehat{\rho}_L, \widehat{\rho}_K) \gneqq (\widetilde{\rho}_L, \widetilde{\rho}_K)$ implies $L_{\widehat{V}} \succ_L L_{\widetilde{V}}$.*
*(3) $L_{\widehat{V}} \succ_L L_{\widetilde{V}}$ if and only if $L_{\widetilde{T}} \succ_L L_{\widehat{T}}$.*

*(4)* $(\widehat{\rho}_L, \widehat{\rho}_K) \gneqq (\widetilde{\rho}_L, \widetilde{\rho}_K)$ *if and only if* $\overrightarrow{\psi_{\widehat{T}}}(x_i, y_i) \geq \overrightarrow{\psi_{\widetilde{T}}}(x_i, y_i)$ *for all tax-payer* $i$, *with at least one inequality.*

*(5)* $(\widehat{\rho}_L, \widehat{\rho}_K) \gneqq (\widetilde{\rho}_L, \widetilde{\rho}_K)$ *if and only if* $\overrightarrow{\alpha_{\widehat{T}}}(x_i, y_i) \leq \overrightarrow{\alpha_{\widetilde{T}}}(x_i, y_i)$ *for all tax-payer* $i$, *with at least one inequality.*

**Proof.** Statements 1, 3, 4 and 5 follow immediate from Proposition 1. Statement 2 follows from Proposition 1, which requires Condition 2, and Lemma 1, which requires Condition 1. ■

Beyond the mathematics, the idea behind this latter result relies on two facts. First, a Lorenz improvement in a one-dimensional income distribution is always obtained if we transfer income from rich tax-payers to poor tax-payers, and the richer the tax payer is the more it loses (or the less it gains). Second, Lemma 1 guarantees that under pairwise yield-equivalent restrictions, Lorenz domination is maintained after the sum of two one-dimensional distributions.

Also observe that a bilinear tax reform is defined from four parameters, namely $\rho_L, \sigma_L, \rho_K, \sigma_K$. Condition 1 forces us to fix $\sigma_L$ and $\sigma_K$ from $\rho_L$ and $\rho_K$, which are left as the relevant variables that determine the bilinear tax reforms. Proposition 2 tells that there exists a partial order to compare different pairs $(\rho_L, \rho_K)$ among them with respect to the Lorenz domination of either post-tax income (Statement 2) or tax liability distributions (Statement 3) of the bilinear tax reforms that they define.

It is important to point out that statement 2 in the above proposition may not hold if either Condition 1 or Condition 2 do not hold. To get the idea why Condition 1 and Condition 2 are 'necessary' for the Lorenz Dominance between the total (labor and capital) corresponding distributions, consider the two following examples. The failure of Condition 1 is illustrated by the first example, in which $x' = (1, 8, 11) \succ_L (0, 4, 6) = x''$ and $y' = (2, 3, 5) \succ_L (7/2, 9/2, 12) = y''$ but $x' + y' = (3, 11, 16) \not\succ_L (7/2, 17/2, 16) = x'' + y''$. The failure of Condition 2 is illustrated by the second example, in which $x' = (2, 7, 8) \succ_L (2, 5, 10) = x''$ and $y' = (5, 4, 8) \succ_L (6, 3, 8) = y''$ but $x' + y' = (7, 11, 16) \not\succ_L (8, 8, 18) = x'' + y''$.[12]

One could argue that the strict conditions (Condition 1 and Condition 2) that we impose in Proposition 2 are driven by the very general structure the dual tax function is allowed to have. Nevertheless, even in the case where one of the tax schedules of the dual tax function $T(x, y) = L(x) + K(y)$ is flat, e.g. $K(y) = \lambda y$, both conditions have to be maintained.

A consequence of Proposition 2 is that the set of bilinear reforms can be arranged in a *lattice*[13] structure when they are restricted to income distributions that

[12] Complete examples can be found in Calonge and Tejada (2008).

[13] A given set $S$ is a *partially ordered set (poset)* if there is a reflexive, antisymmetric and transitive binary relation $\preceq$ that orders some pair of elements in $S$. We say a poset $S$ is a *lattice* if any element in $S$ has supremum and infimum, i.e, there are operations $\vee, \wedge : S \times S \to S$ so that for any $x, y \in S$ there is $x \vee y \in S$ (resp. $x \wedge y \in S$) so that $x, y \preceq x \vee y$ (resp. $x, y \succeq x \wedge y$ ) and for all $z \in S \backslash (x \vee y)$ (resp. $z \in S \backslash (x \wedge y)$) so that

satisfy Condition 2. For those distributions, let $\Lambda := \Lambda\left(T, \underline{\rho}_L, \overline{\rho}_L, \underline{\rho}_K, \overline{\rho}_K, \Delta R_L, \Delta R_K\right)$ denote the set of all pairwise yield-equivalent bilinear tax cuts $\rho_L L(x) + \sigma_L x + \rho_K K(y) + \sigma_L y$ applied to a pairwise-progressive dual tax reform $T(x,y) = \rho_L L(x) + \sigma_L x + \rho_K K(y)$ where $\Delta R_L$ and $\Delta R_K$ are the aggregate tax income changes, $\underline{\rho}_L \leq \rho_L \leq \overline{\rho}_L$ and $\underline{\rho}_K \leq \rho_K \leq \overline{\rho}_K$. As in the one-dimensional framework, the bounds are needed to ensure that no tax-payer pays either a negative amount or more (resp. less) than in the pre-tax reform in the case of a tax cut (resp. tax hike). Let us also define the following join and meet operators $\vee, \wedge : \Lambda \times \Lambda \to \Lambda$:

$$\begin{aligned} \widehat{T} \vee \widetilde{T}(x,y) = &\max\{\widehat{\rho}_L, \widetilde{\rho}_L\} L(x) + \min\{\widehat{\sigma}_L, \widetilde{\sigma}_L\} x \\ &+ \max\{\widehat{\rho}_K, \widetilde{\rho}_K\} K(y) + \min\{\widehat{\sigma}_K, \widetilde{\sigma}_K\} y, \end{aligned} \tag{9}$$

$$\begin{aligned} \widehat{T} \wedge \widetilde{T}(x,y) = &\min\{\widehat{\rho}_L, \widetilde{\rho}_L\} L(x) + \max\{\widehat{\sigma}_L, \widetilde{\sigma}_L\} x \\ &+ \min\{\widehat{\rho}_K, \widetilde{\rho}_K\} K(y) + \max\{\widehat{\sigma}_K, \widetilde{\sigma}_K\} y, \end{aligned} \tag{10}$$

where $\widehat{T}(x,y) = \widehat{\rho}_L L(x) + \widehat{\sigma}_L x + \widehat{\rho}_K K(y) + \widehat{\sigma}_K y$ and $\widetilde{T}(x,y) = \widetilde{\rho}_L L(x) + \widetilde{\sigma}_L x + \widetilde{\rho}_K K(y) + \widetilde{\sigma}_K y$. The following theorem is a direct corollary of Proposition 2.

**Theorem 1** *Under Condition 1 and Condition 2,*

*(1)* *$\Lambda$ is a lattice endowed with the partial order $\preceq$ defined by the Lorenz Dominance of post-tax income distributions and the operators $\vee^V, \wedge^V$ defined in (9) and (10). Moreover, $T_{\overline{\rho}_L, \overline{\rho}_K}(\cdot,\cdot)$ is the unique maximal element of $\Lambda$ and $T_{\underline{\rho}_L, \underline{\rho}_K}(\cdot,\cdot)$ is the unique minimal element of $\Lambda$.*

*(2)* *$\Lambda$ is a lattice endowed with the partial order $\preceq$ defined by the Lorenz Dominance of tax liability income distributions and the operators $\vee^V, \wedge^V$ defined in (9) and (10). Moreover, $T_{\overline{\rho}_L, \overline{\rho}_K}(\cdot,\cdot)$ is the unique minimal element of $\Lambda$ and $T_{\underline{\rho}_L, \underline{\rho}_K}(\cdot,\cdot)$ is the unique maximal element of $\Lambda$.*

To illustrate the above theorem, the discrete lattice structure when we restrict only to Pfhäler-type bilinear tax reforms is shown in Figure 1 below.

Let us interpret the above figure in the four possible scenarios.

- **Case 1:** *labor tax cut/capital tax cut.*
  In this case, Figure 1 has to be read from the top to the bottom, i.e. it is only possible to compare elements bearing a vertical relationship, being the element that is above more redistributive than the element that is below. Hence, the supremum of the lattice (i.e. the most redistributive reform) is $T_{2,2}$ and the infimum of the lattice (i.e. the least redistributive reform) is $T_{1,1}$.
- **Case 2:** *labor tax hike/capital tax cut.*
  In this case, Figure 1 has to be read from the right to the left.
- **Case 3:** *labor tax cut/capital tax hike.*
  In this case, Figure 1 has to be read from the left to the right.
- **Case 4:** *labor tax hike/capital tax hike.*
  In this case, Figure 1 has to be read from the bottom to the top.

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$x, y \preceq z$ (resp. $z \preceq x, y$), $x \vee y \preceq z$ (resp. $z \preceq x \vee y$).

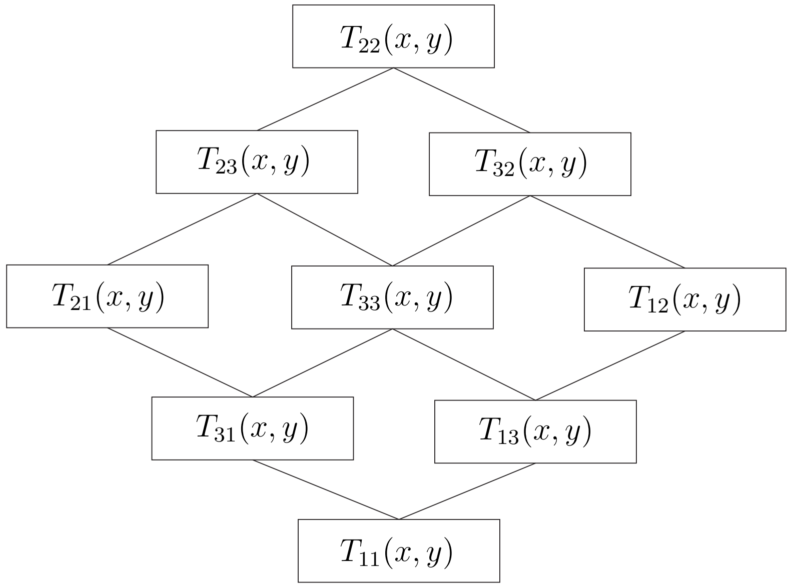


Figure 1. **Lattice structure of the 9 linear dual tax reforms.**

To achieve a certain aggregate change $\Delta R_T$ on the aggregate tax liability (either positive, negative or zero) there are two decisions to be taken. First, we need to decide how to split (or balance) $\Delta R_T$ into $\Delta R_L$ and $\Delta R_K$. Second, we need to decide which of the bilinear tax reforms to pick up. Theorem 1 helps out in this second decision.

We want to stress that our analysis *i*) includes the particular but interesting case of a reform that leaves the public budget unchanged, i.e. $\Delta R_T = 0$, by offsetting an increase in the liability of one tax with either a decrease on the liability of another tax or an increase on the tax benefits and *ii*) can be extended to include an arbitrary finite number of taxes, since Lemma 1 extends naturally to more than two income distributions. It also needs to be pointed out that, since it relies on Proposition 2, the lattice structure may vanish when either Condition 1 or Condition 2 do not hold.

We also want to remark that the lattice setting is a tool to analyze the differential effect of bilinear tax reforms. Hence, although under Condition 1 and Condition 2 we typically could expect the bilinear tax reforms considered so far to be more redistributive than the tax before reform (see e.g. the Spanish case in Section 6), this last assertion may fail to hold in some extreme cases and hence we cannot state in general that bilinear tax reforms are by themselves redistribution-improving. Nevertheless, under the same assumptions of Theorem 1, if we assume additionally that $\Delta R_L = \Delta R_K = 0$, then the post-tax income distribution of the bilinear tax reform $\widehat{T}(x,y)$ defined by $\rho_L$ and $\rho_K$ Lorenz dominates the current post-tax income distribution of $T(x,y)$, as long as $\rho_L \geq 1$ and $\rho_K \geq 1$, with at least one strict inequality. Therefore, one such reform would be redistribution-improving by itself. Furthermore, the greater $\rho_L$ and $\rho_K$ are, the higher the Lorenz improvement would be.

Redistribution of taxes is a relevant equity issue to be considered when adopting tax reforms. Related to this analysis, there is also an empirical issue which is important not only from an equity perspective but from a government point of view when preparing a Budget: the elasticity of tax revenue with respect to income. With progressive taxation, revenue is elastic with respect to a proportional growth of all incomes, and the amount of this elasticity is also crucial for macroeconomic projections. Hutton and Lambert (1979) showed that increasing the average rate progression $\beta(x)$ of a tax at every point of the income distribution raises the elasticity of the revenue function. Similarly, it can be shown (see for instance Calonge and Tejada, 2009) that, given $\widetilde{x} = (x_1, ..., x_n)$ and $\widetilde{y} = (y_1, ..., y_n)$ finite income distributions and $T(x, y) = L(x) + K(y)$ a pairwise-progressive dual tax schedule, if the aggregate total income $Z = \sum_{i=1}^{n} (x_i + y_i)$ changes only due to equiproportional changes in $\widetilde{x}$ and $\widetilde{y}$ [14], then the *elasticity* $E(Z)$ of the aggregate total tax liability $\sum_{i=1}^{n} T(x_i, y_i)$ with respect to $Z$ is

$$E(Z) = 1 + \frac{\sum_{i=1}^{n} \left(\beta^L(x_i) \cdot x_i^2 + \beta^K(y_i) \cdot y_i^2\right)}{\sum_{i=1}^{n} T(x_i, y_i)}.$$

Considering the above result and the fact that $\frac{d}{dx}\left(\frac{\rho T(x)+\sigma x}{x}\right) = \rho$, Theorem 1 is maintained, even when Condition 2 does not hold, if we replace the partial order defined by the Lorenz Dominance of post-tax income distributions by the partial order defined by the elasticity dominance. In particular, as in the one-dimensional tax case, there is no trade-off between the preferences of the Government concerning the aggregate total tax liability and the redistribution of the post-tax income.

### *4.2 Local effects*

We have so far focussed on the differential aggregate effect that bilinear tax reforms have on the income distribution. In the present part, the differential analysis is performed locally, in the sense that we determine who, among all the tax-payers, benefit from one bilinear tax reform being implemented instead of any other pairwise yield-equivalent bilinear tax reform. Our final result (Theorem 2) is stated for a subset of $\Lambda$ that is defined below instead of $\Lambda$. Indeed, assume that the slopes of labor and capital linear transformations that define the bilinear reforms chosen by the Government are linked through the following condition:

$$\rho_L = \varepsilon \rho_K + \tau, \tag{11}$$

where $\varepsilon$ can be interpreted as the elasticity of $\rho_L$ with respect to $\rho_K$. The above condition relates changes in $\rho_L$ to changes in $\rho_L$ and viceversa. We define the subset $\Upsilon^\varepsilon := \Upsilon^\varepsilon(T, \underline{\rho}, \overline{\rho}, \Delta R_L, \Delta R_K)$ of $\Lambda$ where $\rho_K \in \left[\underline{\rho_K}, \overline{\rho_K}\right]$ and $\rho_L$ and $\rho_K$ satisfy (11).

[14] $\widetilde{r} = (r_1, ..., r_n)$ is obtained from a equiproportionate change of $\widetilde{s} = (s_1, ..., s_n)$ if $r_i = ks_i$ for all $i = 1, ..., n$.

Notice that, unlike $\Lambda$, $\Upsilon$ is a curve and not a surface in the space of parameters $\rho_L$, $\sigma_L$, $\rho_K$ and $\sigma_K$ that define a bilinear reform.

The next result proves that, for each tax-payer, $\Upsilon^{\varepsilon}$ can be completely ordered in terms of her local preferences over bilinear tax reforms, and that the preference list is the same for all 'poor' tax-payers and it is reversed for 'rich' tax-payers.

**Theorem 2** *Under Condition 1, assume that $L(x)$ and $K(y)$ are strictly convex functions. Then, there is a continuous piecewise (3 pieces) monotone curve $f(x,y) = 0$ satisfying $f(x_g, y_g) = 0$, where $(x_g, y_g) = (l^{-1}(g_L), k^{-1}(g_K))$, such that*

*(1) if $f(x,y) \leq 0$, the preferred tax reform of $\Upsilon^{\varepsilon}$ by a tax-payer with incomes $(x,y)$ is $T_{\varepsilon\overline{\rho_K},\overline{\rho_K}}(\cdot,\cdot)$ (the most redistributive of $\Upsilon^{\varepsilon}$).*

*(2) if $f(x,y) \geq 0$, the preferred tax reform of $\Upsilon^{\varepsilon}$ by a tax-payer with incomes $(x,y)$ is $T_{\varepsilon\underline{\rho K},\underline{\rho K}}(\cdot,\cdot)$ (the least redistributive of $\Upsilon^{\varepsilon}$).*

**Proof.** First, let $\widehat{T}(x,y) = \widehat{\rho}_L L(x) + \widehat{\sigma}_L x + \widehat{\rho}_K K(y) + \widehat{\sigma}_K y$ and $\widetilde{T}(x,y) = \widetilde{\rho}_L L(x) + \widetilde{\sigma}_L x + \widetilde{\rho}_K K(y) + \widetilde{\sigma}_K y$ be two arbitrary (not necessarily in $\Upsilon$ but just in $\Lambda$) pairwise yield-equivalent bilinear tax reforms such that $(\widetilde{\rho}_L, \widetilde{\rho}_K) \lneqq (\widehat{\rho}_L, \widehat{\rho}_K)$. By condition (7) and (8),

$$\begin{aligned} f(x,y) :&= \widehat{T}(x,y) - \widetilde{T}(x,y) \\ &= (\widehat{\rho}_L - \widetilde{\rho}_L)(L(x) - g_L x) + (\widehat{\rho}_K - \widetilde{\rho}_K)(K(y) - g_K y) \\ &=: g(x) + h(y) = 0, \end{aligned}$$

is the continuous indifference curve between the two different tax cuts. Since, by hypothesis, both $L(x)$ and $K(y)$ are strictly convex, we have that $g'(x) = \frac{\partial f(x,y)}{\partial x} = (\widehat{\rho}_L - \widetilde{\rho}_L)\left(\frac{dL(x)}{dx} - g_L\right)$ and $h'(y) = \frac{\partial f(x,y)}{\partial y} = (\widehat{\rho}_K - \widetilde{\rho}_K)\left(\frac{dK(y)}{dy} - g_L\right)$ are increasing functions of $x$ and $y$ respectively. Further, since $g(0) = g(x_{g_L})$ and $h(0) = h(y_{g_K})$, let $x'_{g_L} \in (0, x_{g_K})$ and $y'_{g_K} \in (0, y_g)$ denote respectively the unique solutions of $\frac{dg(x)}{dx} = 0$ and $\frac{dh(y)}{dy} = 0$. Notice that if $L(x)$ and $K(y)$ are only convex functions then ${x'}_{g_L}$ and ${y'}_{g_K}$ might not be unique.

Second, since $T(x,y)$ is pairwise-progressive, we can apply (2). Thus, if $(x,y) \lneqq (x_{g_L}, y_{g_K})$ then $f(x,y) < 0$, i.e. tax cut $\widehat{T}(\cdot,\cdot)$ is preferred to tax cut $\widetilde{T}(\cdot,\cdot)$ by a tax-payer with pair of incomes $(x,y)$. Similarly, if $(x_{g_L}, y_g) \gneqq (x,y)$ then $f(x,y) > 0$, i.e. tax cut $\widetilde{T}(\cdot,\cdot)$ is preferred to tax cut $\widehat{T}(\cdot,\cdot)$ by a tax-payer with pair of incomes $(x,y)$. Hence, $f(x,y) = 0$ implies that either $x > x_{g_L}$ and $y < y_{g_K}$ or $x < x_{g_L}$ and $y > y_{g_K}$.

Third, to establish the different regions where $f(x,y) = 0$ is a monotone curve, we distinguish three cases. Consider a tax-payer with a pair of incomes $(x_0, y_0)$ that belong to the indifference curve between $\widehat{T}(x,y)$ and $\widetilde{T}(x,y)$, i.e. $f(x_0, y_0) = 0$.

(1) **Case 1:** $(x'_{g_L}, y'_g) < (x_0, y_0)$.

By hypothesis of the case, $\frac{dg(x)}{dx}|_{x=x_0} > 0$ and $\frac{dh(y)}{dy}|_{y=y_0} > 0$. Applying the Theorem of the Implicit Function, $\left(\frac{dy}{dx}\right)|_{(x,y)=(x_0,y_0)} < 0$.

(2) **Case 2:** $x_0 < x'_{g_L}$.

Since $x'_{g_L} < x_{g_L}$, we necessarily have $y \geq y_{g_K} > y''_{g_K}$. Hence, $\frac{dg(x)}{dx} < 0$ and $\frac{dh(y)}{dy} > 0$ . Applying the Theorem of the Implicit Function, $\left(\frac{dy}{dx}\right)|_{(x,y)=(x_0,y_0)} >$ 0.

(3) **Case 3:** $y_0 < y'_{g_K}$.
It is analogous to the above case and hence it is left to the reader to prove that $\left(\frac{dy}{dx}\right)|_{(x,y)=(x_0,y_0)} > 0$.

Notice that under the assumptions of the theorem, the indifference curve of any two pairwise yield-equivalent bilinear tax reforms -not necessarily in $\Upsilon$ but just in $\Lambda$- has always the same piecewise monotone shape, although it might not coincide for different pairs of reforms.

Notice that until here we have not yet exploited the richer structure of $\Upsilon$ with respect to $\Lambda$. In fact, we claim that the indifference curve between any pair of tax cuts from $\Upsilon$ not only has the same shape but is indeed the same curve, since the curve $f(x, y) = 0$ is invariant with respect to changes in $\widehat{\rho}_L, \widehat{\rho}_K, \widetilde{\rho}_L, \widetilde{\rho}_K$ that keep $\frac{\widehat{\rho}_L-\widetilde{\rho}_L}{\widehat{\rho}_K-\widetilde{\rho}_K}$ constant. Then observe that $f(x, y) = 0$ can be rewritten, for any pair of bilinear reforms in $\Upsilon$, as

$$\frac{K(y) - g_K y}{L(x) - g_L x} = -\frac{\widehat{\rho}_L - \widetilde{\rho}_L}{\widehat{\rho}_K - \widetilde{\rho}_K} = \varepsilon,$$

and hence the above claim holds. Moreover, the larger $\rho_K$ -and thus $\rho_L$- is, either the more or the less preferred the associated bilinear reform is, depending on tax-payer's pair of income bases. Therefore, we have proved that the preference list for bilinear tax reforms of $\Upsilon$ in the case of *poor* tax-payers, i.e tax-payers with pair of incomes $(x, y)$ so that $f(x, y) \geq 0$, depends only on $\rho_K$ -or $\rho_L$- and this complete order is reversed for *rich* tax-payers, i.e tax-payers with pair of incomes $(x, y)$ so that $f(x, y) \leq 0$. ■

Figure 2 below illustrates the above Theorem. Besides the figure, some additional comments are needed. First, notice that there is no restriction on the relative order on the capital and labor income distributions. I.e., unlike in Theorem 1, we only require Condition 1 and not Condition 2.

Second, as we already mentioned in the proof of the Theorem 2, the indifference curve of any two pairwise yield-equivalent bilinear tax reforms -not necessarily in $\Upsilon$ but just in $\Lambda$ is composed of three different monotone connected pieces. In the first one, that includes any tax-payer with pair of payoffs $(x, y) > (x_{g_L}, y_{g_K})$, capital and labor incomes are *substitutes*, since the marginal rate of substitution for both 'goods' is negative. In the two remaining pieces, capital and labor incomes are *complements*, since the marginal rate of substitution for both 'goods' is positive. Strict convexity ensures that these three latter pieces are connected. If we only assume convexity, our conclusions do not change essentially but only the indifference curve may be composed of more than three pieces.

Finally, if poor tax-payers account for more than half the population, $T_{\overline{\rho},\overline{\rho}}(\cdot, \cdot)$ would be chosen in an election to decide which bilinear reform out of $\Upsilon$ should be

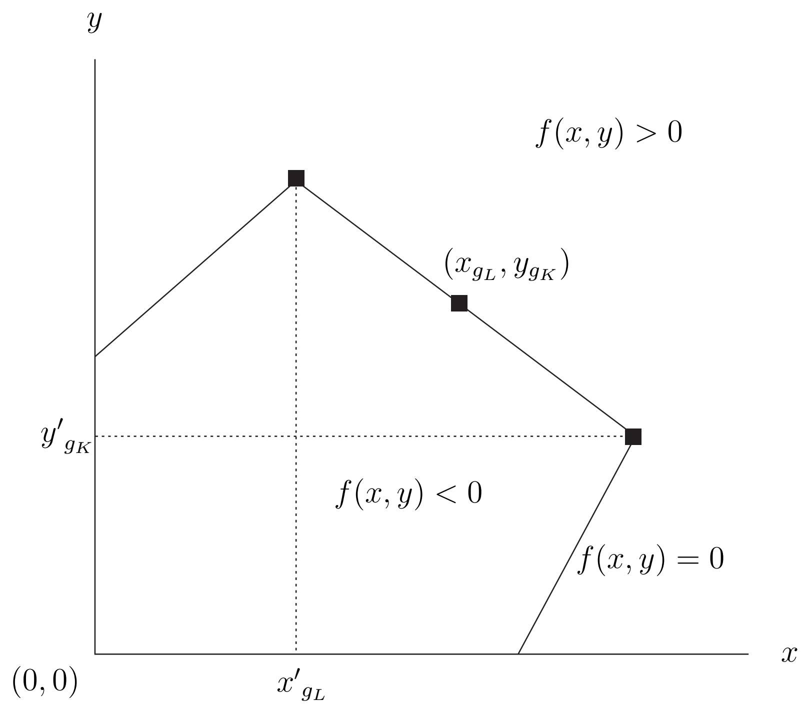


Figure 2. **Rich versus poor bilinear tax reforms preferences.**

carried out, provided that all tax-payers voted rationally. In this case, their interests would be aligned with those of a Government concerned about the redistribution of post-tax incomes.

## 5 Extensions of the lattice setting

In the present section we discuss the robustness of the lattice setting and its redistributional implications when some assumptions of the model are weakened. In Subsection 5.1 we consider the case where Condition 2 might not hold, whereas in Subsection 5.2 we allow agents to be strategic in response to changes in the tax schedule.

### *5.1 The lattice setting under the failure of Condition 2*

As we mentioned in Section 3, under the failure of either Condition 1 or Condition 2, Theorem 1 may vanish. This section is devoted to discuss the robustness of the lattice setting when Condition 2 is not met. Indeed, its failure would most likely give rise to a 'reranking' on the post-tax income distribution which could have a negative influence on income redistribution.

To establish the differential redistributive effect of a bilinear tax cut $\widehat{T}(x,y)$, we define the following distance, for all $p \in \{0, \frac{1}{n}, ..., 1\}$:

$$RE_{\widehat{T}}(p) = C_{V_{\widehat{T}};Z}(p) - L_Z(p), \tag{12}$$

where $L_Z(p)$ is the pre-tax total income Lorenz curve and $C_{V_{\widehat{T}};Z}(p)$ is the post tax-total income concentration curve, ranked by total income $Z = \{z_i = x_i + y_i\}_{i=1}^n$. The redistributive effect $RE_{\widehat{T}}(p)$, which can be easily estimated from the data, might be negative despite the application of progressive tax schedules on labor and capital incomes[15] . Let us express equation (12) as, for all $p \in \{0, \frac{1}{n}, ..., 1\}$:

$$RE_{\widehat{T}}(p) = \left(L^*_{V_{\widehat{T}}}(p) - L_{Z^*}(p)\right) - (L_Z(p) - L_{Z^*}(p)) + \Psi_{\widehat{T}}(p), \tag{13}$$

where $L_{Z^*}$ is the 'counterfactual' Lorenz curve of pre-tax income distributions, obtained by artificially aligning $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ and then adding them up, and $L^*_{V_{\widehat{T}}}$ is the 'counterfactual' Lorenz curve of post-tax income distributions, obtained by artificially aligning $\left\{x_i - \widehat{L}(x_i)\right\}_{i=1}^n$ and $\left\{y_i - \widehat{K}(y_i)\right\}_{i=1}^n$ and then adding them up. The term $\Psi_{\widehat{T}}(p) = C_{V_{\widehat{T}};Z}(p) - L^*_{V_{\widehat{T}}}(p)$ measures the distance between post-tax income distributions depending on whether Condition 2 holds or not and it can be also decomposed into two nonnegative terms:

$$\Psi_{\widehat{T}}(p) = \left(C_{V_{\widehat{T}};Z}(p) - L_{V_{\widehat{T}}}(p)\right) + \left(L_{V_{\widehat{T}}}(p) - L^*_{V_{\widehat{T}}}(p)\right). \tag{14}$$

The first term of (14) measures the reranking effect on post-tax income distribution, whereas the second term of (14) represents a vertical effect on the post-tax income distribution depending on whether Condition 2 holds or not.

Equation (13) relates the redistribution effect of the bilinear tax cut $\widehat{T}(x, y)$ with the benchmark situation where Condition 2 holds. Notice that, for all $p \in \{0, \frac{1}{n}, ..., 1\}$, $L_Z(p) - L_{Z^*}(p) \geq 0$ and $\Psi_{\widehat{T}}(p) \geq 0$, so the redistributive effect of $\widehat{T}(x, y)$ could be negative at some $p \in \{0, \frac{1}{n}, ..., 1\}$ if the difference between the Lorenz curves of $Z$ and $Z^*$, i.e. $L_Z(p) - L_{Z^*}(p)$, were sufficient large to offset $\left(L^*_{V_{\widehat{T}}}(p) - L_{Z^*}(p)\right) + \Psi_{\widehat{T}}(p)$. However, this difference is cancelled out when comparing two different bilinear tax reforms. Indeed, when we compare two different bilinear tax reforms, namely $\widehat{T}(x, y)$ and $\widetilde{T}(x, y)$, from (13) we obtain that, for all $p \in \{0, \frac{1}{n}, ..., 1\}$:

$$\Delta RE^{\widetilde{T}}_{\widehat{T}}(p) = RE_{\widehat{T}}(p) - RE_{\widetilde{T}}(p) = \left(L^*_{V_{\widehat{T}}}(p) - L^*_{V_{\widetilde{T}}}(p)\right) + \left(\Psi_{\widehat{T}}(p) - \Psi_{\widetilde{T}}(p)\right), \tag{15}$$

[15] However, in a real-world scenario we expect taxes (and tax reforms schemes) to achieve a certain level of income redistribution, which is usually a relevant objective for policy-makers when they are designing tax policy reforms. Redistribution indices, e.g. Reynolds-Smolensky index, can be used to compare, on the aggregate, the different tax cut policies defined in the lattice.

To establish that $\widehat{T}(x,y)$ is more redistributive than $\widetilde{T}(x,y)$, we restrict comparisons to those made according to the lattice structure, i.e. $\widehat{T}(x,y)$ dominates $\widetilde{T}(x,y)$ in the lattice, which in the case of labor tax cut/capital tax cut implies that $\widehat{T}(x,y)$ is 'above' $\widehat{T}(x,y)$ in the lattice. Therefore $L^*_{V_{\widehat{T}}}(p) - L^*_{V_{\widetilde{T}}}(p) \geq 0$. Hence, even if Condition 2 does not hold, only in those situations where $\Psi_{\widehat{T}}(p) - \Psi_{\widetilde{T}}(p)$ is negative enough so that $\Delta RE^{\widetilde{T}}_{\widehat{T}}(p) < 0$ the lattice may not be a reasonable model to arrange tax reforms (see empirical tests in Section 6).

### *5.2 The lattice setting when tax-payers are strategic*

The model considered so far only takes into account first-round effects of bilinear tax reforms. In particular, tax-payers have been assumed not to react when the Government announces tax reforms, for instance by changing their reported income bases through income base shifting. This is the same approach that Pfähler (1984) follows in his paper.

Since the lattice setting is a model for redistributive analysis of bilinear tax reforms, it is a natural question to wonder how robust its equity predictions are when agents are capable of responding strategically to announced bilinear tax reforms.

As a consequence of Theorem 1, we know that restructuring tax rates according to the lattice model leads to different post-tax income equity levels. For instance, the discrete lattice setting in Figure 1 tells us that the $T_{2,2}$ dual tax function is the most redistributive policy among the set of bilinear tax cuts, being $T_{1,1}$ the least one. However, it is also true that $T_{2,2}$ imposes higher marginal tax rates on the top of income distribution. If rich tax payers are sensitive to absolute or relative changes in tax rates, we could expect strategic responses in their tax behavior.

Decisions based on the complete range of bilinear tax reforms described in the lattice suppose in general a trade-off between equity and efficiency aspects of the tax design. For instance, one could argue that $T_{2,2}$ tax cut should be chosen in the base of income redistribution goals but, if policy makers were aware of how tax payers respond to tax changes, a second-best alternative could be $T_{2,3}$ or $T_{2,1}$, in the case where taxing capital income is more distortive than taxing labor income. Besides this latter comment, what can we say in general about the lattice redistributive predictions in the presence of tax distorsions?

First, the lattice setting is an appropriate model -at least as a short term analysis- to arrange linear reforms as long as the potential reduction in tax bases due to tax rate changes has not -or little- impact on total revenue. This is a reasonable hypothesis if the strategic reaction of tax payers to announced tax reforms comes basically from labor supply or it is due to shifts from taxable salary compensations to fringe benefits, health insurance policies, pension plans, etc [16].

[16] There is a large empirical evidence with has settled the labor supply response to

Second, there are other type of fiscal strategies used by tax-payers in response to announced tax reforms, for instance tax shifting from personal income to corporate tax revenue as reported by Gordon and Slemrod (2000). This tax base shifting mechanism can be potentially important for affluent people, who have relatively easy access to income shifting opportunities. In fact, Governments usually try to control tax-shifting between the two cases by balancing taxation on both incomes so that no incentives to income shifting is given to rich recipients.

In the remaining of the present subsection we show that under certain mild assumptions the lattice setting of Theorem 1 is essentially maintained in the presence of tax shifting, assuming that other short-term tax-payers reactions are not important. Since tax-payer strategic behavior is incorporated into our model, this can be more relevant for policy analysis.

There are at least two different ways to adapt our model to include tax-shifting when a bilinear tax reform $\widetilde{T}(x,y)$ defined by $\rho_L, \sigma_L, \rho_K, \sigma_K$ is applied to a pairwise-progressive dual tax $T(x,y) = L(x) + K(y)$.

On the one hand, we can update equations (7) and (8) in a way that they incorporate the reaction of tax-payers to the announced tax reform. This approach is relegated to Subsection 8.2 in the Appendix because of its limited results, which are that, under some mild hypothesis, if the changes introduced by the bilinear reforms with respect to the current dual tax are small 'enough'[17], then the lattice structure and their redistributive implications are maintained, at least locally.

On the other hand, our second and more enlightening approach is based upon additional but still realistic assumptions, which guide us to a simpler way to adapt our model to include tax shifting. Instead of updating the yield-equivalent restrictions (7) and (8) to determine the parameters that define the set of bilinear tax reforms, we impose *i*) the yield-equivalent conditions with no tax-shifting (7) and (8) and *ii*) a neutral-incentive equation. This last condition restricts the set of parameters $\rho_L$ and $\rho_K$ that a Government can choose when designing a bilinear tax reform. By shrinking the set of parameters we prove the existence of a complete ordered subset of the lattice -it is in fact a sublattice[18]-, according to the Lorenz domination of either post-tax income distributions or tax liability distributions.

We focus our attention to the usual case -see e.g. Spanish PIT in Section 6- where the current tax $T(x,y) = L(x) + K(y)$ is defined on the two bases from marginal tax rates and marginal incomes. Then the additional assumption that is needed for this approach to be successful is stated as the following condition:

changes in tax rates to be fairly small. Specifically, in the Spanish case, Labeaga et al. (2008) conclude that efficiency effects (measured in terms of hours of work) are small for all scenarios and for each household type examined in their article.

[17] Since our argument uses the Theorem of the Implicit Function, small 'enough' means that any of the results obtained under this approach are no longer global but they hold locally in a neighborhood of bilinear tax reforms close to the current tax.

[18] A sublattice of a lattice $L$ is a nonempty subset of $L$ which is a lattice with the same meet and join operations as $L$.

- **(Condition 3)** Only 'rich' tax-payers act strategically by shifting their income bases, and they do so only in response to relative changes in the marginal tax rates.

Under the above condition, when the Government plans to assess the redistributive effects of a bilinear reform $\widetilde{T}(x,y)$, not only it has to impose the two yield-equivalent conditions (7) and (8) but it must ensure that no incentives to change the reported incomes are given to the rich recipients. Let $\widehat{l}$ (resp. $\widehat{k}$) be the highest marginal tax rate that define the current labor tax $L(x)$ (resp. capital tax $K(y)$). Then let us consider that Condition 3 translates into the following neutral-incentive equation:

$$\left[(\rho_L - 1)\,\widehat{l} + \sigma_L\right] = \epsilon\left[(\rho_K - 1)\,\widehat{k} + \sigma_K\right], \tag{16}$$

where $\epsilon$ is explained as follows. We assume that for any change $(\rho_K - 1)\,\widehat{k} + \sigma_K$ in $\widehat{k}$ there is a change $(\rho_L - 1)\,\widehat{l} + \sigma_L$ in $\widehat{l}$ that offsets, for any 'rich' tax-payer, her incentives to shift income from one base to the other. This level is given by (16) and hence, $\epsilon$ is the elasticity of changes in the highest labor marginal tax rate relative to changes in the highest capital marginal tax rate that leaves any rich tax-payer's incomes bases unchanged.

As we next show, as long as this latter parameter, $\epsilon$, can be estimated the Government, it can properly consider neutral-incentive bilinear tax reforms and preserve the differential redistributive implications of the lattice setting. Indeed observe that, using (7) and (8), then (16) reduces to

$$\rho_K = 1 + \frac{1}{\epsilon}\frac{\widehat{l} - g_L}{\widehat{k} - g_K}\rho_L - \frac{1}{\widehat{k} - g_K}\frac{1}{\epsilon}\left(\widehat{l} - g_L + \frac{\Delta_{R_K}}{\sum_{i=1}^n y_i}\epsilon - \frac{\Delta_{R_L}}{\sum_{i=1}^n x_i}\right). \tag{17}$$

Contrary to the case with no tax shifting, after imposing (7), (8) and (17) we are left with only one degree of freedom, either $\rho_L$ or $\rho_K$, instead of two, $\rho_L$ and $\rho_K$. Still, the lattice structure is preserved, at least for a subset of it. Indeed, let $\lambda := \lambda\left(T, \underline{\rho}_L, \overline{\rho}_L, \Delta R_L, \Delta R_K\right)$ be the subset of $\Lambda$ composed of all yield-equivalent bilinear reforms defined by parameters $\rho_L, \rho_K, \sigma_L, \sigma_K$ satisfying (7), (8) and (17). Since

$$\frac{\widehat{l} - g_L}{\widehat{k} - g_K} > 0, \tag{18}$$

then $\rho_L$ and $\rho_K$ are positively related and hence $\lambda$ is closed for the join and the meet operators defined in (9) and (10). Moreover, since there is only one degree of freedom, $\lambda$ is in fact a totally ordered set, instead of a partially ordered set. That is, under our assumptions and for the general class of current taxes considered, we can completely order a subset of bilinear tax reforms of $\Lambda$, namely $\Upsilon$, according to the Lorenz domination of either post-tax income or tax liability distributions. Like in Theorem 1, the higher the parameter $\rho_L$, the most redistributive the yield-equivalent reform $\widetilde{T}(x,y) = \rho_L L(x) + \sigma_L(\rho_L)x + \rho_K(\rho_L)K(y) + \sigma_K(\rho_L)y$ is.

Finally, it is worth noting that (16) can be relaxed without dropping out this latter conclusion, at least in a local sense. Indeed, let $\widehat{l}' = \rho_L \widehat{l} + \sigma_L$ and $\widehat{k}' = \rho_K \widehat{k} + \sigma_K$ be the highest marginal tax rates of the announced bilinear tax reform. We assume that there is a real-valued continuously differentiable function $h\left(\widehat{l}', \widehat{l}, \widehat{k}', \widehat{k}\right)$ that is increasing in $\widehat{l}'$ and $\widehat{k}$, and it is decreasing in $\widehat{l}$ and $\widehat{k}'$ . Then consider the following neutral-incentive condition

$$h\left(\widehat{l}', \widehat{l}, \widehat{k}', \widehat{k}\right) = 0.$$

This latter equation generalizes (16) and tells that, if we raise the announced $\widehat{l}'$, we need to raise the announced $\widehat{k}'$ as well, in order not to alter the incentives of the rich recipients and not induce them to act strategically on the income bases. If $\rho_L$ and $\rho_K$ are such that, for the bilinear reform that define, (16) holds, the Theorem of the Implicit Function guarantees that $\rho_K$ can be obtained locally as a continuously differentiable function of $\rho_L$. Furthermore, by means of simple calculus it can be checked that

$$\frac{\partial \rho_K}{\partial \rho_L} = -\frac{\frac{\partial h}{\partial \widehat{l}'}\left(\widehat{l} - g_L\right)}{\frac{\partial h}{\partial \widehat{k}'}\left(\widehat{k} - g_K\right)} > 0, \tag{19}$$

i.e. an increase of $\rho_L$ results in an increase of $\rho_K$, and therefore the set $\lambda$ defined by (18) instead of (19) remains closed under the above join and meet operators and it is thus a totally ordered set. As a consequence, $\lambda$ is still a totally ordered.

## 6 An empirical application: the case of the Spanish Income dual Tax

We end up the paper with an empirical example related to the dual income tax introduced in Spain in 2007. By means of the static micro-simulation model SIMESP (Arcarons and Calonge, 2008) and a very large data set of a million of tax payers containing fiscal information from the 2004 Spanish Income Tax File-Return, we assess the redistributive effects of bilinear tax reforms [19] .

The most important feature of the current Spanish Personal Income Tax (PIT) is its dual structure. On the one hand, 'labor' income base includes salary, entrepreneur and professional income, pension plan and rental income. This income base is rated according to a four-bracket tax schedule, ranging from 24% to 43% as the highest marginal tax rate. On the other hand, capital income and realized

[19] Stratitification of the sample (1176 stratas) has been carried out by province, income base and the type of tax-return (individual and joint tax returns). Richest recipients have been oversampled in order to get a better description of the highest part of income distribution. Grossing-up factors are derived from the stratification scheme and they are used to estimate population totals and other statistics from the sample (see Picos et al., 2007). See Calonge and Tejada (2009) for a more detailed explanation of both the SIMESP and the data.

capital gains constitute what is called the 'savings' base -or 'capital' income base- which is levied at 18% fixed rate, with a 1500 € deduction for dividends[20] .

We compare three different linear tax cuts, $T_{1,1}$, $T_{2,2}$ and $T_{3,3}$, according to the neutral revenue hypothesis. Table 1 below describes 'labor' and 'capital' tax schedules for the current Spanish Income Tax –which is the baseline of the simulation – and the proposed linear tax cuts after applying a 10% reduction on both labor and capital gross-tax liabilities. The average tax rate on capital income $g_K$ is calculated before a $1,500$€ allowance for dividends is applied.

| **Tax schedule** | | | | |
|---|---|---|---|---|
| | **Baseline Tax** | $T_{1,1}$ | $T_{2,2}$ | $T_{3,3}$ |
| **'Labor' base Tax Schedule** | | | | |
| 0-$17,360$€ | 24% | $21,60\%$ | $21,25\%$ | $21,34\%$ |
| $17,360-32,360$€ | 28% | $25,20\%$ | $25,40\%$ | $25,34\%$ |
| $32,360-53,360$€ | 37% | $33,30\%$ | $34,72\%$ | $34,34\%$ |
| $> 53,360$€ | 43% | $38,70\%$ | $40,94\%$ | $40,34\%$ |
| **Parameter** | $g_L = 26.56\%$ | $\rho_L = 90\%$ | $\rho_L = 103.61\%$ | $\rho_L = 100\%$ |
| **'Savings' base Tax Schedule** | 18% | $16,20\%$ | $16,29\%$ | $16,27\%$ |
| **Parameter** | $g_K = 17.28\%$ | $\rho_K = 90\%$ | $\rho_K = 102.08\%$ | $\rho_K = 100\%$ |

**Table 1. Baseline tax and simulated tax reforms.**

The differential distributional effects of the three bilinear tax cuts in terms of Lorenz dominance is represented by the concentration curves of tax-payments (or post-tax income) against the cumulated shares of pre-tax income recipients. For instance, to check whether $T_{2,2}$ is more redistributive than $T_{1,1}$ and $T_{3,3}$ we only need to prove the differences $LV_{2,2} - LV_{1,1}$ and $LV_{3,3} - LV_{1,1}$ are always positive through income distribution, where $LV_{1,1}$, $LV_{2,2}$ and $LV_{3,3}$ are the income post-tax distributions. The redistributive profile of PIT reforms $T_{2,2}$ and $T_{3,3}$ with respect to $T_{1,1}$ is represented in Figure 3. Like in the theoretical framework, our simulations are based on the taxable-income variable, but their effects on income distribution are measured as well considering the recipient's pre-tax income. Pre-tax income is

[20] See Calonge and Tejada (2009) for a more detailed explanation of the current Spanish Income Tax.

a more accurate measure of the individual welfare than taxable income. The use of pre-tax income takes into account the impact of allowances and deductions on PIT.

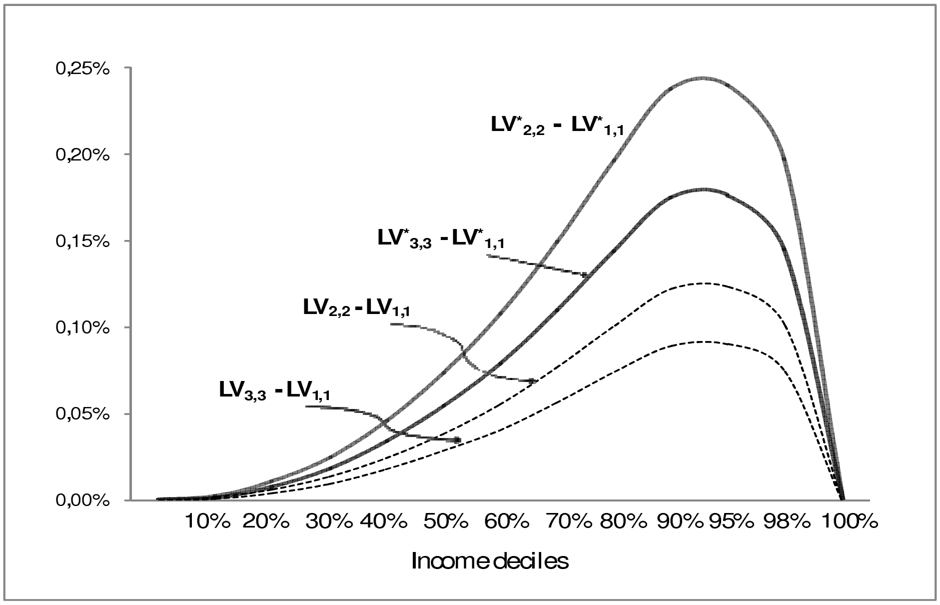


Figure 3. **Difference between post-tax taxable income Lorenz curves (continuous curves refer to taxable income whereas dashed curves refer to pre-tax income).**

The results in Figure 3 bear out the theoretical ones described in Section 4 indicating a clear Lorenz dominance order $T_{2,2} \succ T_{3,3} \succ T_{1,1}$ for post-tax incomes. Similar results not reproduced here, are obtained for tax concentration curves.

## 7 Conclusions

The distributive effects of three yield-equivalent linear tax reforms are established by Pfähler (1984). His main result is that those reforms can be ranked according to an income distribution criterion, namely the Lorenz dominance between either post-tax income distributions or tax liability distributions. When dual taxes are concerned, Pfähler's results need further development.

We show that a lattice based on the Lorenz dominance criterion can be established for the set of bilinear tax reforms applied to a given dual tax or, more generally, to two different one-dimensional taxes. Our analysis is consistent with the median voter model and their redistributional implications are aligned with the interests of a Government primarily concerned with the elasticity of the aggregate tax liability.

We also show that, even in the presence of tax shifting between income bases and when the short-term elasticities of other types of tax-payer strategic behavior, e.g. tax evasion or change on the labor supply, with respect to changes in the tax reform are almost zero, the main distributional implications of the lattice setting are maintained. In any case, whatever the policy adopted, the lattice structure does offer a range of tax reforms with different implications on income redistribution, and efficiency aspects of the tax reforms have also to be considered by policymakers when determining the tax rate structures of each policy.

# 8 Appendix

## *8.1 A technical proof*

**Proof.** [proof of Lemma 1 ] The Lorenz curve of $\widetilde{x^1 + y^1} = (x_1^1 + y_1^1 \leq ... \leq x_n^1 + y_n^1)$ is defined by

$$L_{x^1+y^1}(p) = \frac{\sum_{i=1}^{p} (x_i^1 + y_i^1)}{\sum_{i=1}^{n} (x_i^1 + y_i^1)}$$

for $p = 1, ..., n$. Analogously, the Lorenz curve of $\widetilde{x^2 + y^2} = (x_1^2 + y_1^2 \leq ... \leq x_n^2 + y_n^2)$ is defined by

$$L_{x^2+y^2}(p) = \frac{\sum_{i=1}^{p} (x_i^2 + y_i^2)}{\sum_{i=1}^{n} (x_i^2 + y_i^2)}$$

for $p = 1, ..., n$. Then, from hypothesis, for $p \neq 0, n$,

$$L_{x^1+y^1}(p) > L_{x^2+y^2}(p) \Leftrightarrow \sum_{t=1}^{p} \left(x_i^1 + y_i^1\right) > \sum_{t=1}^{p} \left(x_i^2 + y_i^2\right).$$

Finally, given $0 < p < n$,

$$\sum_{t=1}^{p} \left(x_i^1 + y_i^1\right) = \sum_{t=1}^{p} x_i^1 + \sum_{t=1}^{p} y_i^1 > \sum_{t=1}^{p} x_i^2 + \sum_{t=1}^{p} y_i^2 = \sum_{t=1}^{p} \left(x_i^2 + y_i^2\right)$$

for $\widetilde{x^2}$ is LD by $\widetilde{x^1}$, $\widetilde{y^2}$ is LD by $\widetilde{y^1}$ and the hypothesis of the lemma. ■

## *8.2 Appendix II: the lattice setting under a more general tax shifting behavior*

Suppose that the tax-payer $i$, which plans to declare incomes $x_i$ and $y_i$ under the current dual tax, decides, after a bilinear reform is announced, to change her income bases to ${x'}_i$ and ${y'}_i$. As a consequence of this change, what she eventually pays as tax is no longer $\widetilde{T}(x_i, y_i)$ but $\widetilde{T}({x'}_i, {y'}_i)$, since ${x'}_i$ and ${y'}_i$ are the only data available to the Government.

Recall that the lattice setting in Section 3 relies on Condition 1 and Condition 2. In the absence of tax shifting, Condition 1 is just a dummy design condition that states that dual tax reforms must be pairwise-equivalent. However, if we assume an strategic behavior of tax payers, conditions (7) and (8) no longer yield tax reforms with the same revenue. To find what conditions on $\rho_L, \sigma_L, \rho_K, \sigma_K$ ensure that labor and capital tax reforms are indeed pairwise yield-equivalent, we need to impose the following more general equations on $\rho_L$, $\sigma_L$, $\rho_K$ and $\sigma_K$:

$$\rho_L \sum_{i=1}^{n} L(x_i'(\rho_L, \sigma_L, \rho_K, \sigma_L)) + \sigma_L \sum_{i=1}^{n} x_i'(\rho_L, \sigma_L, \rho_K, \sigma_L) = \sum_{i=1}^{n} L(x_i) - \Delta R_L \quad (20)$$

and

$$\rho_K \sum_{i=1}^{n} K(y_i'(\rho_L, \sigma_L, \rho_K, \sigma_L)) + \sigma_K \sum_{i=1}^{n} y_i'(\rho_L, \sigma_L, \rho_K, \sigma_L) = \sum_{i=1}^{n} K(y_i) - \Delta R_K. \quad (21)$$

Observe that (20) and (21) define implicitly $\sigma_L$ and $\sigma_K$ from $\rho_L$ and $\rho_K$. Solving this system of two nonlinear equations can always be done empirically as long as we can estimate ${x'}_i$ and ${y'}_i$ for each tax-payer $i$ in response to the announced bilinear tax reform defined by $\rho_L$, $\sigma_L$, $\rho_K$ and $\sigma_K$. In the framework with no strategic behavior, this can be done easily using (7) and (8). As we already mentioned, these latter equations leave the designer of the tax reform with two degrees of freedom, namely $(\rho_L, \rho_K)$. Theorem 1 let us arrange all those pairs in a lattice. We wish to do the same in the presence of strategic response from the tax-payers, at least under some mild hypothesis.

Indeed, following the philosophy behind Gordon and Slemrod (2000), we assume that, for each recipient, a raise in the average labor (resp. capital) tax rate relative to the average capital (resp. labor) tax rate results in an increase in the capital (resp. labor) tax base at the expense of the labor (resp. capital) tax base[21]. We also consider an analogous assumption for decreases on the average labor (resp. capital) tax rate. That is to say, the tax income bases in response to an announced reform are

$${x'}_i(\rho_L, \sigma_L, \rho_K, \sigma_L) = x_i \left(1 - \alpha_i \left[\left((\rho_L - 1)\, l(x_i) + \sigma_L\right) - \epsilon_i \left((\rho_K - 1)\, k(y_i) + \sigma_K\right)\right]\right)$$

and

$${y'}_i(\rho_L, \sigma_L, \rho_K, \sigma_L) = y_i \left(1 + \beta_i \left[\left((\rho_L - 1)\, l(x_i) + \sigma_L\right) - \epsilon_i \left((\rho_K - 1)\, k(y_i) + \sigma_K\right)\right]\right),$$

being $\alpha_i$ (resp. $\beta_i$) the elasticity for tax-payer $i$ of the labor income base ${x'}_i$ (resp. capital income base ${y'}_i$) with respect to a weighted difference between the average labor change tax rate and the average capital tax rate change, and $\epsilon_i$ is the elasticity of changes in the average labor tax rate relative to changes in the average capital

[21] It should be pointed out again that labor and capital are used in our theoretical model as names for any pair of taxes, since equation (**??**) may be more meaningful when considering different pairs of taxes rather than labor and capital taxes, e.g. personal and corporate taxes.

tax rate that leaves tax-payer $i$'s reported incomes unchanged. We assume that $\alpha_i$, $\beta_i$ and $\epsilon_i$ may be different among tax-payers[22].

Although the proof is only sketched here[23], we claim that, if both the changes introduced by the bilinear reforms with respect to the current dual tax and the above defined elasticities are small 'enough', then the lattice structure and their redistributive implications are maintained. The argument to prove the claim relies on the following two facts.

First, it is easy to check that the system of nonlinear equations composed of (20) and (21) that define the set of possible yield-equivalent bilinear tax reforms has always a solution close to the current tax schedule if we pick conveniently $\Delta R_L$ and $\Delta R_K$, although this last condition is sufficient but not necessary. Then the Theorem of the Implicit Function ensures that for bilinear tax reforms close 'enough' to the current tax schedule, we can explicitly obtain $\sigma_L$ and $\sigma_K$ as continuously differentiable functions of $\rho_L$ and $\rho_K$. Moreover, the derivative of $\sigma_L$ with respect to $\rho_L$ and the derivative of $\sigma_K$ with respect to $\rho_K$ are negative.

Second, all the above implies that the derivative of both the average labor rate $l(x)$ and the average capital tax rate $k(y)$ with respect to either $\rho_L$ or $\rho_K$ are nondecreasing continuous functions in $x$ and $y$ respectively. That is, given a bilinear tax reform that leaves the tax schedule close 'enough' to the current one then *i*) a small change in either $\rho_L$ or $\rho_K$ that preserves the yield-equivalent conditions (20) and (21) results in a transfer of post-tax income from rich tax-payers to poor tax-payers and *ii*) the richer the tax-payer the larger the increase (or the lower the decrease) in the average tax rate. As a consequence, using the same argument as the one used in the proof of Proposition 1, a small 'enough' increase in either $\rho_L$ or $\rho_K$ results in independent Lorenz criterion improvements of both labor and capital post-tax income distributions.

Finally, Lemma 1 implies that after such small 'enough' increase we obtain a Lorenz criterion improvement of post-tax income distribution. Therefore there is a neighborhood of bilinear reforms close enough to the current tax schedule endowed with the same lattice structure defined in Theorem 1 for the set $\Lambda$. In that neighborhood, analogously to the case with no tax shifting, the higher $\rho_L$ and $\rho_K$ are, the more redistributive the bilinear reformed defined by them is.

[22] Observe that these assumptions include the realistic and limit case in which 'poor' tax-payers have zero elasticities and 'rich' tax-payers have all common elasticities. Moreover, when a tax-payer is rich 'enough', the average tax rate almost coincides with the marginal tax rate, and hence the assumptions imply that one such tax-payer reacts essentially to variations on the marginal tax rates. This is indeed the approach followed in subsection 5.2.

[23] A formal proof can be provided by the authors upon request.

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