Long-run effects of Public Capital rent dissipation*

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* This work is partially supported by the Instituto de Estudios Fiscales and by the Research Grants SA087A08 (Junta Castilla y León) and ECO2009-14457-C04-01 (Ministerio de Ciencia e Innovación).

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Abstract. We consider an overlapping generation model where the consumption commodity is produced by means of a constant returns to scale technology that uses both private and public inputs. Congestion is addressed by sharing the rents of public capital between the private factors (labor and capital). We analyze what happens to the equilibrium when there is an exogenous increase in the stock of public capital. The impact of such an increase is decomposed into three different effects that work through factor prices and the higher level of taxation required for finance the public investment. One key parameter is the fraction of the rents created by the public capital that goes to labor.

JEL Classification: D62, E62,H54.

Keywords: Public inputs, congestion, OLG models.

1 Introduction

It is well known that the public capital stock (for instance, basic scientific research, roads, highways, irrigation systems, bridges, hospitals and streets) plays an important role in determining the productive capacity of an economy. Aschauer (1989) showed the positive effect that public capital stock exerts on the productivity of private capital and argued that the slowdown in productivity growth in the U.S. since the early 1970s is due to a shortage of investment in public infrastructure. In spite of the fact that Aschauer's results were controversial and their robustness has been questioned, economists generally support the notion that public capital is indeed productive. In this trend, early theoretical models as those provided by Arrow and Kurz (1969), Pestieau (1974), Basu (1987) and Baxter and King (1993), have studied several aspects of the importance of public infrastructures in economic growth by considering public capital stock as an additional input.

For both empirical and theoretical analysis, the role that public capital plays as a productive factor relies crucially on the way in which it enters in the production function. On the one hand, when constant returns to scale affects only private inputs, zero-profits are reached under competitive conditions. In such a case the public input is often described as "atmospheric" (Meade, 1952) and more recently has been referred to as pure or factor-augmenting public input. On the other hand, when the production function exhibits constant returns to scale in all inputs, including the public one, the competitive mechanism may lead to economic profits as long as there are decreasing returns to scale regarding private factors of production. Public inputs of this type are usually referred to as unpaid factors or profit-augmenting and may be subject to congestion in use. That is, the zero profit condition does not fulfill when prices of private inputs equates their marginal products. This fact prompts a rent-dissipation phenomenon which leads to congestion provided that firms hire private inputs above the efficient level until potential economic profit is exhausted (see Henderson, 1974).

Most of public capital stock is financed through taxes and public debt issue and it is provided to firms in a free-access basis. However, it is an empirical point whether a public input is either atmospheric or an unpaid factor. As was pointed out by Stiglitz (1988), a large part of public capital stock is subject to congestion, and thus the last case constitutes an interesting situation to be analyzed. In this trend the problem of public capital stock affected by congestion has been modeled in different ways within macroeconomics setups. For instance, Uzawa (1988) states the congestion of infrastructure as an externality which is internalized by means of a service charged on the users. Glomm and Ravikumar (1994), using a Ramsey-Cass-Koopmans growth model, tackle the congestion by adjusting the stock of public capital to the aggregate use of private factors in the production function. Fisher and Turnovsky (1998) incorporate a form of congestion function from the public good literature to a Ramsey type model and analyze the impact of the stock of government infrastructure expenditure on the accumulation of private capital. Their results depend on the degree of congestion and the substitutability properties between public and private capital in production. Recently, Feehan and Batina (2007), using a static partial equilibrium model with perfectly elastic supply of capital, deal with the congestion by posing an equilibrium concept, where the prices of private factors equate their marginal products plus a share of the marginal product of a public input, in such a way that the positive economic profits disappear.

The aim of this paper is to deepen the study of economies with the presence of the so-called unpaid factors which, as we have already remarked, are subject to congestion in use, that is, there are decreasing returns to scale in private inputs. Precisely, our main purpose is to provide a contribution for a better insight into the effects that variations in public capital investment may have in the private capital stock. For this, we consider a productive public capital within an overlapping generations model where the technology exhibits constant returns to scale in both public and private inputs. In this case, whenever the public input is freely available, it will initially generate economic profits and then firms will hire more of the private inputs or factors to capture those profits with the consequence of a congestion problem in the private factors markets. To tackle the congestion issue, we adapt to our model the approach followed by Feehan and Batina (2007) characterizing congestion by means of the rent-dissipation caused by the returns of public capital stock. To keep things simple we assume that labor is supplied inelastically and that the government has access to lump-sum taxes in order to provide public capital. Thus, the notion of equilibrium we consider is an extension to an overlapping generation model of the one given by Feehan and Batina (2007). In this way, we analyze in greater depth the effects and implications of considering the presence of a public input of the unpaid factor variety.

Our economy allows us not only to share the rents of public capital stock between the remain private factors (labor and private capital stock) but also to study the effects of public capital investment in the amount of intensive private capital stock in the steady state. Precisely, we divide such effect into three different effects that we call wage effect, interest rate effect and tax effect, respectively. As it is not surprising, we show that these effects rely crucially not only on preferences over consumption but also on the elasticities of the marginal products of public and private capital stocks and on the parameter that defines the way in which the share of public capital stock rent is dissipated between private factors.

The remainder of this paper is structured as follows. Section 2 presents the model and the equilibrium concept. Section 3 states our main results and deepens the analysis of the role of the rent dissipation parameter regarding the wage, interest rate and tax effects. Section 4 analyzes two examples which illustrate the main results and exhibit particular economic interest. Finally, Section 5 points out some final remarks.

2 The Model

Let us consider an overlapping-generations economy where agents live for two periods. During the first period, when individuals are young, they work and in the second period, when they are old, they are retired from the labor force. An agent born at period t is endowed with one unit of labor that is supplied inelastically. Consumers can save during youth in order to consume when they are old.

In this economy there is a private consumption commodity for every period. The preference relation over consumption of an agent born at time t is represented by an utility function $U(c_t^y, c_{t+1}^o)$, where c_t^y denotes the consumption of a young agent at time t and c_{t+1}^o is the consumption of an old agent at time $t+1$. We assume that the utility function is concave and increasing in both c_t^y and c_{t+1}^o. Then, given a wage w_t and a interest rate r_{t+1}, the individual problem of a consumer born at t is given by:

$$\begin{aligned} Max \quad & U(c_t^y, c_{t+1}^o) \\ s.t. \quad & c_t^y + s_t = w_t - T_t \\ & c_{t+1}^o = (1 + r_{t+1}) s_t, \end{aligned}$$

where s_t denotes savings and T_t is a lump-sum tax. That is, agents behave as price-takers and maximize preferences subject to the intertemporal budget constraint $c_t^y + \dfrac{c_{t+1}^o}{1+r_{t+1}} = w_t - T_t$.

Let $c_t^y(w_t, r_{t+1}, T_t)$ and $c_{t+1}^o(w_t, r_{t+1}, T_t)$ denote the consumption demands when young and old, respectively. We assume that preferences ensure that both consumption when young and consumption when old are normal commodities. This implies $\dfrac{\partial c_t^y}{\partial w_t} > 0$ and $\dfrac{\partial c_{t+1}^o}{\partial w_t} > 0$; and then $1 - \dfrac{\partial c_t^y}{\partial w_t} > 0$. Analogously, $\dfrac{\partial c_t^y}{\partial T_t} < 0$ and $\dfrac{\partial c_{t+1}^o}{\partial T_t} < 0$; and then $1 + \dfrac{\partial c_t^y}{\partial T_t} > 0$. In addition, $\dfrac{\partial c_{t+1}^0}{\partial r_{t+1}} > 0$ provided that c_{t+1}^o is a normal commodity. Moreover, we also state the assumption that $\dfrac{\partial c_t^y}{\partial r_{t+1}} \leq 0$. That is, substitution effects domain income effects; i.e., an increase in interest rate leads to an increase in savings (see Blanchard and Fischer, 1989)

In every period, the private consumption good is produced by a same technology that uses two private factors, capital and labor, denoted by K and L respectively; and a public input, namely, public capital, denoted by G. Private capital is supplied by the old whereas labor is supplied by the young. The technology displays constant returns to scale and is represented by a production function F which is homogeneous of degree one in all inputs. That is, an equiproportionate increase in the private factors and in the public input increases output of the consumption commodity in the same proportion. As we have remarked in the Introduction, this type of public input is typically referred to as unpaid factor and is subject to congestion, i.e., there are decreasing returns in the private factors.

We assume that the population grows at a constant rate n, that is, there are $L_t = (1+n)L_{t-1}$ consumers at every date $t \geq 1$, with L_0 given. Thus, $k_t = K_t/L_t$ units of capital per capita joint with the unit of labor per capita and with an amount $g_t = G_t/L_t$ of public input per capita produce $f(k_t, g_t)$ units of the private good per capita, where $f(k_t, g_t) = F(K_t, L_t, G_t)/L_t = F(k_t, 1, g_t)$

Regarding this technology with constant returns to scale we state the following assumptions[1] that we will use along this paper:

[1] We remark that these assumptions are also considered in Heijdra and van der Ploeg (2002, ch. 17) within an atmospheric public capital framework.

(A.1) $f(0, g_t) = f(k_t, 0) = 0$,

(A.2) $f_k = \frac{\partial f}{\partial k_t} > 0$ and $f_g = \frac{\partial f}{\partial g_t} > 0$,

(A.3) $f_{kk} = \frac{\partial^2 f}{\partial k_t^2} < 0$ and $f_{gg} = \frac{\partial^2 f}{\partial g_t^2} < 0$;

(A.4) $f_{kg} = \frac{\partial^2 f}{\partial k_t \partial g_t} > 0$ and $f_g - k f_{kg} > 0$.

Assumption (A.1) states that both private and public capital are essential in production, i.e., output is null if either input is zero. Assumption (A.2) means that the marginal product of both private and public capital are positive whereas (A.3) ensures that the marginal products are decreasing for both private capital and public capital. Finally, assumption (A.4) implies that $0 < \eta_k^{f_g}(f, g) < 1$, for every pair (f, g), where $\eta_k^{f_g} = f_{gk} \frac{k}{f_g}$ denotes the elasticity of f_g with respect to k. This condition will be used in Section 3 but will be dropped in Section 4, where f_g will not be required to be inelastic with respect k.

In order to provide the public input at every date t, the agents in the economy devise a government that finances the public investment $I_t = G_{t+1} - (1 - \delta_t) G_t$ by means of lump-sum taxes during the youth age, where δ_t denotes the depreciation rate of the public capital. Thus, a government policy is an infinite sequence of taxes T_t which fulfills the budget constraint and leads to a sequence of levels of public investment I_t such that $T_t L_t = I_t$, equivalently, in per capita terms $(1 + n) g_{t+1} = T_t + (1 - \delta) g_t$, for each period t. For simplicity, we consider that both public and private capital fully depreciate after use.

As we have already remarked, the production function considered in this paper is linearly homogeneous. Then, applying Euler's formula, $F(K, L, G) = F_K K + F_L L + F_G G$. Thus, if firms would hire private capital and labor according to their marginal product the null profit equilibrium does not fulfil. Following Feehan and Batina (2007), let us consider an exogenous parameter $\gamma \in (0, 1)$ that specifies the share of the contribution to output from the public capital, $F_G G$, that goes to capital income. Then $1 - \gamma$ is the share of the contribution to output from the public capital that goes to private labor income[2]. Thus, the mechanism of price formation is defined by the following equalities which state the distribution

[2] We recall that, analyzing a different issue, namely, the problem of golden rule of public investment, Kellermann (2007) considers an overlapping generations model where the possible public factor rents are fully appropriated by the factor labor (i.e., $\gamma = 0$).

of the rents of public capital between the two private inputs:

$$(1+r_t)K_t = F_K(K_t, L_t, G_t)K_t + \gamma F_G(K_t, L_t, G_t)G_t \quad \text{and}$$
$$w_t L_t = F_L(K_t, L_t, G_t)L_t + (1-\gamma)F_G(K_t, L_t, G_t)G_t,$$

This price formation rule relies on the fact that public input rents are dissipated between private factors. This dissipation phenomenon leads to congestion in private factor markets since they are hired above their marginal products level. In addition, the characterization of rent-dissipation which precises the distribution of the public input rent between private capital and labor income allows us to get null profit equilibrium.

Since $F_k = f_k, F_G = f_g$ and $F_L = f - f_k k - \gamma f_g g$, in per-capita terms, the above expressions can be written as follows:

$$1 + r_t = f_k(k_t, g_t) + \gamma f_g(k_t, g_t)\tfrac{g_t}{k_t} \quad \text{and}$$
$$w_t = f(k_t, g_t) - f_k(k_t, g_t)k_t - \gamma f_g(k_t, g_t)g_t.$$

Note that the equalities above imply that $w_t + r_t k_t = f(k_t, g_t)$ and the impact that a modification of g_t has on prices r_t and w_t, respectively, is given by:

$$\frac{dr_t(\cdot)}{dg_t} = f_{kg}(\cdot) + \gamma f_{gg}(\cdot)\tfrac{g_t}{k_t} + \tfrac{\gamma}{k_t} f_g(\cdot) + \left(f_{kk}(\cdot) + \gamma f_{gk}(\cdot)\tfrac{g_t}{k_t} - \gamma f_g(\cdot)\tfrac{g_t}{k_t^2}\right)\frac{dk_t}{dg_t},$$
$$\frac{dw_t(\cdot)}{dg_t} = (1-\gamma)f_g(\cdot) - f_{gk}(\cdot)k_t - \gamma f_{gg} g_t - \left(f_{kk}(\cdot)k_t + \gamma f_{gk}(\cdot)g_t\right)\frac{dk_t}{dg_t}.$$

Definition 2.1 *Given $\gamma \in (0,1)$, an initial private capital level K_0 and an initial public capital level G_0, an equilibrium is defined as a sequence of allocation $\{c_t^y, c_{t+1}^o, k_t, g_t\}_{t=0}^{\infty}$, factor prices $\{w_t, r_t\}_{t=0}^{\infty}$ and lump-sum taxes $\{T_t\}_{t=0}^{\infty}$ such that:*

(i) Given the factor prices and the taxes, the allocation solves the maximization problem of each consumer;

(ii) given the allocation and taking into account γ, the factor prices are consistent with the firms' profit maximization,

(iii) the market for the consumption commodity clears at every date t; and

(iv) $s_t L_t = K_{t+1}$ and $T_t L_t = G_{t+1}$.

We now have a complete description of the economy and in which follows we will consider the corresponding variables in per-capita terms.

Observe that the accumulation expression for capital provided by the notion of equilibrium we address can be written as follows:

$$w_t - c_t^y(w_t, r_{t+1}, T_t) - (1+n)g_{t+1} = (1+n)k_{t+1}. \qquad (1)$$

Note that the equilibrium notion above is actually described by a sequence of $\{k_t, g_t\}_{t=0}^{\infty}$ under which $T_t = g_{t+1}(1+n)$ and $k_{t+1} = \dfrac{s_t\left(w_t, r_{t+1}\right)}{1+n}$. A steady-state equilibrium is a stationary private capital-labor ratio $\bar{k}$ and a stationary public capital-labor ratio $\bar{g}$ such that:

$$\bar{k} = \frac{s\left(f(\bar{k},\bar{g}) - f_k(\bar{k},\bar{g})\bar{k} - \gamma f_g(\bar{k},\bar{g})\bar{g}, f_k(\bar{k},\bar{g}) + \gamma f_g(\bar{k},\bar{g})\frac{\bar{g}}{\bar{k}}\right)}{1+n}$$

In order to analyze the effect that a modification in public capital investment may have in the stock of private capital we will address steady states $(\bar{k}, \bar{g})$ such that the absolute value of the point derivative of k_{t+1} with respect to k_t evaluated at $(\bar{k}, \bar{g})$ is smaller than one, that is, the next stability[3] property holds:

Stability Property (S): $\left|\dfrac{\partial k_{t+1}}{\partial k_t}(\bar{k}, \bar{g})\right| < 1$

By calculating the derivative of equation (1), which defines the accumulation for capital, with respect to k_t we can deduce:

$$\frac{\partial k_{t+1}}{\partial k_t}(\cdot) = -\frac{\left(1 - \frac{\partial c_t^y(\cdot)}{\partial w_t}\right)\left(f_{kk}(\cdot)k_t + \gamma f_{gk}(\cdot)g_t\right)}{1 + n + \frac{\partial c_t^y(\cdot)}{\partial r_{t+1}}\left(f_{kk}(\cdot) + \gamma \frac{g_{t+1}}{k_{t+1}^2}\left(f_{gk}(\cdot)k_{t+1} - f_g(\cdot)\right)\right)}$$

Let $\mathcal{D}$ denote the following function:

$$\mathcal{D}(\cdot) = \ 1 + n + \frac{\partial c_t^y(\cdot)}{\partial r_{t+1}}\left(f_{kk}(\cdot) + \gamma \frac{g_{t+1}}{k_{t+1}^2}\left(f_{gk}(\cdot)k_{t+1} - f_g(\cdot)\right)\right) + \left(1 - \frac{\partial c_t^y(\cdot)}{\partial w_t}\right)\left(f_{kk}(\cdot)k_t + \gamma f_{gk}(\cdot)g_t\right).$$

Note that if $1 + n > \left|\frac{\partial c_t^y(\cdot)}{\partial r_{t+1}}\right|\left(f_{kk}(\cdot) + \gamma \frac{g_{t+1}}{k_{t+1}^2}\left(f_{gk}(\cdot)k_{t+1} - f_g(\cdot)\right)\right)$, then $\mathcal{D}(\cdot)$ is greater than zero if and only if $\frac{\partial k_{t+1}}{\partial k_t}(\cdot) < 1$, provided that $\frac{\partial c_t^y(\cdot)}{\partial r_{t+1}} < 0$.

[3]See Galor and Ryder (1989) for a rigorous analysis of stability of equilibrium in an overlapping generations model with productive capital.

3 Main Results

Our aim now is to analyze how public capital investment affects private capital regarding the steady state.

For this, given a function $\mathcal{F}$ depending on (k,g), let $\eta_k^{\mathcal{F}}(\cdot)$ and $\eta_g^{\mathcal{F}}(\cdot)$ denote the elasticity of $\mathcal{F}$ with respect to k and g, respectively, i.e., $\eta_k^{\mathcal{F}}(\cdot) = \frac{\partial \mathcal{F}(\cdot)}{\partial k}\frac{k}{\mathcal{F}(\cdot)}$ and $\eta_g^{\mathcal{F}}(\cdot) = \frac{\partial \mathcal{F}(\cdot)}{\partial g}\frac{g}{\mathcal{F}(\cdot)}$. To simplify the notation, we write $\eta_k^{\mathcal{F}}$ $\eta_g^{\mathcal{F}}$ when the elasticities are evaluated in a steady state. Finally, let $\mathcal{A} = \eta_k^{f_g} + \gamma\left(1+\eta_g^{f_g}\right)$.

Consider the economy defined in the Section 2 under the hypotheses on consumption demands previously stated and assumptions (A.1)-(A.4). The next result provides a condition which is equivalent to ensure that if our starting point is a stable steady state, then an increase in public capital investment results in a decrease of private capital stock. As we will see below, this condition can be easily interpreted from an economic point of view.

Theorem 3.1 *Consider a steady state where the stability property (S) holds. Then, an increase of public capital stock depresses private capital stock if and only if the following condition holds:*

$$(1+n)k\left(1+\frac{\partial c^y}{\partial T}\right) > \left(1-\frac{\partial c^y(\cdot)}{\partial w}\right) k f_g \ (1-\mathcal{A}) + \left|\frac{\partial c^y(\cdot)}{\partial r}\right| f_g \mathcal{A}.$$

Proof. Let us consider a stable steady state equilibrium $\{w,r,k,g,T\}$. In this case, expression (1) becomes $w - c^y(w,r,T) - (1+n)g = (1+n)k$. Taking the derivative with respect the public capital g we obtain:

$$\frac{dk}{dg} = \frac{1}{\mathcal{D}} \ \Big[\left(1-\tfrac{\partial c^y}{\partial w}\right)\left(f_g - f_{kg}k - \gamma\left(f_g + f_{gg}g\right)\right) - \tfrac{\partial c^y}{\partial r}\left(f_{gk} + \gamma f_{gg}\tfrac{g}{k} + \tfrac{\gamma}{k} f_g\right) - (1+n)\left(1+\tfrac{\partial c^y}{\partial T}\right)\Big]$$

where $\mathcal{D}$ is given by

$$\mathcal{D} = 1+n+\frac{\partial c^y}{\partial r}\frac{f_k}{k}\left(\eta_k^{f_k} + \gamma\frac{g}{k}\frac{f_g}{f_k}\left(\eta_k^{f_g}-1\right)\right) + \left(1-\frac{\partial c^y}{\partial w}\right) f_k\left(\eta_k^{f_k} + \gamma\eta_g^{f_k}\right). \qquad (2)$$

Assumption (A.4) guarantees $f_{kk} + \gamma\frac{g}{k^2}\left(f_{gk}k - f_g\right) < 0$. Then, the stability condition (S) allows us to conclude that $\mathcal{D} > 0$ provided that $\frac{\partial c^y}{\partial r} < 0$.

Note that, using the notation $\mathcal{A} = \eta_k^{f_g} + \gamma\left(1 + \eta_g^{f_g}\right)$ previously stated, $\dfrac{dk}{dg}$ can be written as follows:

$$\frac{dk}{dg} = \frac{1}{\mathcal{D}}\left[\left(1 - \frac{\partial c^y}{\partial w}\right) f_g(1 - \mathcal{A}) + \left|\frac{\partial c^y}{\partial r}\right| \frac{f_g}{k}\mathcal{A} - (1+n)\left(1 + \frac{\partial c^y}{\partial T}\right)\right] \qquad (3)$$

Therefore, it is immediate to conclude that $\frac{dk}{dg} < 0$ if and only if the inequality in the statement of this theorem holds.

Q.E.D.

Note that equilibrium wages, interest rates and taxes depend crucially on the amounts of private and public capital stocks and on the parameter γ. Thus, a variation of public capital has an impact on the wage, on the interest rate and on the taxes which affect the consumption demands that in turn determine the amount of private capital stock. All these effects are collected in the equation (3) in the proof of Theorem 3.1. Actually, this expression (3) states that the impact of public capital investment on private capital stock can be separated into the following effects:

1. A *wage effect* (WE), given by $\frac{\left(1-\frac{\partial c^y}{\partial w}\right) f_g (1-\mathcal{A})}{\mathcal{D}}$. (4)
2. An *interest rate effect* (IE), given by $\frac{\left|\frac{\partial c^y}{\partial r}\right| \frac{f_g}{k}\mathcal{A}}{\mathcal{D}}$. (5)
3. A *tax effect* (TE), given by $-\frac{(1+n)\left(1+\frac{\partial c^y}{\partial T}\right)}{\mathcal{D}}$. (6)

The assumptions stated on preferences and technology joint with the stability condition allow us to deduce that $\mathcal{D}$ is positive. This implies that the tax effect is always negative. Therefore, the sufficient and necessary condition in Theorem 3.1 is just requiring the absolute value of the tax effect to be greater than the sum of the wage and interest rate effect. That is, the condition $|TE| > IE + WE$ is equivalent to the fact that a public capital investment leads to a reduction of the private capital stock.

The stability property (S) allows us to obtain that the wage effect is negative if and only if $\frac{\partial w}{\partial g} < 0$ which is equivalent to $1 < \mathcal{A}$. Moreover, the sign of IE is given by the sign of $\mathcal{A}$ and then IE is negative if and only if $\frac{\partial r}{\partial g}$ which is equivalent to $\mathcal{A} < 0$.

The sign of the tax effect leads us to deepen the analysis if we focus attention on wage and interest effects. Indeed, next we show how the equivalence result in Theorem 3.1 can be lightened by stating sufficient conditions, with interesting economic interpretations, which ensure that $\frac{dk}{dg} < 0$

Proposition 3.1 *Assume that the following conditions hold:*

(i) $\min\{\mathcal{A}, 1-\mathcal{A}\} < 0$ *and*

(ii) $\left(1 - \frac{\partial c^y(\cdot)}{\partial w}\right) k \ \ (\mathcal{A}-1) > \left|\frac{\partial c^y(\cdot)}{\partial r}\right| \mathcal{A}.$

Then, an increase of the public capital investment depresses the private capital stock whenever the starting point is a steady state where the stability property (S) holds.

Proof. Note that $\left(1 - \frac{\partial c^y}{\partial w}\right) f_g(1-\mathcal{A}) + \left|\frac{\partial c^y}{\partial r}\right| \frac{f_g}{k}\mathcal{A} < 0$ if and only if both condition (i) and condition (ii) hold. Therefore, from equation (3) in the proof of Theorem 3.1, it is immediate to conclude that, whenever *(i)* and *(ii)* hold then $\frac{dk}{dg} < 0$.

Q.E.D.

As we have already remarked, an investment in public capital induces a tax effect which is negative. However, the wage and interest rate effect induced by such investment cannot be negative at the same time. In spite of this, if either the wage effect or the interest rate effect is negative and has an absolute value greater than the other one, then we can conclude that the total effect is negative. This is the point which Proposition 3.1 shows. Note that the condition *(i)* in the statement of Proposition 3.1. is equivalent to require the ratio $(\mathcal{A}-1)/\mathcal{A}$ to be positive, which holds if and only if either $\mathcal{A} < 0$ or $\mathcal{A} > 1$. If $\mathcal{A} < 0$ then the interest rate effect is negative and the wage effect is positive whereas if $\mathcal{A} > 1$ then wage effect is negative and the interest rate effect is positive instead. Therefore, condition (i) ensures that either the wage effect or the interest rate effect is negative. On the other hand, once (i) is satisfied, condition (ii) guarantees that the wage effect plus the interest rate effect is negative.

We also remark that condition (i) depends only on the parameter γ and on the elasticities of the marginal product of the public capital regarding both private and public capital. This allow us to deepen the interpretation of this first condition. Precisely, since $\eta_g^{f_g} < 0, \ 0 < \eta_k^{f_g} < 1$ and $\gamma \in (0,1)$, we have that the following statements hold:

- $\mathcal{A} < 0$ (equivalently, $\frac{\partial r}{\partial g} < 0$) if and only if $\eta_k^{f_g} < \gamma\left(\left|\eta_g^{f_g}\right| - 1\right)$ which implies that $\left|\eta_g^{f_g}\right| > 1 + \eta_k^{f_g}$ (in particular, f_g is elastic with respect to the public input) for any $\gamma \in (0,1)$. Note also that given a technology f such that $\eta_k^{f_g} + 1 < \left|\eta_g^{f_g}\right|$, we have $\mathcal{A} < 0$ for any γ close enough to 1. We remark that the condition $\mathcal{A} < 0$ is equivalent to the fact that the ratio $\eta_k^{f_g} / \left(\left|\eta_g^{f_g}\right| - 1)\right)$ is a lower bound for γ. Actually, given γ, when either $\eta_k^{f_g}$ is small enough or $\left|\eta_g^{f_g}\right|$ is large enough we have $\mathcal{A} < 0$.

  $\mathcal{A} > 1$ (equivalently, $\frac{\partial w}{\partial g} < 0$) if and only if $1 - \eta_k^{f_g} < \gamma(1 + \eta_g^{f_g})$, which, independently of the value of γ, implies $\left|\eta_g^{f_g}\right| < \eta_k^{f_g}$ and in particular, the inelasticity property of the marginal product of public input with respect itself. Note also that given a technology f such that $\left|\eta_g^{f_g}\right| < \eta_k^{f_g}$, we have $\mathcal{A} > 1$ for any γ close enough to 1. We remark that $\mathcal{A} > 1$ whenever $\left|\eta_g^{f_g}\right| < \eta_k^{f_g} < 1$ and the ratio $\left(1 - \eta_k^{f_g}\right) / \left(1 - \left|\eta_g^{f_g}\right|\right)$ is a lower bound for γ. Actually, given γ, when $\eta_k^{f_g}$ is close enough to 1 we have $\mathcal{A} > 1$.

Therefore, $\mathcal{A} < 0$ requires f_g to be both elastic regarding g and more sensible to changes in the public input than to changes in the private capital whereas $\mathcal{A} > 1$ requires f_g to be both inelastic regarding g and more sensible to k than to g instead. Both $\mathcal{A} < 0$ and $\mathcal{A} > 1$ require a strictly positive lower bound for the parameter γ. Note also that the closer is γ to the unit the less demanding is either $\mathcal{A} < 0$ (when $\left|\eta_g^{f_g}\right| > 1$) or $\mathcal{A} > 1$ (when $\left|\eta_g^{f_g}\right| < 1$). That is, closer is γ to the unit the less demanding is condition *(i)* in the statement of Proposition 3.1. Next figure illustrate these observations for the case $\eta_k^{f_g} < 1$, which is implied by the assumption (A.4).

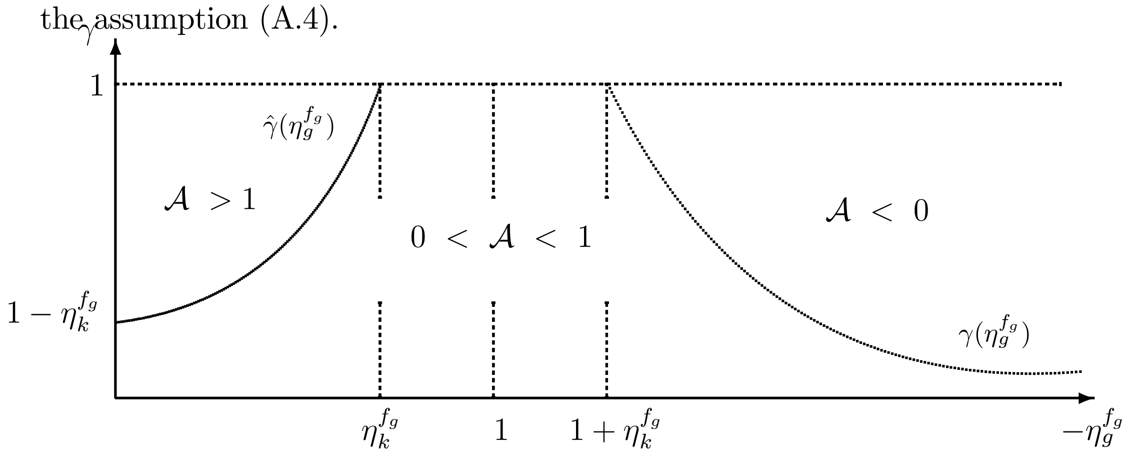

Figure 1. Illustration of the requirements on $\mathcal{A}$, with $\hat{\gamma}(\eta_g^{f_g}) = \frac{1-\eta_k^{f_g}}{1-|\eta_g^{f_g}|}$ and $\gamma(\eta_g^{f_g}) = \frac{\eta_k^{f_g}}{|\eta_g^{f_g}|-1}$

Note that when $\mathcal{A} < 0$, which implies $\eta_k^{f_g} < 1 < \left|\eta_g^{f_g}\right|$ and $\gamma > \frac{\eta_k^{f_g}}{|\eta_g^{f_g}|-1}$, any of the next situations ensures that *(ii)* holds:

- The private capital stock k is small enough.
- $\left|\dfrac{\partial c^y}{\partial r}\right|$ is large enough.
- $\dfrac{\partial c^y}{\partial w}$ is close enough to 1.

In contrast, when $\mathcal{A} > 1$, which implies $\eta_k^{f_g} > \left|\eta_g^{f_g}\right|$ and $\gamma > \frac{1-\eta_k^{f_g}}{1-|\eta_g^{f_g}|}$, any of the next situations ensures that *(ii)* holds:

- The private capital stock k is large enough.
- $\left|\dfrac{\partial c^y}{\partial r}\right|$ is small enough.

In this case, we may argue that the decreasing marginal products property support the intuition.

In short, the conditions stated in both Theorem 3.1 and Proposition 3.1, support the intuition that, the impact that investment in public input may have in the private capital stock depends crucially not only on consumption demands (which in turn are given by preferences) but also on γ (the parameter that defines the distribution of public capital stock rent between the private factors income) and on the elasticities of the marginal product of public capital with respect to both private and public capital (which are given by the technology). Actually, taking into account all the remarks above, we can deduce from Theorem 3.1 that precise conditions on demands, precise requirements on technology joint with conditions on the parameter γ defining the distribution of the public capital rents allow us to characterized the fact that $\dfrac{dk}{dg} < 0$. Moreover, Proposition 3.1 shows sufficient conditions which enlighten the equivalence result previously stated.

As a consequence of the sufficient conditions stated in Proposition 3.1., we can state the next necessary condition for $\dfrac{dk}{dg} > 0$.

Corollary 3.1 *If an increase of the public capital investment results in an increment of the private capital stock when the starting point is a steady state where the stability property (S) holds, then* $\left(1 - \frac{\partial c^y(\cdot)}{\partial w}\right) k(\mathcal{A} - 1) < \left|\frac{\partial c^y(\cdot)}{\partial r}\right| \mathcal{A}$.

Proof. It is an immediate consequence of the expression (3) for $\dfrac{dk}{dg}$ in the proof of Theorem 3.1.

Q.E.D.

Note that the above necessary condition is precisely $WE+IE > 0$. We remark that, under our assumptions $\mathcal{A} > 0$ (resp. $\mathcal{A} < 1$) implies $\frac{\partial r}{\partial g} > 0$ (resp. $\frac{\partial w}{\partial g} > 0$) and then $IR > 0$ (resp. $WE > 0$). Therefore, the positivity of $WE + IE$ holds whenever one of the following statement is satisfied:

- $0 < \mathcal{A} < 1$, equivalently $(\mathcal{A} - 1)/\mathcal{A} < 0$.
- $\mathcal{A} < 0$ and k large enough.
- $\mathcal{A} > 1$ and k small enough.

To sum up, observe that this necessary condition ensures that if either the interest rate effect or the wage effect is negative then the one that is positive has a greater impact than the negative one in absolute terms.

Remark. Let $y = f(k, g)$ be the per-capita output. Note that the sign of $\dfrac{dy}{dg}$ is equal to the sign of $\dfrac{dk}{dg} + \dfrac{f_g}{f_k}$. Therefore, the impact that public capital changes have on the private capital has important implications regarding how the per-capita output is affected by public investment. In particular, as it should be expected, when public investment results in an increase of the private capital stock we also have that the per-capita output increases. On the contrary, the fact that the per-capita output decreases with public investment requires a reduction of the private capital as well.

To finish this section we analyze the impact that the rent dissipation parameter parameter γ has on the steady state stock of capital. Precisely, we show that the sign of $\frac{dk}{d\gamma}$ is equal to the sign of $\left|\frac{\partial c^y}{\partial r}\right| - \left(1 - \frac{\partial c^y}{\partial w}\right) k$.

Proposition 3.2 *Consider a steady state where the stability property (S) holds. Then* $\frac{dk}{d\gamma} \geq 0$ *if and only if* $\left|\frac{\partial c^y}{\partial r}\right| \geq \left(1 - \frac{\partial c^y}{\partial w}\right) k$.

Proof. Given de equations that define the mechanism of price formation, the impact that a modification of γ has on prices is given by:

$$\begin{aligned} \frac{dr_t(\cdot)}{d\gamma} &= f_{kk}(\cdot)\frac{dk_t}{d\gamma} + f_g(\cdot)\frac{g_t}{k_t} + \gamma f_{gk}(\cdot)\frac{dk_t}{\gamma}\frac{g_t}{k_t} - \gamma f_g(\cdot)\frac{g_t}{k_t^2}\frac{dk_t}{d\gamma}, \\ \frac{dw_t(\cdot)}{d\gamma} &= -f_{kk}(\cdot)\frac{dk_t}{d\gamma}k - f_g(\cdot)g_t - \gamma f_{gk}\frac{dk_t}{d\gamma}g_t. \end{aligned}$$

Considering a steady state and taking the derivatives with respect γ in equation (1), we have

$$\frac{dw}{d\gamma} - \frac{\partial c^y}{\partial w}\frac{dw}{d\gamma} - \frac{\partial c^y}{\partial r}\frac{dr}{d\gamma} = (1+n)\frac{\partial k}{\partial \gamma}$$

Thus, we obtain

$$\frac{dk}{d\gamma} = -\frac{f_g g}{k\mathcal{D}}\left[\left(1 - \frac{\partial c^y}{\partial w}\right)k + \frac{\partial c^y}{\partial r}\right],$$

where $\mathcal{D}$ is defined as in expression (2).

As in the proof of Theorem 3.1, since $\frac{\partial c^y}{\partial r} < 0$, assumption (A.4) and the stability condition allow us to conclude that $\mathcal{D}$ is positive.

Q.E.D.

Therefore, from the above proposition we can conclude that $\frac{dk}{d\gamma} \leq 0$ for all k large enough whereas $\frac{dk}{d\gamma} \geq 0$ for all k small enough instead. That is, when we start from a small (resp. large) capital level, the impact that γ has on the capital stock via the wage dominates (resp. is lower than) the impact that it has via the interest rate.

4 Some Examples

In this Section, we state some examples which illustrate our main results and are useful for a better understanding of the assumptions.

Precisely, we analyze the model for a technology with constant elasticity of substitution (CES technology) and then we address the particular case of a Cobb-Douglas production function. In both situations, we consider preferences relation represented by the canonical utility function $U(c_t^y, c_{t+1}^o) = \ln c_t^y + \beta \ln c_{t+1}^o$, with $0 < \beta < 1$. This preferences lead to demands which satisfy the requirements stated in Section 2. Actually, given the houshold's budget constraints, the optimal choices are the following consumptions and savings:

$$c_t^y = \tfrac{1}{1+\beta}\left(w_t - T_t\right)$$

$$s_t = \tfrac{\beta}{1+\beta}\left(w_t - T_t\right)$$

Then, in this particular case, an interest rate modification does not alter consumption demand when young and then the interest rate effect is null. Therefore, in this scenario, the impact of public capital investment on private capital stock becomes explained just by a wage and a tax effect.

We remark that, as it might not be surprising, the second condition in assumption (A.4), which is used to obtain Theorem 3.1, is not required in the examples, neither for the CES nor for the Cobb Douglas production function. Thus, the fact that the interest rate effect is null allows us to relax requirements on technology and consider a larger set of production functions.

Let us consider the CES technology given in per-capita terms by the production function $f(k_t, g_t) = F(k_t, 1, g_t) = \left(ak_t^\rho + b + cg_t^\rho\right)^{1/\rho}$, with $0 \neq \rho < 1$ and $a + b + c = 1$. Some calculations show that both (i) $\dfrac{1-\gamma\rho}{\gamma - 1} > \dfrac{c}{b}g^\rho + \dfrac{a}{b}\rho k^\rho$ and (ii) $\rho < 0$ and $\dfrac{c}{a}\left(\dfrac{k}{g}\right)^{1-\rho} < \dfrac{\beta}{1+\beta}$ ensure that an increase of the public capital investment depresses the private capital stock whenever the starting point is a steady state where the stability property (S) holds.

Note that condition (i) guarantees a negative wage effect. Actually, this condition implicitly requires the elasticity of substitution between private and public capital has to be less than one implying a certain degree of complementarity between both types of capital.

On the other hand condition (ii) implies that the wage effect is lower than the absolute value of the tax effect. Indeed, *(ii)* requires $\rho < 0$ explicitly and does not depend on γ. Actually, this condition requires the marginal rate of technical substitution between private and public capital to be less than $\dfrac{\beta}{1+\beta}$, which is a ratio depending only on the discount factor defining the preference relation for consumption. Thus, in this case, given the discount factor β, it suffices to have a ratio k/g small enough in order obtain that an increase in public investment reduces the private capital stock. Alternatively, when $k < g$, in accordance with the economic intuition, a large enough $|\rho|$, which implies a sufficiently high degree of complementarity between both types of capital, ensures that $\dfrac{dk}{dg} < 0$.

Moreover, the larger is the discount factor for the consumption when old the less demanding is the condition *(ii)*. We remark that this particular case allows us to obtain a sufficient condition independently of the parameter γ but depending on preferences instead

Now, consider a Cobb-Douglas technology $F(K_t, L_t, G_t) = K_t^a L_t^b G_t^c$ with a, b, c strictly positive and $a + b + c = 1$. In per-capita terms: $f(k_t, g_t) = k_t^a g_t^c$.

In this case, taking into account the expressions of accumulation identity for the public capital stock $T_t = ((1+n)g_t$ and the accumulation rule for the private capital (1), the dynamic of both capital stocks is described as

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta}\left[(1-a-\gamma c)\, k_t^a g_t^c - T_t\right]$$

Figure 2 below illustrates a phase diagram for a constant public investment policy, represented by the line TT, which is given by $T_t = T$ for every t. By considering the constant policy and the steady state condition for private capital stock $k_{t+1} = k_t$, we an write the equation

$$g_t = \left(\frac{1}{1-a-\gamma c}\left((1+n)\frac{1+\beta}{\beta}k_t^{1-a} T k_t^{-a}\right)\right)^{1/c}$$

which is represented by te!'he line KK. Note that $\lim_{k_t \to 0} g_t = \lim_{k_t \to \infty} g_t = \infty$ and the minimum is attained at $k_m = \frac{aT(1+\beta)}{\beta(1-a)(1+n)}$.

In the Figure 2 we find two steady-state equilibria, namely, A and B. The dynamic patterns are illustrated by the arrows and lead us to ensure that the high private capital equilibrium B is the unique which is stable. Thus, k denotes the stable steady-state level of private capital stock.

To exemplify Proposition 3.2, note that if the contribution of the public capital stock that goes to the private capital income, γ, increases (declines) the TT line remains constant but the KK line shifts up (down). Then, in this particular situation we have $\frac{dk}{d\gamma} < 0$. This is so provided that, as we have already remarked, for the canonical utility function conidered in this example, the interest rate is null.

On the other hand, an increase in taxes (public investment) shifts up both TT and KK lines. Hence, the long run effect on the private capital stock of an increase in the public capital stock depends on the balance between the (positive) wage effect and the (negative) tax effect on savings. Actually, we obtain that if

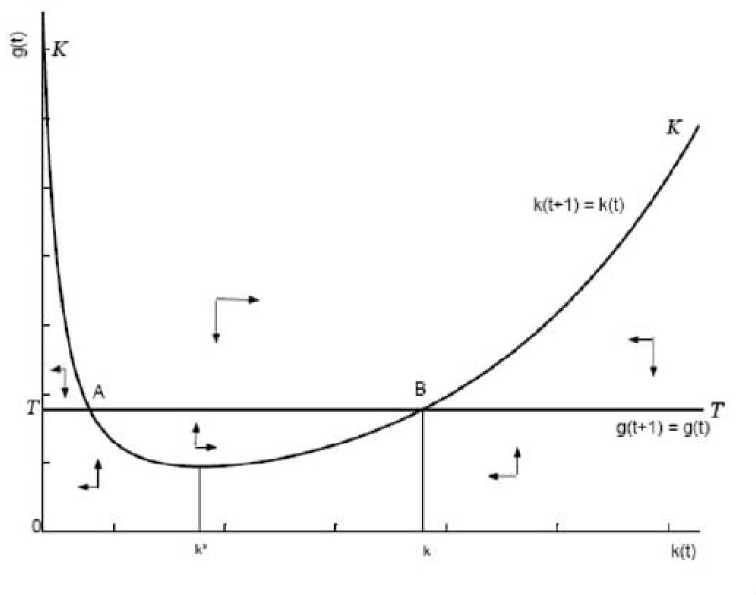

Figure 1: Phase diagram for Cobb-Douglas technology and canonical utility function

we consider a steady state which holds the stability condition (S) as a starting point, then $\beta(1-c)g > (1+\beta)ck$ if and only WE +TE is negative, which characterizes the fact that an increase of the public capital investment depresses the private capital stock.

Finally, regarding the remark stated at the end of section 3, we have that that public investment leads to a decrease of the per-capita output if and only if $\beta ag > (1+\beta)ck$, which implies the condition stated in the previous characterization[4].

In short, we conclude that the specification of our model to some preferences and technologies leads us to go into detail about the analysis of our main results.

5 Final Remarks

This paper is concerned with the issue of how public investment affects in the long run the amount of private capital stock. For that purpose we have considered an overlapping generations model where public capital stock enters in the production function in such way that causes a congestion problem. We have characterized

[4]Note that by taking the limit when ρ goes to zero the sufficient condition (ii) for the CES technology becomes $\beta ag > (1+\beta)ck$ which implies $\beta(1-c)g > (1+\beta)ck$.

this congestion with a rent dissipation scenario. This approach has allowed us to analyze the impact that a change in the amount of public capital investment has on the amount of private capital stock. The decomposition of this impact into wage, interest rate and tax effect makes clear the analysis.

The conditions stated in our main results support the intuition that, the total that investment in public input may have in the private capital stock depends crucially on the following issues:

- Consumption demands which in turn are given by preferences;
- Elasticities of the marginal product of public capital with respect to both private and public capital, which are given by the technology.
- The parameter γ that defines the distribution of public capita stock rent between the private factors income.

Actually, as we have shown, the rent dissipation parameter plays an important role in the different effects in which we have divided the total impact of public capital investment on the private capital stock.

Moreover, the examples presented in this manuscript not only illustrate the results but also strengthen them, since we can obtain the precise expressions of consumption demand for particular cases of preferences and the corresponding elasticities of the marginal product of inputs for the concrete technologies we have addressed.

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