Logic TK: Algebraic Notions from Tarski's Consequence Operator

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Abstract. Tarski presented his definition of consequence operator to explain the most important notions which any logical consequence concept must contemplate. A Tarski space is a pair constituted by a nonempty set and a consequence operator. This structure characterizes an almost topological space. This paper presents an algebraic view of the Tarski spaces and introduces a modal propositional logic which has as a model exactly the closed sets of a Tarski space.

Keywords: Tarski space, almost topological space, consequence operator, modal logic, algebraic model.

Introduction

This work is inserted into the tradition of the algebraic logic, particularly, in the Lindenbaum-Tarski style.

The contribution of this paper to the algebraic logic field is to present the concept of Tarski's consequence operator in an algebraic structure, the TK-algebra, and as well as to introduce a subnormal modal logic whose algebraic models are the counterpart of Tarski's consequence operator.

Thus, a new propositional logic is generated and its adequacy in relation to TK-algebras is shown.

1. Tarski spaces

As follows, we adopt the concept of consequence operator in a slightly more general way than it was introduced by Tarski, in 1935.

Definition 1.1. A *consequence operator* on E is a function $^{-} : \mathscr{P}(E) \to \mathscr{P}(E)$ such that, for every $A, B \in \mathscr{P}(E)$:

(i) $A \subseteq \overline{A}$;

(ii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$;

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(iii) $\overline{\overline{A}} \subseteq \overline{A}$.

From (i) and (iii), it follows that $\overline{\overline{A}} = \overline{A}$ holds, for every $A \subseteq E$.

Definition 1.2. A consequence operator $^{-} : \mathscr{P}(E) \to \mathscr{P}(E)$ is *finitary* when, for every $A \subseteq E$:

$$\overline{A} = \cup\{\overline{A_0} : A_0 \text{ is a finite subset of } A\}.$$

Definition 1.3. A *Tarski space* (*Tarski's deductive system* or *Closure space*) is a pair $(E,^{-})$ such that E is a nonempty set and $^{-}$ is a consequence operator on E.

Definition 1.4. Let $(E,^{-})$ be a Tarski space. The set A is *closed* in $(E,^{-})$ when $\overline{A} = A$, and A is *open* when its complement relative to E, denoted by A^C, is closed in $(E,^{-})$.

Proposition 1.5. *In $(E,^{-})$ any intersection of closed sets is also a closed set.*

Proof. If $\{A_i\}$ is a collection of closed sets, then $\cap_i A_i \subseteq \overline{\cap_i A_i} \subseteq \cap_i \overline{A_i} = \cap_i A_i$. Hence, $\cap_i A_i = \overline{\cap_i A_i}$. □

Clearly, $\overline{\varnothing}$ and E correspond to the least and the greatest closed sets, respectively, associated to the consequence operator $^{-}$.

Proposition 1.6. *In a structure of sets, the following conditions are equivalent:*

(i) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$;

(ii) $\overline{A} \subseteq \overline{A \cup B}$.

Proof. $(i) \Rightarrow (ii)$: As $A \subseteq A \cup B$, by (i), it follows that $\overline{A} \subseteq \overline{A \cup B}$. $(ii) \Rightarrow (i)$: If $A \subseteq B$, then $A \cup B = B$. So, by (ii), $\overline{A} \subseteq \overline{B}$. □

Definition 1.7. A Tarski space $(E,^{-})$ is *vacuous* when $\overline{\varnothing} = \varnothing$.

Definition 1.8. An *almost topological space* is a pair (S, Ω) such that S is a nonempty set and $\Omega \subseteq \mathscr{P}(S)$ satisfies the following condition:

$$B \subseteq \Omega \Rightarrow \cup B \in \Omega.$$

The collection Ω is called *almost topology* and each member of Ω is an *open* of (S, Ω). A set $A \in \mathscr{P}(S)$ is *closed* when its complement relative to S is an open of (S, Ω).

Proposition 1.9. *In an almost topological space (S, Ω) the set $\varnothing$ is open and S is closed.*

Proposition 1.10. *In an almost topological space (S, Ω), any intersection of closed sets is still a closed set.*

Definition 1.11. Let (S,Ω) be an almost topological space. The *closure* of A is the set:

$$\overline{A} =_{\mathrm{df}} \cap\{X \subseteq S : X \text{ is closed and } A \subseteq X\}.$$

Proposition 1.12. *Let (S,Ω) be an almost topological space. For every $A \subseteq S$, $\overline{A}$ is closed.*

Proposition 1.13. *Let (S,Ω) be an almost topological space and let $\overline{A}$ be defined as above, for every $A \subseteq S$. Then $(S,^{-})$ is a Tarski space.*

On the other hand, if $(E,^{-})$ is a Tarski space, let us consider $\Omega = \{X \subseteq E : X\ is\ open\}$.

Proposition 1.14. *If $(E,^{-})$ is a Tarski space, then (E,Ω) is an almost topological space.*

It follows from Propositions 1.13 and 1.14 that that given a Tarski space an almost topological space is obtained and in another direction given an almost topological space we can define a Tarski space. So naturally we can make an interrelation between the two concepts.

Definition 1.15. An almost topological space (S,Ω) is *0-closed* when it holds:

(iv) $\overline{\varnothing} = \varnothing$.

Definition 1.16. A *topological space* (S,Ω) is an almost topological space 0-closed such that it holds:

(v) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The previous definition of topological space was given by the Kuratowski's closure. Naturally, every topological space is a Tarski space, but there are several Tarski spaces which are not topological spaces. Each topological space is an instance of a vacuous Tarski space.

2. TK-algebras

The definition of a TK-algebra introduces the notions of consequence operator in the context of the algebraic structures.

Definition 2.1. A *TK-algebra* is a sextuple $\mathscr{A} = (A,0,1,\vee,\sim,\bullet)$ such that $(A,0,1,\vee,\sim)$ is a Boolean algebra and $\bullet$ is a new operator, called *operator of Tarski*, such that:

(i) $a \vee \bullet a = \bullet a$;

(ii) $\bullet a \vee \bullet(a \vee b) = \bullet(a \vee b)$;

(iii) $\bullet(\bullet a) = \bullet a$.

Examples:

(a) The space of sets $\mathscr{P}(A)$ with $A \neq \varnothing$ and $\bullet a = a$, for all $a \in A$, is a TK-algebra.
(b) The space of sets $\mathscr{P}(\mathbb{R})$ with $\bullet X = X \cup \{0\}$ is a TK-algebra.
(c) The space of sets $\mathscr{P}(\mathbb{R})$ with $\bullet X = \cap\{I \mid I$ is an interval and $X \subseteq I\}$ is a TK-algebra.

Since we are working with Boolean algebras, the item (i) of the Definition 2.1 asserts that, for every $a \in A$, $a \leq \bullet a$.

We define in a TK-algebra:

$$a \rightarrowtail b =_{\text{df}} \sim a \vee b;$$
$$a - b =_{\text{df}} a \wedge \sim b.$$

Proposition 2.2. *In any TK-algebra the following conditions are valid:*

(i) $\sim\bullet a \leq \sim a \leq \bullet\sim a$;

(ii) $a \leq b \Rightarrow \bullet a \leq \bullet b$.

Proof. (ii) $a \leq b \Rightarrow a \vee b = b \Rightarrow \bullet(a \vee b) = \bullet b \Rightarrow \bullet a \vee \bullet b = \bullet a \vee \bullet(a \vee b) = \bullet(a \vee b) = \bullet b \Rightarrow \bullet a \leq \bullet b$. □

Proposition 2.3. *In any TK-algebra the following assertions are valid:*

(i) $\bullet(a \wedge b) \leq \bullet a \wedge \bullet b$;

(ii) $\bullet a \vee \bullet b \leq \bullet(a \vee b)$.

Proof. (i) $a \wedge b \leq a$ and $a \wedge b \leq b \Rightarrow \bullet(a \wedge b) \leq \bullet a$ and $\bullet(a \wedge b) \leq \bullet b \Rightarrow \bullet(a \wedge b) \leq \bullet a \wedge \bullet b$.
(ii) is similar to (i). □

Example:

(a) Let $E = \{a, b, c\}$. The space of sets $(\mathscr{P}(E), \varnothing, E, \cap, \cup, {}^C)$ can be extended to a TK-algebra in the following way: $\bullet X = X$, for all $X \subseteq E$ such that $X \neq \{a, b\}$, and $\bullet\{a, b\} = E$.

If $X = \{a\}$ and $Y = \{b\}$, then $\bullet X = \bullet\{a\} = \{a\}$ and $\bullet Y = \bullet\{b\} = \{b\}$; $\bullet X \cup \bullet Y = \{a, b\}$; $\bullet(X \cup Y) = \bullet\{a, b\} = E$. Hence, $\bullet X \cup \bullet Y \subset \bullet(X \cup Y)$.

If $X = \{a, b\}$ and $Y = \{c\}$, then $\bullet X = \bullet\{a, b\} = E$, $\bullet Y = \bullet\{c\} = \{c\}$, $\bullet X \cap \bullet Y = \{c\}$, $\bullet(X \cap Y) = \bullet\varnothing = \varnothing$. Hence, $\bullet(X \cap Y) \subset \bullet X \cap \bullet Y$.

Proposition 2.4. *In any TK-algebra, it holds:*

(i) $\bullet(\bullet a \wedge \bullet b) = \bullet a \wedge \bullet b$;

(ii) $\bullet(\bullet a \vee \bullet b) = \bullet(a \vee b)$;

(iii) $\bullet a \rightarrowtail \bullet b \leq \bullet(a \rightarrowtail b)$.

Proof. (i) It is enough to verify that $\bullet(\bullet a \wedge \bullet b) \leq \bullet a \wedge \bullet b$. But, $\bullet(\bullet a \wedge \bullet b) \leq \bullet\bullet a \wedge \bullet\bullet b = \bullet a \wedge \bullet b$.
(ii) $\bullet(a \vee b) \leq \bullet(\bullet a \vee \bullet b) \leq \bullet\bullet(a \vee b) = \bullet(a \vee b)$. Hence, $\bullet(a \vee b) = \bullet(\bullet a \vee \bullet b)$.
(iii) $\bullet a \rightarrowtail \bullet b = \sim\bullet a \vee \bullet b \leq \bullet\sim a \vee \bullet b \leq \bullet(\sim a \vee b) = \bullet(a \rightarrowtail b)$. □

We have a new operation in a TK-algebra, dual of $\bullet$:

$$\circ a =_{\text{df}} \sim\bullet\sim a.$$

Proposition 2.5. *In a TK-algebra, the following conditions are valid:*

(i) $\circ a \leqslant a$;

(ii) $a \leq b \Rightarrow \circ a \leq \circ b$;

(iii) $\circ(a \wedge b) \leq \circ a$;

(iv) $\circ a \leq \circ\circ a$.

Proof. (i) $\circ a = \sim\bullet\sim a \Rightarrow \sim a \leq \bullet\sim a = \sim\circ a \Rightarrow \circ a \leq \sim\sim a = a$.
(ii) $a \leq b \Rightarrow \sim b \leq \sim a \Rightarrow \bullet\sim b \leq \bullet\sim a \Rightarrow \sim\bullet\sim a \leq \sim\bullet\sim b \Rightarrow \circ a \leq \circ b$.
(iii) It follows from (ii).
(iv) $\circ a \leq a \Rightarrow \sim a \leq \sim\circ a \Rightarrow \bullet\sim a \leq \bullet\sim\circ a \Rightarrow \sim\bullet\sim\circ a \leq \sim\bullet\sim a \Rightarrow \circ\circ a \leq \circ a$. □

Definition 2.6. An element $a \in A$ is *closed* when $\bullet a = a$, and $a \in A$ is *open* when $\circ a = a$.

Proposition 2.7. *In any TK-algebra:*

(i) *If a is open, then:* $a \leq b \Leftrightarrow a \leq \circ b$;

(ii) *If b is closed, then:* $a \leq b \Leftrightarrow \bullet a \leq b$.

Definition 2.8. An algebra $\mathscr{A}$ is *non-degenerate* when its universe A has at least two elements.

Definition 2.9. Let $\mathscr{A} = (A, 0, 1, \vee, \sim, \bullet)$ and $\mathscr{B} = (B, 0, 1, \vee, \sim, \bullet)$ be TK-algebras. A *homomorphism* between $\mathscr{A}$ and $\mathscr{B}$ is a function $h : A \longrightarrow B$ that preserves the TK-operations. The *kernel* of h is the set $Ker(h) =_{\text{df}} \{x \in A : h(x) = 0\} = h^{-1}(0)$.

Theorem 2.10. *For each TK-algebra $\mathscr{A} = (A, 0, 1, \vee, \sim, \bullet)$ there is a monomorphism h from A into a Tarski space of sets defined in $\mathscr{P}(\mathscr{P}(A))$.*

Proof. Through Stone's isomorphism, we know that for each Boolean algebra $\mathscr{A} = (A, 0, 1, \vee, \sim)$ there is a monomorphism h from A into a field of subsets of $\mathscr{P}(A)$.

Next, we introduce a Tarski space in $\mathscr{P}(A)$ in the following way:
For each set $X \subseteq \mathscr{P}(A)$, we define:

$$\overline{X} = \cap_{a \in A}\{h(a) : X \subseteq h(a) \text{ and } a = \bullet a\}.$$

Then, we must show that:

(i) $X \subseteq \overline{X}$

(ii) $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$

(iii) $\overline{\overline{X}} \subseteq \overline{X}$

(iv) $h(\bullet a) = \overline{h(a)}$.

We can see that (i) - (iv) are valid by:

(i) By definition of $\overline{X}$.

(ii) Suppose $\overline{X} \nsubseteq \overline{Y}$. Then there is z such that $z \in \overline{X}$ and $z \notin \overline{Y}$. So, for some $a \in A$, $z \notin h(a)$ with $Y \subseteq h(a)$ and $\bullet a = a$. Since $X \subseteq Y \subseteq h(a)$ and $\bullet a = a$, then $z \notin \overline{X}$, and it contradicts $z \in \overline{X}$.

(iii) $x \in \overline{\overline{X}} \Rightarrow x \in \cap_{a \in A}\{h(a) : \overline{X} \subseteq h(a) \text{ and } a = \bullet a\}$. As $X \subseteq \overline{X}$, it follows that $x \in \cap_{a \in A}\{h(a) : X \subseteq h(a) \text{ and } a = \bullet a\} = \overline{X}$ and, therefore, $\overline{\overline{X}} \subseteq \overline{X}$.

(iv) On one hand, $h(\bullet a) \subseteq \overline{h(a)}$ is valid: Consider that $h(a) \subseteq h(b)$ and $b = \bullet b$. Since h is a Boolean monomorphism, $a \leq b$ and $\bullet a \leq \bullet b$. But, since $\bullet b = b$, then $\bullet a \leq b$ and $h(\bullet a) \subseteq h(b)$. Concluding, for each $b \in A$ such that $\bullet b = b$ and $h(a) \subseteq h(b)$, it results that $h(\bullet a) \subseteq h(b)$, that is, $h(\bullet a) \subseteq \overline{h(a)}$.

On the other hand, $\overline{h(a)} \subseteq h(\bullet a)$ is also valid: $a \leq \bullet a \Rightarrow h(a) \subseteq h(\bullet a) \Rightarrow \overline{h(a)} \subseteq \overline{h(\bullet a)}$. Since $h(\bullet a) \subseteq h(\bullet a)$ and $\bullet a = \bullet\bullet a$, then $\overline{h(\bullet a)} = h(\bullet a)$ and $\overline{h(a)} \subseteq h(\bullet a)$. □

3. Ideals in TK-algebras

As in Boolean algebras, we can define an ideal into TK-algebras and use it to analyze aspects of the consequence in the next sections.

Definition 3.1. Let $\mathscr{A} = (A, 0, 1, \vee, \sim, \bullet)$ be a TK-algebra. An *ideal* in A is a nonempty set $I \subseteq A$ such that, for all $x, y \in A$:

(i) $x, y \in I \Rightarrow x \vee y \in I$;

(ii) $x \in I$ and $y \leq x \Rightarrow y \in I$.

If I is an ideal and $a_1, a_2, \cdots, a_n \in I$, by induction on n, we have that $a_1 \vee a_2 \vee \cdots \vee a_n \in I$.

Definition 3.2. The ideal I is a *TK-ideal* when $x \in I \Rightarrow \bullet x \in I$.

Examples:

(a) The set A is a TK-Ideal in $\mathscr{A}$.
(b) The single $\{0\}$ is a TK-ideal if, and only if, $\bullet 0 = 0$.
(c) Given $a \in A$, the set $[a] = \{x \in A : x \leq \bullet a\}$ is a TK-Ideal. The set $[a]$ is the TK-Ideal *generated* by a.

Proposition 3.3. *Let $\mathscr{A}$ be a TK-algebra and I be an ideal in $\mathscr{A}$. The following conditions are equivalent:*

(i) $a \in I \Rightarrow \bullet a \in I$;

(ii) $a \rightarrowtail b \in I \Rightarrow \bullet a \rightarrowtail \bullet b \in I$.

Proof. ($\Rightarrow$) $a \rightarrowtail b \in I \Rightarrow \bullet(a \rightarrowtail b) \in I$. Since $\bullet a \rightarrowtail \bullet b \leq \bullet(a \rightarrowtail b)$, then $\bullet a \rightarrowtail \bullet b \in I$.
($\Leftarrow$) If $a \in I$, as $a = 1 \rightarrowtail a$, then $1 \rightarrowtail a \in I$. By (ii), $\bullet 1 \rightarrowtail \bullet a \in I \Rightarrow 1 \rightarrowtail \bullet a \in I \Rightarrow \bullet a \in I$. □

Proposition 3.4. *Let $\mathscr{A}$ be a TK-algebra and $\emptyset \neq B \subseteq A$. The set $[B] = \{x \in A : (\exists a_1, \ldots, a_n \in B)(x \leq \bullet(a_1 \vee \ldots \vee a_n)\}$ is a TK-ideal.*

Proof. (i) If $x, y \in [B]$, then there are $a_1, \ldots, a_n, b_1, \ldots, b_n \in B$ such that $x \leq \bullet(a_1 \vee \ldots \vee a_n)$ and $y \leq \bullet(b_1 \vee \ldots \vee b_n)$. So $x \vee y \leq \bullet(a_1 \vee \ldots \vee a_n) \vee \bullet(b_1 \vee \ldots \vee b_n) \leq \bullet(a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_n)$ and $x \vee y \in [B]$. (ii) If $x \in [B]$ and $y \leq x$, then $y \in [B]$. (iii) If $x \in [B]$, then there exists $a_1, \ldots, a_n \in B$ such that $x \leq \bullet(a_1 \vee \ldots \vee a_n)$ and since $\bullet(a_1 \vee \ldots \vee a_n)$ is closed, by Proposition 2.7, $\bullet x \leq \bullet(a_1 \vee \ldots \vee a_n)$ and $\bullet x \in [B]$. □

Definition 3.5. TK-ideal $[B]$ defined in the Proposition 3.4 is the TK-ideal *generated* by B.

Proposition 3.6. *Let $\mathscr{A}$ be a TK-algebra. If I is a TK-Ideal in $\mathscr{A}$ and $b \in A$, then*

$$[I, b] = \{x \in A : (\exists c \in I)(\bullet x \leq \bullet(b \vee c))\}$$

is a TK-Ideal.

Proof. Of course, if $x \in [I, b]$ and $y \leq x$, then $y \in [I, b]$; and $x \in [I, b] \Rightarrow \bullet x \in [I, b]$.

Now, if $x, y \in [I, b]$, there are $c, d \in I$ such that $\bullet x \leq \bullet(b \vee c)$ and $\bullet y \leq \bullet(b \vee d)$. Hence, $\bullet(x \vee y) = \bullet(\bullet x \vee \bullet y) \leq \bullet(\bullet(b \vee c) \vee \bullet(b \vee d)) = \bullet((b \vee c) \vee (b \vee d)) = \bullet(b \vee (c \vee d))$. Therefore, $x \vee y \in [I, b]$. □

Definition 3.7. The ideal $[I, b]$ is the TK-ideal *generated* by I and b.

Definition 3.8. Let I be an ideal in a TK-algebra $\mathscr{A}$. The ideal I is *proper* when $I \neq A$. The ideal I is *maximal* when it is proper and it is not included in any proper ideal distinct of I. The ideal I is *prime* when it is proper and for all $a, b \in A$ it holds: $a \wedge b \in I \Rightarrow a \in I \text{ or } b \in I$.

Of course, if I is proper, then there is an $a \in A$ such that $a \notin I$ and therefore $1 \notin I$.

Proposition 3.9. *Let I be an ideal in a TK-algebra $\mathscr{A}$. The following statements are equivalent:*

(i) *I is maximal;*

(ii) *for every $a \in A$: $a \in I \veebar {\sim} a \in I$;*

(iii) *I is prime;*

(iv) *for every $a, b \in A$, $a - b \in I$ or $b - a \in I$.*

Proof. As $\mathscr{A}$ is a Boolean algebra (i), (ii) and (iii) are equivalent. Let's show the equivalence between (iii) and (iv). (iii) $\Rightarrow$ (iv) Let $a, b \in A$. As $(a - b) \wedge (b - a) = 0 \in I$ and I is prime, then $a - b \in I$ or $b - a \in I$. (iv) $\Rightarrow$ (iii) Let $a \wedge b \in I$. If $a \wedge {\sim} b = a - b \in I$, then $a = (a \wedge b) \vee (a \wedge {\sim} b) \in I$. If $b \wedge {\sim} a = b - a \in I$, then $b \in I$. □

The next definitions are specific for TK-ideals.

Definition 3.10. A TK-Ideal I is *TK-irreducible* when it is proper and for any two TK-Ideals I_1 and I_2:

$$I = I_1 \cap I_2 \Rightarrow I = I_1 \text{ or } I = I_2.$$

Definition 3.11. A TK-Ideal I is *TK-maximal* when it is proper and it is not included in any proper TK-Ideal distinct of I.

Definition 3.12. Let I be a TK-ideal in a TK-algebra $\mathscr{A}$. The ideal I is *TK-prime* when it is a proper ideal and for all $a, b \in A$ it holds:

$$\bullet a \wedge \bullet b \in I \Rightarrow \bullet a \in I \text{ or } \bullet b \in I.$$

Let I be a TK-ideal. If I is a prime ideal then I is a TK-prime ideal. However, it is possible that I is a TK-prime ideal that is not prime. The same holds to TK-maximal ideals. Besides, being maximal does not imply to be TK-ideal and hence to be TK-maximal ideal.

Example:

(a) Let $E = \{a, b, c\}$. Consider the TK-algebra $\mathscr{A} = (\mathscr{P}(E), \varnothing, E, \cap, \cup, {}^{C}, \bullet)$, such that $\bullet\varnothing = \varnothing, \bullet\{a\} = \{a\}, \bullet\{b\} = \bullet\{c\} = \bullet\{a, b\} = \bullet\{a, c\} = \bullet\{b, c\} = \bullet\{a, b, c\} = \{a, b, c\}$. The TK-ideal $\{\varnothing, \{a\}\}$ is TK-prime but it is not prime.

Proposition 3.13. *If a TK-ideal is TK-maximal, then it is TK-irreducible.*

Proof. Let I be a TK-maximal ideal. So I is proper. Now, if I_1 and I_2 are two proper TK-ideals such that $I = I_1 \cap I_2$, then $I \subseteq I_1$ and $I \subseteq I_2$. As I is TK-maximal, so $I = I_1 = I_2$. □

Example:

(a) Let $E = \{a, b\}$. Consider the TK-algebra $\mathscr{A} = (\mathscr{P}(E), \varnothing, E, \cap, \cup, {}^{C}, \bullet)$, such that $\bullet\varnothing = \varnothing, \bullet\{a\} = \{a\}, \bullet\{b\} = \bullet\{a, b\} = \{a, b\}$. The TK-ideal $\{\varnothing\}$ is TK-irreducible and it is contained in the TK-ideal maximal $\{\varnothing, \{a\}\}$.

It follows from the previous example that we have a TK-irreducible ideal which is not TK-maximal.

Proposition 3.14. *Let $\mathscr{A}$ be a TK-algebra and I a TK-ideal in $\mathscr{A}$. The following statements are equivalent:*

(i) *I is a TK-prime ideal;*

(ii) *for every $a \in A$: either $\bullet a \in I$ or $\sim\bullet a \in I$.*

Proof. (i) ⇒ (ii) If I is TK-prime, then I is proper. Now, if $\bullet a \in I$ and $\sim\bullet a \in I$, then $\bullet a \vee \sim\bullet a = 1 \in I$, which is a contradiction. Besides, as I is prime and $\bullet a \wedge \sim\bullet a = 0 \in I$, then $\bullet a \in I$ or $\sim\bullet a \in I$.
(ii) ⇒ (i) By hypothesis, I is proper. If $\bullet a \wedge \bullet b \in I$ and $\bullet a \notin I$, then, by hypothesis, $\sim\bullet a \in I$. Hence, $\bullet b \leq \bullet b \vee \sim\bullet a = 1 \wedge (\bullet b \vee \sim\bullet a) = (\bullet a \vee \sim\bullet a) \wedge (\bullet b \vee \sim\bullet a) = (\bullet a \wedge \bullet b) \vee \sim\bullet a \in I$. Since I is a TK-ideal, then $\bullet b \in I$. Therefore, I is a TK-prime ideal. □

Proposition 3.15. *If a TK-ideal is TK-prime, then it is TK-maximal.*

Proof. Consider I as a TK-prime ideal which is properly included in a TK ideal M and take x such that $x \in M$, but $x \notin I$. Since I is a TK-ideal, then $\bullet x \notin I$ and since M is a TK-ideal, $\bullet x \in M$. As I is a TK-prime ideal, by Proposition 3.14, $\sim\bullet x \in I$. Hence, $\bullet x \in M$, $\sim\bullet x \in I \subseteq M$, therefore $1 = \bullet x \vee \sim\bullet x \in M$, that is, $M = A$. □

Corollary 3.16. *If an TK-ideal is TK-prime, then it is TK-irreducible.*

Proof. It follows from Propositions 3.15 and 3.13. □

Example:

(a) Let $E = \{a, b, c\}$. Consider the TK-algebra $\mathscr{A} = (\mathscr{P}(E), \varnothing, E, \cap, \cup, {}^{C}, \bullet)$, such that $\bullet\varnothing = \varnothing$, $\bullet\{a\} = \{a\}$, $\bullet\{b\} = \{b\}$, $\bullet\{c\} = \{c\}$, $\bullet\{a, b\} = \{a,\ b\}$, $\bullet\{a,\ c\} = \bullet\{b,\ c\} = \bullet\{a,\ b,\ c\} = \{a,\ b,\ c\}$. The TK-ideal $I = \{\varnothing, \{c\}\}$ is TK-maximal, but it is not TK-prime:

(i) I is TK-maximal: if $I \subset J$ with J a TK-ideal, then J has an element x such that $x \notin I$. If $x = \{a\}$, then $\{a, c\} = \{a\} \cup \{c\} \in J$, and like J is a TK-ideal, then $\{a, b, c\} = \bullet\{a, c\} \in J$ and therefore $J = A$. If $x = \{b\}$ or $x = \{a, b\}$ or $x = \{a, c\}$ or $x = \{b, c\}$ or even $x = \{a, b, c\}$, we show that $J = A$.

(ii) I is not TK-prime: $\bullet\{a\} \wedge \bullet\{b\} = \varnothing \in I$, but $\bullet\{a\} \notin I$ and $\bullet\{b\} \notin I$.

Definition 3.17. A *chain* of ideals is a sequence $(I_1, I_2, I_3, \dots)$ of ideals such that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$.

Lemma 3.18. *Let $\mathscr{A}$ be a TK-algebra and $(I_1, I_2, I_3, \dots)$ be a chain of proper TK-ideals of $\mathscr{A}$. The union $\cup I_n$ is a proper TK-ideal.*

Proof. Let $x \in I$ and $y \in A$, with $y \leq x$. Since $I = \cup I_n$, there is $n \in \mathbb{N}$ such that $x \in I_n$, and since I_n is a TK-ideal, then $y \in I_n \subseteq I$. Let $x, y \in I = \cup I_n$. Since $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$, there is $n \in \mathbb{N}$ such that $x, y \in I_n$, and as I_n is a TK-ideal, so $x \vee y \in I_n \subseteq I$. Hence, I is an ideal. Since $1 \notin I_n$, for every n, hence $1 \notin I$. Therefore, I is a proper ideal. It is immediate that I is a TK-ideal. □

Theorem 3.19. *Each proper TK-ideal in a TK-algebra is contained in a TK-maximal TK-ideal.*

Proof. The result follows from the previous lemma and by Zorn's Lemma. □

Corollary 3.20. *Let $\mathscr{A}$ be a TK-algebra and $a \in A$ such that $\bullet\, a \neq 1$. Then, there is a maximal TK-ideal I for which $a \in I$.*

Proof. Since $\bullet\, a \neq 1$, the TK-ideal generated by a is proper, then by previous theorem, $[a]$ is included in a TK-maximal TK-ideal I and $a \in I$. □

Proposition 3.21. *Let $\mathscr{A}$ be a TK-algebra and I be a TK-ideal in $\mathscr{A}$. If $a \notin I$, then there is a TK-irreducible TK-ideal I^* such that $I \subseteq I^*$ and $a \notin I^*$.*

Proof. Let S be the ordered set of all TK-ideals J of $\mathscr{A}$ such that $I \subseteq J$ and $a \notin J$. By Zorn's Lemma and Lemma 3.18, there is a TK-maximal element M in S. If $M = I_1 \cap I_2$, considering that $a \notin M$, then $a \notin I_1$ or $a \notin I_2$. If $a \notin I_1$, since $I \subseteq M \subseteq I_1$, then $I_1 \in S$ and as M is TK-maximal in S, $M = I_1$. Besides, if $a \notin I_2$, $M = I_2$. Hence, M is TK-irreducible. □

Theorem 3.22. *Let $h : A \longrightarrow B$ be a surjective homomorphism between $\mathscr{A} = (A, 0, 1, \vee, \sim, \bullet)$ and $\mathscr{B} = (B, 0, 1, \vee, \sim, \bullet)$. Then:*

(i) $h(a) = h(b) \Leftrightarrow a \equiv_{Ker(h)} b \Leftrightarrow_{df} a - b \in Ker(h)$ *and* $b - a \in Ker(h)$*;*

(ii) *the relation* $\equiv_{Ker(h)}$ *is a congruence;*

(iii) $\mathscr{A}|_{Ker(h)}$ *is a TK-algebra;*

(iv) $\mathscr{A}|_{Ker(h)} \approx \mathscr{B}$.

Proof. (i) ($\Rightarrow$) $h(a) = h(b) \Rightarrow h(a) \wedge \sim h(b) = 0 \Rightarrow h(a \wedge \sim b) = 0 \Rightarrow a \wedge \sim b \in ker(h) \Rightarrow a - b \in ker(h)$. In the same way, $b - a \in ker(h)$. ($\Leftarrow$) $a - b \in ker(h) \Rightarrow a \wedge \sim b \in ker(h) \Rightarrow h(a \wedge \sim b) = 0 \Rightarrow h(a) \wedge \sim h(b) = 0 \Rightarrow h(a) \vee h(b) = h(b) \Rightarrow h(a) \leq h(b)$. Also, $b - a \in ker(h) \Rightarrow h(b) \leq h(a)$. So, $h(a) = h(b)$.
(ii) Clearly $\equiv_{Ker(h)}$ is an equivalence relation. We only need to show the details about how to extend to a TK-algebra, that is, $a \equiv_{Ker(h)} b \Rightarrow \bullet a \equiv_{Ker(h)} \bullet b$. So $a \equiv_{Ker(h)} b \Leftrightarrow h(a) = h(b) \Rightarrow \bullet h(a) = \bullet h(b) \Rightarrow h(\bullet a) = h(\bullet b) \Leftrightarrow \bullet a \equiv_{Ker(h)} \bullet b$.
(iii) Since $\equiv_{Ker(h)}$ is an equivalence we have a partition on $\mathscr{A}|_{Ker(h)}$. Now, let $^{-} : \mathscr{A} \longrightarrow \mathscr{A}|_{Ker(h)}$ such that $\overline{a} = \{b \in A : a \equiv_{Ker(h)} b\}$. We must produce a Tarski operator in $\mathscr{A}|_{Ker(h)}$. Let $\bullet\overline{a} = \overline{\bullet a}$. This is the looked operator: (1) $\overline{a} \vee \overline{\bullet a} = \overline{a \vee \bullet a} = \overline{\bullet a} = \bullet\overline{a}$; (2) $\bullet\overline{a} \vee \bullet(\overline{a} \vee \overline{b}) = \bullet\overline{a} \vee \bullet\overline{(a \vee b)} = \overline{\bullet a} \vee \overline{\bullet(a \vee b)} = \overline{\bullet a \vee \bullet(a \vee b)} = \overline{\bullet(a \vee b)} = \bullet\overline{a \vee b} = \bullet(\overline{a} \vee \overline{b})$; (3) $\bullet\bullet\overline{a} = \bullet\overline{\bullet a} = \overline{\bullet\bullet a} = \overline{\bullet a} = \bullet\overline{a}$.
(iv) As usual, we define $\tilde{h} : \mathscr{A}|_{Ker(h)} \longrightarrow \mathscr{B}$ by $\tilde{h}(\overline{a}) = h(a)$. Naturally $\tilde{h}$ is well defined and bijective. We will show that $\tilde{h}$ preserves the operation $\bullet$: $\tilde{h}(\bullet\overline{a}) = \tilde{h}(\overline{\bullet a}) = h(\overline{\bullet a}) = \bullet h(a) = \bullet\tilde{h}(\overline{a})$. □

Definition 3.23. Let $\mathscr{A}$ be a TK-algebra and I a maximal (or prime) TK-ideal in $\mathscr{A}$. Consider the following (equivalence) relation $\equiv_I$ in $\mathscr{A}$:

$$a \equiv_I b \Leftrightarrow=_{\text{df}} a - b \in I \text{ and } b - a \in I.$$

In the above definition, I more than TK-maximal is maximal.

The next theorem will show that the relation $\equiv_I$ is a congruence in $\mathscr{A}$ with respect to $\wedge, \vee, \sim$ and $\bullet$, that is, $[a] = [b]$ implies $[\sim a] = [\sim b]$ and $[\bullet a] = [\bullet b]$; and $[a] = [b], [c] = [d]$ implies $[a \wedge c] = [b \wedge d]$ and $[a \vee c] = [b \vee d]$. Also, $[1]$ is the unit, $[0]$ is the zero element of $\mathscr{A}|_I$.

Proposition 3.24. *Let $\mathscr{A}$ be a TK-algebra and I a maximal TK-ideal in $\mathscr{A}$.*

(i) *the relation $\equiv_I$ determined by I is a congruence relation;*

(ii) *considering, for every $x \in A$, $\bullet[x] = [\bullet x]$, the quotient algebra $\mathscr{A}|_I$ is a TK-algebra;*

(iii) *the function $h : \mathscr{A} \longrightarrow \mathscr{A}|_I$ defined by $h(a) = \overline{a}$ is a surjective homomorphism;*

(iv) *I is the kernel of h, and for every $a \in A$, $a \in I \Leftrightarrow a \equiv_I 0$;*

(v) *for every $a \in A$, $a \notin I \Leftrightarrow a \equiv_I 1$.*

Proof. (i) Clearly $\equiv_I$ is an equivalence relation. Now:
(1) $[a] = [b] \Leftrightarrow a - b \in I$ and $b - a \in I \Leftrightarrow a \wedge \sim b \in I$ and $b \wedge \sim a \in I \Leftrightarrow \sim a \wedge b \in I$ and $\sim b \wedge a \in I \Leftrightarrow \sim a - \sim b \in I$ and $\sim b - \sim a \in I \Leftrightarrow [\sim a] = [\sim b]$.
(2) $[a] = [b] \Leftrightarrow a - b \in I$ and $b - a \in I \Leftrightarrow a \wedge \sim b \in I$ and $b \wedge \sim a \in I$. If $a \in I$, then $\bullet a \in I$ and $\sim a \notin I \Rightarrow \sim \bullet a \notin I$ and $b \in I \Rightarrow \bullet b \in I \Rightarrow \bullet a - \bullet b \in I$ and $\bullet b - \bullet a \in I \Leftrightarrow [\bullet a] = [\bullet b]$. If $a \notin I$, then $\bullet a \notin I$ and $\sim b \in I \Rightarrow \sim \bullet a \in I$ and $b \notin I \Rightarrow \bullet b \notin I \Rightarrow \sim \bullet b \in I \Rightarrow \bullet a - \bullet b \in I$ and $\bullet b - \bullet a \in I \Leftrightarrow [\bullet a] = [\bullet b]$. In any case, we have $[\bullet a] = [\bullet b]$.
(3) If $[a] = [b]$ and $[c] = [d]$, then $a \wedge \sim b \in I$, $b \wedge \sim a \in I$, $c \wedge \sim d \in I$ and $d \wedge \sim c \in I$. Since I is an ideal, and $a \wedge \sim b$, $c \wedge \sim d \in I$, then $a \wedge c \wedge \sim b$, $a \wedge c \wedge \sim d \in I$, therefore $(a \wedge c) \wedge \sim (b \wedge d) = (a \wedge c) \wedge (\sim b \vee \sim d) = (a \wedge c \wedge \sim b) \vee (a \wedge c \wedge \sim d) \in I$. That is, $(a \wedge c) - (b \wedge d) \in I$. Analogously, $(b \wedge d) - (a \wedge c) \in I$. Hence, $[a \wedge c] = [b \wedge d]$.
(4) If $[a] = [b]$ and $[c] = [d]$ then $a \wedge \sim b \in I$, $b \wedge \sim a \in I$, $c \wedge \sim d \in I$ and $d \wedge \sim c \in I$. Since I is an ideal, and $a \wedge \sim b$, $c \wedge \sim d \in I$, then $a \wedge \sim b \wedge \sim d$, $c \wedge \sim b \wedge \sim d \in I$, therefore $(a \vee c) \wedge \sim (b \vee d) = (a \vee c) \wedge (\sim b \wedge \sim d) = (a \wedge \sim b \wedge \sim d) \vee (c \wedge \sim b \wedge \sim d) \in I$. That is, $(a \vee c) - (b \vee d) \in I$. Analogously, $(b \vee d) - (a \vee c) \in I$. Hence, $[a \vee c] = [b \vee d]$.

(ii) We need to verify that $\bullet|_{\equiv_I}$ preserves the properties of operator $\bullet$.
(1) $[x] \vee \bullet[x] = [x] \vee [\bullet x] = [x \vee \bullet x] = [\bullet x] = \bullet[x]$.
(2) $\bullet[x] \vee \bullet[x \vee y] = [\bullet x] \vee [\bullet(x \vee y)] = [\bullet x \vee \bullet(x \vee y)] = [\bullet(x \vee y)] = \bullet[(x \vee y)]$.
(3) $\bullet\bullet[x] = [\bullet\bullet x] = [\bullet x] = \bullet[x]$.

(iii) It is immediate.

(iv) ($\Rightarrow$) If $a \in I$, as $a = a \wedge 1 = a \wedge \sim 0 = a - 0 \in I$ and $0 = 0 \wedge \sim a = 0 - a \in I$. So $a \equiv_I 0$. ($\Leftarrow$) If $a \equiv_I 0$ then $a = a \wedge 1 = a \wedge \sim 0 = a - 0 \in I$.

(v) $a \notin I \Leftrightarrow \sim a \in I \Leftrightarrow \sim a \equiv_I 0 \Leftrightarrow a \equiv_I 1$. □

We observe that if I is an ideal then $\mathscr{A}|_I$ is degenerate $\Leftrightarrow I = A$.

Proposition 3.25. *If I is a prime TK-ideal, then the quotient algebra $\mathscr{A}|_{\equiv_I}$ is linearly ordered.*

Proof. Let I be a prime TK-ideal and take $a, b \in A$. From this we have that $a - b \in I$ or $b - a \in I$, that is, $[a] \leq [b]$ or $[b] \leq [a]$ and, therefore $\mathscr{A}|_I$ is linearly ordered. □

In the next section, a new logic associated with Tarski's consequence operator is introduced.

4. Logic TK

The propositional logic TK is the logical system associated to the TK-algebras. The propositional language of TK is $L = (\neg, \vee, \to, \blacklozenge, p_1, p_2, p_3, \ldots)$ and TK is presented as follows:

Axioms:

(CPC) φ, if φ is a tautology;

(TK_1) $\varphi \to \blacklozenge\varphi$;

(TK_2) $\blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi$.

Deduction Rules:

(MP) $\dfrac{\varphi \to \psi,\ \varphi}{\psi}$;

($RM^{\blacklozenge}$) $\dfrac{\vdash \varphi \to \psi}{\vdash \blacklozenge\varphi \to \blacklozenge\psi}$.

As usual, we write $\vdash_{\mathbf{S}} \varphi$ to indicate that φ is a theorem of some axiomatic system **S**, and we drop the subscript when there is no possibility of misunderstanding.

Definition 4.1. Let Γ be a set of formulas, and φ a formula of some system **S**. We say that Γ *deduces* φ (notation: $\Gamma \vdash_{\mathbf{S}} \varphi$) if there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and, for every φ_i, $1 \le i \le n$,

(i) φ_i is an axiom; or

(ii) $\varphi_i \in \Gamma$; or

(iii) φ_i is obtained from previous formulas of the sequence by some of the deduction rules.

Notice that the notion of syntactic consequence (deduction) presented here is *global*. Accordingly, we will have, for instance, $p \to q \vdash \blacklozenge p \to \blacklozenge q$. However, $(p \to q) \to (\blacklozenge p \to \blacklozenge q)$ is not a theorem (what we can show semantically after proving completeness), and so the Deduction Theorem does not hold[1].

Proposition 4.2. $\vdash \blacklozenge\varphi \to \blacklozenge(\varphi \vee \psi)$.

Proof.

1. $\varphi \to (\varphi \vee \psi)$	Tautology
2. $\blacklozenge\varphi \to \blacklozenge(\varphi \vee \psi)$	$R^{\blacklozenge}$ in 1

□

Proposition 4.3. $\vdash \varphi \Rightarrow \vdash \blacklozenge\varphi$.

Proof.

1. φ	Premise	
2. $\varphi \to \blacklozenge\varphi$	Ax_{TK1}	□
3. $\blacklozenge\varphi$	MP in 1 and 2.	

Proposition 4.4. $\Gamma \vdash \blacklozenge\varphi \vee \blacklozenge\psi \to \blacklozenge(\varphi \vee \psi)$

Proof.

1. $\blacklozenge\varphi \to \blacklozenge(\varphi \vee \psi)$	Proposition 4.2	
2. $\blacklozenge\psi \to \blacklozenge(\varphi \vee \psi)$	Proposition 4.2	□
3. $\blacklozenge\varphi \vee \blacklozenge\psi \to \blacklozenge(\varphi \vee \psi)$	CPC.	

As in the case of a TK-algebra, we can define the dual operator of $\blacklozenge$ in the following way:

$$\lozenge\varphi =_{df} \neg\blacklozenge\neg\varphi.$$

Proposition 4.5. $\vdash \varphi \to \psi \Rightarrow \vdash \lozenge\varphi \to \lozenge\psi$.

Corollary 4.6. $\vdash \varphi \leftrightarrow \psi \Rightarrow \vdash \lozenge\varphi \leftrightarrow \lozenge\psi$.

Proposition 4.7. $\vdash \lozenge\varphi \to \varphi$.

Proposition 4.8. $\vdash \lozenge\varphi \to \lozenge\lozenge\varphi$.

Proposition 4.9. $\vdash \lozenge(\varphi \wedge \psi) \to \lozenge\varphi$.

Corollary 4.10. $\vdash \lozenge(\varphi \wedge \psi) \to (\lozenge\varphi \wedge \lozenge\psi)$.

We could, alternatively, consider the operator $\lozenge$ as primitive and substitute the axioms TK_1 and TK_2 for the following ones:

(TK_1^*) $\lozenge\varphi \to \varphi$,

(TK_2^*) $\lozenge\varphi \to \lozenge\lozenge\varphi$,

and the rule $RM^{\blacklozenge}$ by the rule $RM^{\lozenge}$:

$$(RM^{\lozenge}) \quad \frac{\varphi \to \psi}{\lozenge\varphi \to \lozenge\psi}.$$

In the following section, the algebraic adequacy of the Logic TK relative to TK-algebras is shown.

5. The algebraic adequacy

Below, we indicate the set of propositional variables of TK by Var(TK), the set of its formulas by For(TK) and a generic TK-algebra by $\mathscr{A}$. The propositional logical system TK is determined by a pair $(L, ^{-})$, where L is the propositional language of TK and $^{-}$ is a consequence operator on For(TK) given by axioms and deduction rules of TK.

Thus, for $\Gamma \subseteq$ For(TK), denoting the set of axioms of TK by Ax, then $\overline{\Gamma} = \{\psi : \Gamma \cup Ax \vdash \psi\}$. We say that ψ is derivable in TK or is a theorem of TK when $\psi \in \overline{\varnothing}$, or, $\Gamma = \varnothing$.

Definition 5.1. A *TK-theory* is a set $\Delta \subseteq$ For(TK), such that $\overline{\Delta} = \Delta$.

When $\Delta = \varnothing$, we have the theorems of TK, that is, $\psi \in \overline{\varnothing} \Leftrightarrow \vdash \psi$.

Definition 5.2. A formula $\psi \in$ For(TK) is *refutable* in Γ when $\Gamma \vdash \neg\psi$. Otherwise, ψ is *irrefutable*.

Definition 5.3. A set $\Gamma \subseteq$ For(TK) is *irreducible* if it is consistent and for any two sets $\Delta_1, \Delta_2 \subseteq$ For(TK):

$$\overline{\Gamma} = \overline{\Delta}_1 \cap \overline{\Delta}_2 \Rightarrow \overline{\Gamma} = \overline{\Delta}_1 \ \vee \ \overline{\Gamma} = \overline{\Delta}_2.$$

Definition 5.4. A set $\Gamma \subseteq$ For(TK) is *maximal* if it is consistent and for any consistent set $\Delta \subseteq$ For(TK):

$$\overline{\Gamma} \subseteq \overline{\Delta} \Rightarrow \overline{\Gamma} = \overline{\Delta}.$$

Definition 5.5. A *restrict valuation* is a function $\check{v}$: Var(TK) $\longrightarrow \mathscr{A}$ that interprets each variable of TK in an element of $\mathscr{A}$.

Definition 5.6. A *valuation* is a function v: For(TK) $\longrightarrow \mathscr{A}$ that extends natural and uniquely v as follows:

(i) $v(p) = \check{v}(p)$;
(ii) $v(\neg\varphi) = \sim v(\varphi)$;
(iii) $v(\blacklozenge\varphi) = \bullet\, v(\varphi)$;
(iv) $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$;
(v) $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$.

As usual, operator symbols of left members represent logical operators and the right ones represent algebraic operators.

Definition 5.7. Let $\mathscr{A}$ be a TK-algebra. A valuation v: For(TK) $\longrightarrow \mathscr{A}$ is a *model* for a set $\Gamma \subseteq$ For(TK) when $v(\gamma) = 1$, for each formula $\gamma \in \Gamma$.

In particular, a valuation v: For(TK) $\longrightarrow \mathscr{A}$ is a model for $\varphi \in$ For(TK) when $v(\varphi) = 1$.

Definition 5.8. A formula φ is valid in a TK-algebra $\mathscr{A}$ when every valuation v: For(TK) $\longrightarrow \mathscr{A}$ is a model for φ.

Definition 5.9. A formula φ is TK-valid, what is denoted by $\vDash \varphi$, when it is valid in every TK-algebra.

If we consider the set of formulas For(TK), naturally we have an algebra on For(TK), $\mathscr{B} = (For(TK), \wedge, \vee, \sim, \blacklozenge)$ such that $\wedge$ and $\vee$ are binary operators, $\neg$ and $\blacklozenge$ are unary operators.

Now, we define the Lindembaum algebra of TK.

Definition 5.10. Let $\Gamma \cup \{\varphi, \psi\} \subseteq$ For(TK) and $\simeq$ the equivalence relation defined by:

$$\varphi \simeq \psi \Leftrightarrow_{df} \ \Gamma \vdash \varphi \to \psi \ and \ \Gamma \vdash \psi \to \varphi.$$

The relation $\simeq$, more than an equivalence, is a congruence, since by the rule $R^{\blacklozenge}$: $\varphi \simeq \psi \Rightarrow \ \Gamma \vdash \varphi \to \psi \ and \ \Gamma \vdash \psi \to \varphi \Rightarrow \ \Gamma \vdash \blacklozenge\varphi \to \blacklozenge\psi \ and \ \Gamma \vdash \blacklozenge\psi \to \blacklozenge\varphi \Rightarrow \blacklozenge\varphi \simeq \blacklozenge\psi$.

For each $\psi \in$ For(TK), we denote the class of equivalence of ψ modulo $\simeq$ and Γ by $[\psi]_\Gamma = \{\sigma \in For(TK) : \sigma \simeq \psi\}$.

The *Lindembaum algebra* of TK, denoted by $\mathscr{A}_\Gamma(TK)$, is the quotient algebra $\mathscr{B}|_\simeq$, defined by:

$\mathscr{A}_\Gamma(TK) = (For(TK)|_\simeq, 0, 1, \neg_\simeq, \blacklozenge_\simeq, \wedge_\simeq, \vee_\simeq)$, such that:

$$\begin{aligned}
&0 = [\varphi \wedge \neg\varphi],\\
&1 = [\varphi \vee \neg\varphi],\\
&\neg_\simeq[\varphi] = [\neg\varphi],\\
&\blacklozenge_\simeq[\varphi] = [\blacklozenge\varphi],\\
&[\varphi] \wedge_\simeq [\psi] = [\varphi \wedge \psi],\\
&[\varphi] \vee_\simeq [\psi] = [\varphi \vee \psi].
\end{aligned}$$

In general, we do not indicate the index $_\simeq$ of operations. When $\Gamma = \varnothing$ we just write $\mathscr{A}(TK)$.

Proposition 5.11. *In $\mathscr{A}_\Gamma(TK)$ it is valid: $[\varphi] \leq [\psi] \Leftrightarrow \Gamma \vdash \varphi \to \psi$.*

Proof. $[\varphi] \leq [\psi] \Leftrightarrow [\varphi] \vee [\psi] = [\psi] \Leftrightarrow [\varphi \vee \psi] = [\psi] \Leftrightarrow \Gamma \vdash \varphi \vee \psi \leftrightarrow \psi \Leftrightarrow \Gamma \vdash \varphi \to \psi$. □

Proposition 5.12. *The algebra $\mathscr{A}_\Gamma(TK)$ is a TK-algebra.*

Proof. Ax_{TK1}: $\Gamma \vdash \varphi \to \blacklozenge\varphi \Rightarrow [\varphi] \le [\blacklozenge\varphi] \Rightarrow [\varphi] \le \blacklozenge[\varphi]$;
Proposition 4.2: $\Gamma \vdash \blacklozenge\varphi \to \blacklozenge(\varphi \vee \psi) \Rightarrow [\blacklozenge\varphi] \le [\blacklozenge(\varphi \vee \psi)] \Rightarrow \blacklozenge[\varphi] \le \blacklozenge[\varphi \vee \psi]$;
Ax_{TK2}: $\Gamma \vdash \blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi \Rightarrow [\blacklozenge\blacklozenge\varphi] \le [\blacklozenge\varphi] \Rightarrow \blacklozenge\blacklozenge[\varphi] \le \blacklozenge[\varphi]$. □

Definition 5.13. The algebra $\mathscr{A}_\Gamma(TK)$ is the *canonical model* of $\Gamma \subseteq$ For(TK).

As an immediate consequence, $\mathscr{A}(TK)$ is the canonical model for the theorems of TK.

Corollary 5.14. *Let $\Gamma \cup \{\varphi\} \subseteq$ For(TK):*

(i) *If $\Gamma \vdash \varphi$, then $[\varphi] = 1$ in $\mathscr{A}_\Gamma(TK)$;*

(ii) *If $\Gamma \vdash \neg\varphi$ (φ is refutable in Γ), then $[\varphi] = 0$ in $\mathscr{A}_\Gamma(TK)$;*

(iii) *If $\mathscr{A}_\Gamma(TK)$ is non degenerate, then there is a formula that is not a theorem of Γ.*

Proof. Since $\mathscr{A}_\Gamma(TK)$ always has an identity element 1, then for every $\varphi \in$ For(TK), $[\varphi] \le 1$.
(i) The formula $\varphi \to (\psi \to \varphi)$ is a tautology and, hence, a theorem of TK. With a substitution we have $\vdash \varphi \to ((\varphi \to \varphi) \to \varphi)$. Now, if $\Gamma \vdash \varphi$, by MP, it follows that $\Gamma \vdash (\varphi \to \varphi) \to \varphi$, that is, $1 = [\varphi \to \varphi] \le [\varphi]$ and $[\varphi] = 1$.
(ii) Let φ be refutable in Γ, that is, $\Gamma \vdash \neg\varphi$. But $\Gamma \vdash \neg\varphi$ iff $[\neg\varphi] = 1$ iff $\sim[\varphi] = 1$ iff $[\varphi] = 0$.
(iii) Finally, $[\varphi] = 1$ iff $\Gamma \vdash \varphi$ and, therefore, $\mathscr{A}_\Gamma(TK)$ has a different element of 1 iff there is $\varphi \in$ For(TK) such that $\Gamma \nvdash \varphi$. □

Naturally, if $[\varphi] = 1$, then $\Gamma \vdash \varphi$ and if $[\varphi] = 0$, then $\Gamma \vdash \neg\varphi$. So, it results from preceding propositions that for every formula φ: $[\varphi] = 1$ iff $\Gamma \vdash \varphi$, $[\varphi] = 0$ iff $\Gamma \vdash \neg\varphi$, and $[\varphi] \neq 0$ iff φ is irrefutable in Γ, and, since $\mathscr{A}_\Gamma(TK)$ is non-degenerate, then Γ has some non theorem.

Theorem 5.15 (Soundness)**.** *The TK-algebras are correct models for the Logic TK.*

Proof. Let $\mathscr{A} = (A, 0, 1, \vee, \sim, \bullet)$ be a TK-algebra. It remains to prove that the axioms Ax_{TK1} and Ax_{TK2} are valid and the rule $R^{\blacklozenge}$ preserves validity:
Ax_{TK1}: $v(\varphi \to \blacklozenge\varphi) = v(\varphi) \rightarrowtail v(\blacklozenge\varphi) = \sim v(\varphi) \vee v(\blacklozenge\varphi) = \sim v(\varphi) \vee (v(\varphi) \vee v(\blacklozenge\varphi))$ $= (\sim v(\varphi) \vee v(\varphi)) \vee v(\blacklozenge\varphi) = 1 \vee v(\blacklozenge\varphi) = 1$.
Ax_{TK12}: $v(\blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi) = \bullet\bullet v(\varphi) \rightarrowtail \bullet v(\varphi) = \sim\bullet\bullet v(\varphi) \vee \bullet v(\varphi) = \sim\bullet v(\varphi) \vee \bullet v(\varphi) = 1$.
$R^{\blacklozenge}$: Using Proposition 2.2 (ii): $v(\varphi \to \psi) = 1 \Leftrightarrow v(\varphi) \le v(\psi) \Rightarrow \blacklozenge v(\varphi) \le \blacklozenge v(\psi)$ $\Rightarrow v(\blacklozenge\varphi) \le v(\blacklozenge\psi) \Leftrightarrow v(\blacklozenge\varphi \to \blacklozenge\psi) = 1$. □

Theorem 5.16 (Adequacy)**.** *Let $\varphi \in$ For(TK). The following assertions are equivalent:*

(i) $\vdash \varphi$;

(ii) $\vDash \varphi$;

(iii) *φ is valid in every TK-algebra of closed subsets of a Tarski space $(S, ^{-})$;*

(iv) $v_0(\varphi) = 1$, *where v_0 is the valuation defined at the canonical model.*

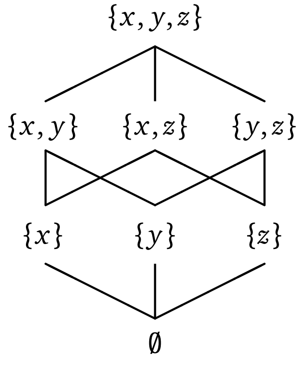
Proof. (i) $\Rightarrow$ (ii) it follows from Soundness Theorem. (ii) $\Rightarrow$ (iii) it suffices to observe that the algebra of closed subsets of any Tarski space is a TK-algebra. (iii) $\Rightarrow$ (iv) since every TK-algebra is isomorphic to a subalgebra of closed subsets of a Tarski space $(S, ^{-})$ and $\mathscr{A}(TK)$ is a TK-algebra, the result follows. (iv) $\Rightarrow$ (i) if $\varphi \in$ For(TK) and it is not derivable in TK, by Corollary 5.12, $[\varphi] \neq 1$ in $\mathscr{A}(TK)$ and thus $v_0(\varphi) \neq 1$. □

Corollary 5.17 (Completeness). *If $\varphi \in$ For(TK), then: $\vDash \varphi \Leftrightarrow \vdash \varphi$.*

In the next proposition it is proved that the formula $(\varphi \to \psi) \to (\blacklozenge\varphi \to \blacklozenge\psi)$ is not TK valid by a counter example.

Proposition 5.18. $\nvDash (\varphi \to \psi) \to (\blacklozenge\varphi \to \blacklozenge\psi)$.

Proof. There is a TK-algebra in which does not hold the above formula. Let $E = \{x, y, z\}$ and take the Boolean algebra $(\mathscr{P}(E), ^{C}, \cap, \cup, \varnothing, E)$. Now, define the following consequence operator over $(\mathscr{P}(E), ^{C}, \cap, \cup, \varnothing, E)$: $\bullet\{x\} = \{x, y\}$, $\bullet\{x, z\} = \{x, y, z\}$, and $\bullet X = X$, for all the other sets in $\mathscr{P}(E)$. Then $(\mathscr{P}(E), ^{C}, \bullet, \cap, \cup, \varnothing, E)$ is a TK-algebra, but the formula $(\varphi \to \psi) \to (\blacklozenge\varphi \to \blacklozenge\psi)$ is not valid in it. Below we show that $v(\varphi \to \psi) \nsubseteq v(\bullet\varphi \to \bullet\psi)$, when φ is interpreted by $\{x\}$ and ψ by $\{z\}$:



$\{x\} \rightarrowtail \{z\} = \{x\}^C \cup \{z\} = \{y, z\} \cup \{z\} = \{y, z\}$ and
$\bullet\{x\} \rightarrowtail \bullet\{z\} = (\bullet\{x\})^C \cup \bullet\{z\} = \{z\} \cup \{z\} = \{z\}$. □

As a consequence of the previous proposition, follows that the Deduction Theorem is not valid for the TK Logic when it was applied the rule $R^{\blacklozenge}$ in a deduction.

The next results involve the models of set of formulas with the concept of ideals in a TK-algebra.

6. Strong adequacy

Definition 6.1. Let $\mathscr{A}$ be a TK-algebra and $\Gamma \subseteq$ For(TK). An *algebraic model* for Γ is a valuation v: For(L) $\longrightarrow \mathscr{A}$, such that for every $\gamma \in \Gamma$, $v(\gamma) = 1$.

We denote that $\mathscr{A}$ is an algebraic model for $\Gamma \subseteq$ For(TK) by $\mathscr{A} \vDash \Gamma$, that is, $\mathscr{A} \vDash \Gamma$ if, and only if, for every $\gamma \in \Gamma$, $v_{\mathscr{A}}(\gamma) = 1$.

Definition 6.2. Let $\Gamma \cup \{\gamma\} \subseteq$ For(TK). The set Γ *implies* γ (or γ is a *semantic consequence* of Γ) when, for every model $\mathscr{A}$, if $\mathscr{A} \vDash \Gamma$, then $\mathscr{A} \vDash \{\gamma\}$.

We denote that Γ *implies* γ by $\Gamma \vDash \gamma$.

Proposition 6.3. *For* $\Gamma \subseteq$ *For(TK), we have:* $\Gamma \vdash \gamma \Rightarrow \Gamma \vDash \gamma$.

Proof. Let $v : For(L) \longrightarrow \mathscr{A}$ be a model for Γ. Since $\mathscr{A}$ is a TK-algebra, then v is a model for every axiom of TK and for every $\gamma \in \Gamma$. As in the Soundness Theorem, the rules of TK preserve validity and if $\Gamma \vdash \gamma$, then $v_{\mathscr{A}}(\gamma) = 1$. □

Definition 6.4. Let $\Gamma \subseteq For(TK)$. A model v: For(TK) $\longrightarrow \mathscr{A}$ is *strongly adequate* for Γ when, for every $\gamma \in$ For(TK):

$$\Gamma \vdash \gamma \Leftrightarrow \Gamma \vDash \gamma.$$

Proposition 6.5. *Let* $\Gamma \subseteq$ *For(TK) be consistent. Then:*

(i) *the algebra* $\mathscr{A}_\Gamma(TK)$ *is non degenerate;*

(ii) *the canonical valuation* v_0 *is an adequate model for* Γ *in* $\mathscr{A}_\Gamma(TK)$ *such that* $\Gamma \vdash \sigma \Leftrightarrow$ *if* $v_0(\gamma) = 1$, *for every* $\gamma \in \Gamma$, *then* $v_0(\sigma) = 1$.

Proof. (i) As Γ is consistent, there is σ such that $\Gamma \nvdash \sigma$. So $[\sigma] = 0 \in \mathscr{A}_\Gamma(TK)$ and for any axiom φ of TK it follows that $[\varphi] = 1 \in \mathscr{A}_\Gamma(TK)$. Then $\mathscr{A}_\Gamma(TK)$ is non degenerate.
(ii) $\mathscr{A}_\Gamma(TK)$ is a TK algebra and by construction $v_0 = [\ \ ]$ and $\Gamma \vdash \gamma \Leftrightarrow [\gamma] = 1$, for any $\gamma \in \Gamma \Rightarrow [\sigma] = 1$. □

Theorem 6.6. *Let* $\Gamma \subseteq$ *For(TK). The following conditions are equivalent:*

(i) Γ *is consistent;*

(ii) *there is an adequate model for* Γ*;*

(iii) *there is an adequate model for* Γ *in a TK-algebra* $\mathscr{A}$ *of all closed subsets of a Tarski space* $(S,^-)$*;*

(iv) *there is a model to* Γ.

Proof. $(i) \Rightarrow (ii)$ It follows of preceding proposition.
$(ii) \Rightarrow (iii)$ Since $\mathscr{A}_\Gamma(TK)$ is a TK-algebra and every TK-algebra is isomorphic to a subalgebra of closed sets of a Tarski space $(S,^-)$, then the result follows.
$(iii) \Rightarrow (iv)$ It is an immediate consequence.
$(vi) \Rightarrow (i)$ Let $\mathscr{A}$ be a model for Γ and suppose that Γ is not consistent. Then $\Gamma \vdash \gamma$ and $\Gamma \vdash \neg\gamma$. So $v_\mathscr{A}(\gamma) = 1$ and $v_\mathscr{A}(\neg\gamma) = 1 = \sim v_\mathscr{A}(\gamma)$. But if $v_\mathscr{A}(\gamma) = 1$, then $\sim v_\mathscr{A}(\gamma) = 0$ and therefore we have a contradiction. □

Corollary 6.7. *Let* $\Gamma \cup \{\gamma\} \subseteq For(TK)$ *consistent. The following conditions are equivalent:*

(i) $\Gamma \vdash \gamma$*;*

(ii) $\Gamma \vDash \gamma$*;*

(iii) *every model of* Γ *in a TK-algebra of all closed subsets of a Tarski space* $(S,^-)$ *is a model to* γ*;*

(iv) $v_0(\gamma) = 1$*, for the canonical valuation* v_0.

7. Some meta-theorems

In this section, some other syntactic and semantic consequent results from previous sections are established.

Theorem 7.1. *Propositional calculus TK is consistent.*

Proof. Suppose that TK is not consistent. Then there is $\varphi \in$ For(TK) such that $\vdash \varphi$ and $\vdash \neg\varphi$. By Soundness Theorem, φ and $\neg\varphi$ are valid. Let v be a valuation in a TK-algebra with two elements $2 = \{0, 1\}$. Since φ is valid, then $v(\varphi) = 1$ and therefore $v(\neg\varphi) = \sim v(\varphi) = 0$. This contradicts the fact that $\neg\varphi$ is valid. □

Proposition 7.2. *Let* $\Gamma \subseteq$ *For(TK). The set* $I_\Gamma = \{[\varphi] : \exists k \in \mathbb{N}, \gamma_1, \ldots, \gamma_k \in \overline{\Gamma} \vdash \varphi \to \blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k)\}$ *is a TK-ideal in the TK-algebra* $\mathscr{A}(TK)$*.*

Proof. (1) $[\varphi], [\psi] \in I_\Gamma \Rightarrow \vdash \varphi \to \blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k)$ and $\vdash \psi \to \blacklozenge(\sigma_1 \vee \ldots \vee \sigma_m) \Rightarrow \vdash (\varphi \vee \psi) \to \blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k) \vee \blacklozenge(\sigma_1 \vee \ldots \vee \sigma_m) \Rightarrow$ (by Proposition 4.4) $\vdash (\varphi \vee \psi) \to \blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k \vee \sigma_1 \vee \ldots \vee \sigma_m) \Rightarrow [\varphi \vee \psi] = [\varphi] \vee [\psi] \in I_\Gamma$.

(2) $[\varphi] \leq [\psi]$ and $[\psi] \in I_\Gamma \Rightarrow \vdash \varphi \to \psi$ and $\vdash \psi \to \blacklozenge(\sigma_1 \vee \ldots \vee \sigma_m) \Rightarrow \vdash \varphi \to \blacklozenge(\sigma_1 \vee \ldots \vee \sigma_m) \Rightarrow [\varphi] \in I_\Gamma$.
(3) $[\varphi] \in I_\Gamma \Rightarrow \vdash \varphi \to \blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k) \Rightarrow [\varphi] \leq [\blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k)] \Rightarrow [\varphi] \leq \blacklozenge[(\gamma_1 \vee \ldots \vee \gamma_k)] \Rightarrow \blacklozenge[\varphi] \leq \blacklozenge[(\gamma_1 \vee \ldots \vee \gamma_k)] \Rightarrow [\blacklozenge\varphi] \leq [\blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k)] \Rightarrow \vdash \blacklozenge\varphi \to \blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k) \Rightarrow [\blacklozenge\varphi] = \blacklozenge[\varphi] \in I_\Gamma$. □

In the logical context, this proposition is similar to the Proposition 3.4 about ideals. In both we could take $\gamma \in \overline{\Gamma}$ because if $\gamma_1, \ldots, \gamma_k \in \overline{\Gamma}$, then $\gamma_1 \vee \ldots \vee \gamma_k \in \overline{\Gamma}$ and also $\blacklozenge(\gamma_1 \vee \ldots \vee \gamma_k) \in \overline{\Gamma}$. So, the above TK-ideal I_Γ is the TK-ideal generated by $\{[\gamma] : \gamma \in \overline{\Gamma}\}$.

The set $\Gamma \subseteq$ For(TK) is consistent if, and only if, I_Γ is a proper ideal of $\mathscr{A}(TK)$, because Γ is not consistent iff $1 \in \overline{\Gamma}$ iff $[1] \in I_\Gamma$ iff I is not a proper ideal.

Proposition 7.3. *If J is a TK-ideal in $\mathscr{A}(TK)$ and $\Gamma = \{\psi \in For(TK) : [\psi] \in J\}$, then $J \subseteq I_\Gamma$.*

Proof. Let $\Gamma = \{\psi \in For(TK) : [\psi] \in J\}$. Hence $[\varphi] \in J \Rightarrow \varphi \in \Gamma \Rightarrow [\varphi] \in I_\Gamma$. □

Proposition 7.4. *Let $\Gamma \subseteq$ For(TK). The TK-ideal I_Γ is maximal if, and only if, the set Γ is maximal*

Proof. ($\Rightarrow$) Suppose that Γ is not maximal. In this case, there is $\psi \in$ For(TK) such that $\Gamma_1 = \Gamma \cup \{\psi\}$ and $\Gamma_2 = \Gamma \cup \{\neg\psi\}$ are consistent. So $\overline{\Gamma} \subseteq \overline{\Gamma}_1 \cap \overline{\Gamma}_2$, with $\overline{\Gamma}_1 \neq \overline{\Gamma}_2$. Therefore, $I_\Gamma \subseteq I_{\Gamma_1} \cap I_{\Gamma_2}$ and since I_Γ is maximal, then $I_\Gamma = I_{\Gamma_1} = I_{\Gamma_2}$ what is a contradiction, because $[\psi] \in I_{\Gamma_1}$, $[\neg\psi] \in I_{\Gamma_2}$ and $1 \notin I_\Gamma$.
($\Leftarrow$) If Γ is maximal, for each $\psi \in$ For(TK), either $\psi \in \overline{\Gamma}$ or $\neg\psi \in \overline{\Gamma}$. Then for each $\psi \in$ For(TK), $[\psi] \in I_\Gamma$ or $[\neg\psi] \in I_\Gamma$. Since Γ is consistent, then I_Γ is proper and it is not the case that $1 = [\psi] \vee [\neg\psi] \in I_\Gamma$. Hence, the TK-ideal I_Γ is maximal. □

Proposition 7.5. *If $\Gamma \subseteq$ For(TK) is consistent, then there is a maximal set $\Delta \subseteq$ For(TK) such that $\Gamma \subseteq \Delta$.*

Proof. The result follows from Zorn's Lemma. □

Proposition 7.6. *If $\Gamma \subseteq$ For(TK) is consistent, then there is $\Delta \subseteq$ For(TK) irreducible such that $\Gamma \subseteq \Delta$.*

Proof. If Γ is consistent, by previous proposition, there is a maximal set Δ such that $\Gamma \subseteq \Delta$. Now, since each maximal set is an irreducible set, then Δ is irreducible. □

Definition 7.7. A model for Γ in a non degenerate TK-algebra $\mathscr{A}$ is a $\bullet$-*semantic model* of Γ when for every $a \in A$: either $\bullet a = 1$ or $\bullet a = 0$.

Proposition 7.8. *If $\Gamma \subseteq$ For(TK) is consistent, then Γ has a $\bullet$-semantic model.*

Proof. If Γ is consistent, it is included in a maximal set Γ^* and it has a model $\mathscr{A}$. By Proposition 7.4, I_{Γ^*} is maximal and from Proposition 3.24, $\mathscr{A}|_{J^*}$ is a non degenerate TK-algebra with the property that for every $\bar{a} \in \mathscr{A}|_{J^*}$, either $\bullet\bar{a} \equiv 0$ or $\bullet\bar{a} \equiv 1$. If h is the surjective homomorphism $h : \mathscr{A} \longrightarrow \mathscr{A}|_{J^*}$, so the composition hov is a $\bullet$-semantic model for Γ. □

Given $\Gamma \subseteq$ For(TK), consider the function $h : \mathscr{A}(TK) \longrightarrow \mathscr{A}_\Gamma(TK)$, defined by $h([\psi]) = [\psi]_\Gamma$. We are going to denote a member of $\mathscr{A}(TK)$ by $[\psi]$, and a member of $\mathscr{A}_\Gamma(TK)$ by $[\psi]_\Gamma$. Naturally h is a surjective homomorphism. Now, the kernel of h is:

Ker(h) $= \{[\psi] \in \mathscr{A}(TK) : h([\psi]) = [0]_\Gamma\} = \{[\psi] \in \mathscr{A}(TK) : [\psi]_\Gamma = [0]_\Gamma\} = \{[\psi] \in \mathscr{A}(TK) : \Gamma \vdash \neg\psi\}$, because $[\psi]_\Gamma = [0]_\Gamma \Leftrightarrow \Gamma \vdash \psi \to 0$ and $\Gamma \vdash 0 \to \psi \Leftrightarrow \Gamma \vdash \neg\psi$ and $\Gamma \vdash 1 \Leftrightarrow \Gamma \vdash \neg\psi$.

By Theorem 3.22, $\mathscr{A}(TK)|_{Ker(h)}$ is isomorphic to $\mathscr{A}_\Gamma(TK)$

Theorem 7.9. *Given $\Gamma \subseteq$ For(TK), the following statements are equivalent:*

(i) *for every $\varphi \in$ For(TK) exactly one holds: $\Gamma \vdash \blacklozenge\varphi$ or $\Gamma \vdash \neg\blacklozenge\varphi$;*

(ii) Γ *is maximal;*

(iii) *$\mathscr{A}_\Gamma(TK)$ is isomorphic to a non degenerate TK-algebra of sets (Tarski space) $\mathscr{A}$ which is a $\bullet$-semantic model of Γ;*

(iv) Γ *is consistent and each $\bullet$-semantic model for Γ is adequate.*

Proof. (i) $\Rightarrow$ (ii) Suppose that Γ is not maximal. So there is Δ maximal such that $\overline{\Gamma} \subset \overline{\Delta}$. From Proposition 7.5, there is $[\psi] \in I_\Delta$ but $[\psi] \notin I_\Gamma$. Then $[\blacklozenge\psi] \notin I_\Gamma$ and $\Gamma \nvdash \blacklozenge\psi$. By (i) $\Gamma \vdash \neg\blacklozenge\psi$, then $\Delta \vdash \neg\blacklozenge\psi$ and $\Delta \vdash \blacklozenge\psi$, what contradicts the maximality of Δ.

(ii) $\Rightarrow$ (iii) By previous analysis $\mathscr{A}_\Gamma(TK) \approx \mathscr{A}(TK)|_{Ker(h)}$. Now, since Γ is maximal, $\mathscr{A}_\Gamma(TK)$ is not degenerate and, for every $\psi \in$ For(TK), $\Gamma \vdash \psi \Leftrightarrow \psi \in \overline{\Gamma}$. So, for every $\psi \in$ For(TK), either $[\blacklozenge\psi] = 0$ or $[\blacklozenge\psi] = 1$, that is, $\mathscr{A}(TK)|_{Ker(h)}$ is $\bullet$-semantic. From Theorem 2.10, this TK-algebra is isomorphic to a TK-algebra of sets (or Tarski space) $\mathscr{A}$.

(iii) $\Rightarrow$ (iv) Considering (iii), the set Γ is consistent. Now let $\mathscr{A}$ be an arbitrary $\bullet$-semantic model for Γ. Given $\varphi \in$ For(TK), either $v_\mathscr{A}(\varphi) = 0$ or $v_\mathscr{A}(\varphi) = 1$. If $v_\mathscr{A}(\varphi) = 1$, then $\Gamma \vdash \blacklozenge\varphi$ and if $v_\mathscr{A}(\varphi) = 0$, then $\Gamma \nvdash \blacklozenge\varphi$, and thus $\Gamma \vdash \neg\blacklozenge\varphi$.

(iv) $\Rightarrow$ (i) Since Γ is consistent, by Proposition 7.8, there is a $\bullet$-semantic model $\mathscr{A}$ for it. By (iv) this model is adequate. So for any $\varphi \in$ For(TK), either $v_\mathscr{A}(\varphi) = 0$ or $v_\mathscr{A}(\varphi) = 1$, that is, either $\Gamma \vdash \blacklozenge\varphi$ or $\Gamma \vdash \neg\blacklozenge\varphi$. □

Corollary 7.10 (Decidability). *The Logic TK is decidable.*

Proof. Consider the •-semantic model $\mathbf{2} = \{0, 1\}$, such that $\bullet 0 = 0$ and $\bullet 1 = 1$. Since from previous theorem any •-semantic model is adequate, therefore $\mathbf{2}$ is adequate and, in this way, for any formula $\psi \in \text{For(TK)}$, $\vdash \psi \Leftrightarrow v_{\mathbf{2}}(\psi) = 1$. □

8. Final considerations

Logic TK is a kind of modal logic. The operator ⧫ has an intuitive algebraic interpretation and another one given by the Tarski spaces.

As it is known from the modal logics, in the presence of the necessitation rule (NR - Proposition 4.3) the *K* axiom: $\blacklozenge(\varphi \to \psi) \to (\blacklozenge\varphi \to \blacklozenge\psi)$ is equivalent to $\blacklozenge(\varphi \wedge \psi) \leftrightarrow (\blacklozenge\varphi \wedge \blacklozenge\psi)$. However, only $\blacklozenge(\varphi \wedge \psi) \to (\blacklozenge\varphi \wedge \blacklozenge\psi)$ holds in TK. Therefore we can observe that TK is a subnormal modal logic. It is easier to see that by analyzing the dual operator $\lozenge$.

Of course, we can investigate other kind of semantics for TK, particularly, some relational semantic kind.

Maybe some variations on TK-algebras can provide natural and simple algebraic models to other modal logics.

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Resumo. Tarski apresentou sua definição de operador de consequência com a intenção de expor as concepções fundamentais da consequência lógica. Um espaço de Tarski é um par ordenado determinado por um conjunto não vazio e um operador de consequência sobre este conjunto. Esta estrutura matemática caracteriza um espaço quase topológico. Este artigo mostra uma visão algébrica dos espaços de Tarski e introduz uma lógica proposicional modal que interpreta o seu operador modal nos conjuntos fechados de algum espaço de Tarski.

Palavras-chave: Espaço de Tarski, espaço quase topológico, operador de consequência, lógica modal, modelo algébrico.