

# Solution of the Schrödinger equation with Hulthén plus Manning-Rosen potential



Sanjib Meyur<sup>1</sup>, S Debnath<sup>2</sup>

<sup>1</sup>TRGR Khemka High School, 23, Rabindrasarani, Liluah, Howrah, West Bengal, India, Pin-711204.

<sup>2</sup>Department of Mathematics, Jadavpur University, Kolkata, West Bengal, India, Pin -700032.

E-mail: upasana.sd@gmail.com

(Received 24 February 2009; accepted 10 April 2009)

## Abstract

The general solutions of the Schrödinger equation with a generalized Hulthén plus Manning-Rosen potential are obtained in terms of the Jacobi polynomials by using the Nikiforov-Uvarov method.

**Keywords:** Schrödinger equation, Hulthén potential, Manning-Rosen potential.

## Resumen

Las soluciones generales de la ecuación de Schrödinger bajo un potencial generalizado de Hulthén combinado con uno de Manning-Rosen son obtenidas en términos de los polinomios de Jacobi mediante el uso del método de Nikiforov-Uvarov.

**Palabras clave:** Ecuación de Schrödinger, potencial de Hulthén, potencial de Manning-Rosen.

PACS: 03.65.Ge, 03.65.-w

ISSN 1870-9095

## I. INTRODUCTION

Recently, there has been a growing interest in the study of the Schrödinger equation for complex potentials. The increasing interest in central potential models aims at an extension of their physical and mathematical background. Recently, considerable efforts have been paid to obtain the exact solution of central problems. Different techniques have been used in solving the above mentioned potentials such as variational methods [1], numerical approaches [2], Fourier analysis [3], semi-classical estimates [4] and Lie group theoretical approaches [5, 6, 7, 8]. Recently, an alternative method which is known as the Nikiforov-Uvarov method has been introduced in solving the Schrödinger equation.

In this present paper we solve the Schrödinger equation for a generalized Hulthén plus Manning-Rosen potential via the Nikiforov-Uvarov method. We discuss the  $PT$ -symmetric and non  $PT$ -symmetric solutions. Mathematically, the combination of these two potentials is highly significant, because one can easily deduce the energy eigenvalues of the Hulthén potential [9, 10] and the Manning-Rosen potential [11] from combined energy separately.

The plan of the paper is as follows. To make it self-contained we give a brief review of the Nikiforov-Uvarov method in Sec. II. In Sec. III, we discuss the Schrödinger equation for Hulthén plus Manning-Rosen potential and obtain its eigenvalue and eigenfunctions for the corresponding Hamiltonian by the Nikiforov-Uvarov

method. In Secs. IV and V, we discuss respectively the solution of  $PT$ -symmetric [12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and non  $PT$ -symmetric Hulthén potential plus Manning-Rosen potential. In Secs. VI and VII, we discuss the eigenvalues and eigenfunctions of the Hulthén potential and the Manning-Rosen potential respectively. Sec. VIII, is kept for conclusions and discussions.

## II. NIKIFOROV-UVAROV METHOD

According to a brief description of Nikiforov-Uvarov method [22], the Schrödinger equation  $\psi''(x) + (E - V(x))\psi(x) = 0$  can be transformed to the following generalized equation of hypergeometric type with appropriate coordinate transformation,  $s = s(x)$ ,

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \quad (1)$$

where  $\sigma(s)$  and  $\tilde{\sigma}(s)$  are polynomials, at most of second degree, and  $\tilde{\tau}(s)$  is a polynomial, at most of first degree. For find the particular solution to Eq. (1), we set the following wave function as a multiple of two independent parts

$$\psi(s) = \varphi(s)y(s). \quad (2)$$

With this substitution Eq. (1) reduces to an equation of hypergeometric type

$$\sigma(s)y''\psi(s) + \tau(s)y'(s) + \lambda y(s) = 0, \quad (3)$$

provided the following conditions be satisfied:

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)}, \quad (4)$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad \tau'(s) < 0, \quad (5)$$

$\lambda$  satisfies the following second-order differential equation

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s). \quad (6)$$

The polynomial  $\tau(s)$  with the parameter  $s$  and prime factors show the differentials at first degree be negative. It is worthwhile to note that  $\lambda$  or  $\lambda_n$  are obtained from a particular solution of the form  $y(s) = y_n(s)$  which is a polynomial of degree  $n$ . The other part  $y_n(s)$  of the wavefunction (2) is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation [22, 23]

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (7)$$

where  $C_n$  being the normalization constant and the weight function  $\rho(s)$  satisfies the relation as

$$\frac{d}{ds} [\sigma(s)\rho(s) = \tau(s)\rho(s)]. \quad (8)$$

On the other hand, in order to find the eigenfunctions,  $\varphi_n(s)$  and  $y_n(s)$  in Eqs. (4) and (7) and eigenvalues  $\lambda_n$  in Eq. (6), we need to calculate the functions:

$$\pi(s) = \left( \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \right) \pm \sqrt{\left( \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \quad (9)$$

$$k = \lambda - \pi'(s). \quad (10)$$

In principle, since  $\pi(s)$  has to be a polynomial of degree at most one, the expression under the square root sign in (9) can be arranged to be the square of a polynomial of first degree [22]. This is possible only if its discriminant is zero. Thus, the value of  $k$  obtained from the equation (9) can be substituted in equation (10). The energy eigenvalues are obtained from equations (6) and (10).

### III. HULTHÉN PLUS MANNING-ROSEN POTENTIAL

The generalized Hulthén potential plus Manning-Rosen potential is given by

$$V(x) = -V_1 \frac{e^{-2ax}}{1 - qe^{-2ax}} + V_2 \operatorname{cosech}_q^2(ax) + V_3 \operatorname{coth}_q(ax). \quad (11)$$

Where the deformed hyperbolic functions is defined as:

$$\sinh_q x = \frac{e^x - qe^{-x}}{2}, \quad \cosh_q x = \frac{e^x + qe^{-x}}{2},$$

$$\tanh_q x = \frac{\sinh_q x}{\cosh_q x}.$$

The Schrödinger equation becomes

$$\frac{d^2\psi}{dx^2} + \left[ E + V_1 \frac{e^{-2ax}}{1 - qe^{-2ax}} - V_2 \operatorname{cosech}_q^2(ax) - V_3 \operatorname{coth}_q(ax) \right] \psi = 0, \quad (12)$$

where  $\hbar = 2m = 1$ . Setting the following notations

$$\varepsilon = -\frac{E}{4a^2}, \quad \beta_i = \frac{V_i}{4a^2} (> 0), \quad i = 1, 2, 3, \quad s = e^{-2ax}, \quad (13)$$

with  $\varepsilon > 0 (E < 0)$  for bound states, Eq. (12) becomes

$$\frac{d^2\psi}{ds^2} + \frac{1-qs}{s-qs^2} \frac{d\psi}{ds} + \frac{1}{(s-qs^2)^2} \left[ -\{q^2(\varepsilon + \beta_3) + q(\beta_1 - 2q\beta_3)\}s^2 + \{2q(\varepsilon + \beta_3) + (\beta_1 - 2q\beta_3 - 4\beta_2)\}s - (\varepsilon + \beta_3) \right] \psi = 0. \quad (14)$$

After the comparison of Eq. (14) with Eq. (1), we have

$$\tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s - qs^2 \quad \text{and}$$

$$\tilde{\sigma}(s) = -\{q^2(\varepsilon + \beta_3) + q(\beta_1 - 2q\beta_3)\}s^2 + \{2q(\varepsilon + \beta_3) + (\beta_1 - 2q\beta_3 - 4\beta_2)\}s - (\varepsilon + \beta_3). \quad (15)$$

Substituting these polynomials into Eq. (9), we have

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \left[ (2\sqrt{\varepsilon + \beta_3} - \mu P)qs - 2\sqrt{\varepsilon + \beta_3} \right]$$

$$\text{If } k = (\beta_1 - 2q\beta_3 - 4\beta_2) + \mu q \sqrt{\varepsilon + \beta_3} P. \quad (16)$$

$$\text{Where } \mu = +1, -1 \quad \text{and } P = \sqrt{1 + \frac{16\beta_2}{q}} = \sqrt{1 + \frac{4V_2}{qa^2}}$$

For bound state solutions, it is necessary to choose

$$\pi(s) = -\frac{qs}{2} - \frac{1}{2} \left[ 2\sqrt{\varepsilon + \beta_3} - \mu P \right] qs - 2\sqrt{\varepsilon + \beta_3} \quad (17)$$

If  $k = (\beta_1 - 2q\beta_3 - 4\beta_2) + \mu q \sqrt{\varepsilon + \beta_3} P$ . (17)

The following track in this selection is to achieve the condition  $\tau'(s) < 0$ . Therefore  $\tau(s)$  becomes

$$\tau(s) = 1 + 2\sqrt{\varepsilon + \beta_3} - [2 + 2\sqrt{\varepsilon + \beta_3} - \mu P] qs, \quad (18)$$

and then its negative derivatives become

$$\tau'(s) = -[2 + 2\sqrt{\varepsilon + \beta_3} - \mu P] qs.$$

Therefore from Eqs. (6) and (10) we have

$$\lambda = \lambda = n[2 + 2\sqrt{\varepsilon + \beta_3} - \mu P]q + n(n+1)q, \quad (19)$$

and

$$\lambda = (\beta_1 - 2q\beta_3 - 4\beta_2) + \mu q \sqrt{\varepsilon + \beta_3} P - \frac{q}{2} - \frac{q}{2} (2\sqrt{\varepsilon + \beta_3} - \mu P). \quad (20)$$

Comparing Eqs. (19) and (20) we have

$$\begin{aligned} n[2 + 2\sqrt{\varepsilon + \beta_3} - \mu P]q + n(n+1)q = \\ (\beta_1 - 2q\beta_3 - 4\beta_2) + \mu q \sqrt{\varepsilon + \beta_3} P - \frac{q}{2} - \frac{q}{2} (2\sqrt{\varepsilon + \beta_3} - \mu P), \end{aligned}$$

$$\Rightarrow (2n+1 - \mu P)\sqrt{\varepsilon + \beta_3} + n(n+1) + \frac{1}{2} - \left( n + \frac{1}{2} \right) \mu P = \frac{\beta_1 - 2q\beta_3 - 4\beta_2}{q},$$

$$\Rightarrow 4(2n+1 - \mu P)\sqrt{\varepsilon + \beta_3} + (2n+1 - \mu P)^2 = \frac{4(\beta_1 - 2q\beta_3 - 4\beta_2)}{q} - 1 + P^2.$$

Substituting the values of  $\beta_1, \beta_2, \beta_3, \varepsilon$  and  $P$  we obtain the energy eigenvalues:

$$E_n = -\frac{a^2}{4} \left[ \left( 2n+1 - \mu \sqrt{1 + \frac{4V_2}{qa^2}} \right) - \frac{V_1 - 2qV_3}{qa^2 \left( 2n+1 - \mu \sqrt{1 + \frac{4V_2}{qa^2}} \right)} \right]^2 + V_3, \quad (21)$$

$n \geq 0, q \geq 1$ . From the Eqs. (5), (8), and (15) we obtain the weight function

$$\rho(s) = s^{2\sqrt{\varepsilon + \beta_3}} (1 - qs)^{-\mu P}, \quad (22)$$

and from Eqs. (4), (15), and (17) we have

$$\rho(s) = s^{\sqrt{\varepsilon + \beta_3}} (1 - qs)^{\frac{1}{2}(1 - \mu P)}. \quad (23)$$

Now using the properties of Jacobi Polynomial [22, 23]

$$P_n^{(c,d)}(x) = \frac{(-1)^n (1-x)^{-c} (1+x)^{-d}}{2^n n!} \times \frac{d^n}{dx^n} \left[ (1-x)^{n+c} (1+x)^{n+d} \right].$$

Hence

$$P_n^{(2\sqrt{\varepsilon + \beta_3}, -\mu P)}(1 - 2qs) = \frac{(-2q)^n s^{2\sqrt{\varepsilon + \beta_3}} (1 - qs)^{-\mu P}}{n!} \times \frac{d^n}{dx^n} \left[ s^{n+2\sqrt{\varepsilon + \beta_3}} (1 - qs)^{n - \mu P} \right]. \quad (24)$$

The wave functions are obtain from Eqs. (2), (7), (22-24)

$$\psi_n(x) = N_n s^{\sqrt{\varepsilon + \beta_3}} (1 - qs)^{\frac{1}{2} \left( 1 - \mu \sqrt{1 + \frac{4V_3}{qa^2}} \right)} \times P_n^{(2\sqrt{\varepsilon + \beta_3}, -\mu \sqrt{1 + \frac{4V_2}{qa^2}})}(1 - 2qs) \quad (25)$$

where  $N_n$  is normalization constant.

#### IV. PT-SYMMETRIC POTENTIAL

In this case, we set the potential parameters in Eq. (11) as  $V_1, V_2, V_3, q \in \mathbf{R}$  and  $a \in \mathbf{IR}$  ( $a \rightarrow ia$ ) then Eq. (11) becomes

$$\begin{aligned} V(x) = -V_1 \frac{\cos(2ax) - q - i \sin(2ax)}{1 + q^2 - 2q \cos(2ax)} + \\ 4V_2 \frac{-2q + (1 + q^2) \cos(2ax) - i(1 - q^2) \sin(2ax)}{(1 + q^2 - 2q \cos(2ax))^2} + \\ V_3 \frac{1 - q^2 - 2iq \sin(2ax)}{1 + q^2 - 2q \cos(2ax)}. \quad (26) \end{aligned}$$

Then  $V(x)$  satisfies the relation  $(PT)V(x)(PT)^{-1} = V(x)$  where  $PxP^{-1} = -x, PpP^{-1} = -p = TpT^{-1}, TAT^{-1} = -iA,$

The energy eigenvalues of the potential (26) are

$$E_n = \frac{a^2}{4} \left[ \left( 2n+1 - \mu \sqrt{1 + \frac{4V_2}{qa^2}} \right) + \right]$$

$$\left. \frac{V_1 - 2qV_3}{qa^2 \left( 2n + 1 - \mu \sqrt{1 - \frac{4V_2}{qa^2}} \right)} \right]^2 + V_3, \quad (27)$$

condition for  $n$  is

$$n < \frac{1}{2} \sqrt{\frac{V_1 - 2V_3}{qa^2} - \frac{2V_3}{a^2}} + \frac{\mu}{2} \sqrt{1 - \frac{4V_2}{qa^2}} - \frac{1}{2}.$$

Eq. (27) is not similar to Eq. (21). Hence eigenvalues are always real for  $V_2 \leq \frac{qa^2}{4}$  and complex for  $V_2 > \frac{qa^2}{4}$ . The corresponding eigenfunctions are

$$\psi_n(x) = A_n s^{\sqrt{\varepsilon+\beta_3}} (1-qs) \frac{1}{2} \left( 1 - \mu \sqrt{1 - \frac{4V_3}{qa^2}} \right) \times P_n^{(2\sqrt{\varepsilon+\beta_3}, -\mu \sqrt{1 - \frac{4V_2}{qa^2}})} (1-2qs), \quad (28)$$

where  $A_n$  is normalization constant.

### V. NON $PT$ -SYMMETRIC POTENTIAL

Now let us take the potential parameters as consider the case,  $V_2, q \in \mathbf{R}$ , and  $V_1, V_3, a \in \mathbf{IR}$  ( $V_1 \rightarrow iV_1, V_3 \rightarrow iV_3, a \rightarrow ia$ ) Then potential (11) takes the form

$$V(x) = -V_1 \frac{\sin(2ax) + i(\cos(2ax) - q)}{1 + q^2 - 2q \cos(2ax)} + 4V_2 \frac{-2q + (1 + q^2) \cos(2ax) - i(1 - q^2) \sin(2ax)}{(1 + q^2 - 2q \cos(2ax))^2} + V_3 \frac{2q \sin(2ax) + i(1 - q^2)}{1 + q^2 - 2q \cos(2ax)}. \quad (29)$$

Then  $V(x)$  is non  $PT$ -symmetric. The energy eigenvalues are

$$E_n = \frac{a^2}{4} \left( 2n + 1 - \mu \sqrt{1 - \frac{4V_2}{qa^2}} \right)^2 - \frac{1}{4a^2 q^2} \left( \frac{V_1 - 2qV_3}{2n + 1 - \mu \sqrt{1 - \frac{4V_2}{qa^2}}} \right)^2 - 2i \frac{V_1}{q}. \quad (30)$$

But it has real and imaginary parts. However, the energy spectrum is not seen at the imaginary part of the energy eigenvalues, since it is independent of  $n$ . The corresponding eigenfunctions are

$$\psi_n(x) = B_n s^{\sqrt{\varepsilon-i\beta_3}} (1-qs) \frac{1}{2} \left( 1 - \mu \sqrt{1 - \frac{4V_3}{qa^2}} \right) \times P_n^{(2\sqrt{\varepsilon-i\beta_3}, -\mu \sqrt{1 - \frac{4V_2}{qa^2}})} (1-2qs), \quad (31)$$

where  $B_n$  is normalization constant.

### VI. HULTHÉN POTENTIAL

Setting  $V_2 = V_3 = 0$ , the potential (11) becomes the Hulthén potential [9, 10]

$$V(x) = -V_1 \frac{e^{-2ax}}{1 - qe^{-2ax}}. \quad (32)$$

The energy eigenvalues and wave functions of this potential are obtained from Eqs. (21), (25) setting  $\mu = -1$

$$E_n = -a^2 \left[ n + 1 - \frac{V_1}{4qa^2(n+1)} \right]^2, \quad n \geq 0, q \geq 1, \quad (33)$$

$$\psi_n(s) = C_n s^{\sqrt{\varepsilon}} (1-qs) P_n^{(2\sqrt{\varepsilon}, 1)} (1-2qs), \quad (34)$$

where  $C_n$  is normalization constant. Eqs. (33) and (34) are consistent with [16]. For  $\delta = 2a$  and  $V_1 = Z^2 e \delta$  Eq. (33) is consistent with [24].

We are now going to consider different forms of generalized Hulthén potential, viz at least one of the parameters is purely imaginary. When  $a \rightarrow ia$  and  $V_1, q \in \mathbf{R}$  then Eq. (32) becomes

$$V(x) = -V_1 \frac{\cos(2ax) - q}{1 + q^2 - 2q \cos(2ax)} + iV_1 \frac{\sin(2ax)}{1 + q^2 - 2q \cos(2ax)}. \quad (35)$$

Then  $(PT)V(x)(PT)^{-1} = V(x)$ . The real positive energy eigenvalues are given by

$$E_n = a^2 \left[ n + 1 + \frac{V_1}{4qa^2(n+1)} \right]^2, \quad q \geq 1, \quad (36)$$

if and only if

$$n = 0, 1, 2, \dots < \frac{1}{2} \sqrt{\frac{V_1}{qa^2}} - 1. \quad (37)$$

Next we set  $V_1 \rightarrow iV_1, a \rightarrow ia$  and  $q \in \mathbf{R}$ , then Eq. (32) takes the form

$$V(x) = -V_1 \frac{\sin(2ax)}{1+q^2-2q\cos(2ax)} + iV_1 \frac{q-\cos(2ax)}{1+q^2-2q\cos(2ax)}. \quad (38)$$

Such a potential is non- $PT$ -symmetric potential. The complex energy eigenvalues are given by

$$E_n = -a^2 \left[ \frac{V_1^2}{16q^2 a^4 (n+1)^2} - (n+1)^2 - i \frac{V_1}{2qa^2} \right]. \quad (39)$$

But it has real plus imaginary energy spectra. We consider the real part of energy eigenvalues an acceptable result is

obtained when  $n < \frac{1}{2} \sqrt{\frac{V_1}{qa^2}} - 1$ . However, the energy

spectrum is not seen at the imaginary part of energy eigenvalues, since it is independent of  $n$ .

## VII. MANNING-ROSEN POTENTIAL

Setting  $V_1 = 0$ , the potential (11) becomes the Manning-Rosen potential

$$V(x) = V_2 \operatorname{cosech}_q^2(ax) + V_3 \operatorname{coth}_q(ax). \quad (40)$$

The energy eigenvalues and wave functions of this potential by setting  $\mu = 1$

$$E_n = -\frac{a^2}{4} \left( 2n+1 - \sqrt{1 + \frac{4V_2}{qa^2}} \right)^2 - \frac{V_3^2}{4} \left( 2n+1 - \sqrt{1 + \frac{4V_2}{qa^2}} \right)^{-2}, \quad n \geq 0, q \geq 1, \quad (41)$$

$$\psi_n(x) = D_n s^{\sqrt{\varepsilon+\beta_3}} (1-qs)^{\frac{1}{2} \left( 1 - \sqrt{1 + \frac{4V_3}{qa^2}} \right)} \times P_n^{(2\sqrt{\varepsilon+\beta_3}, -\sqrt{1 + \frac{4V_2}{qa^2}})}(1-2qs) \quad (42)$$

where  $D_n$  is normalization constant. When  $a \rightarrow ia$  and  $V_2, V_3, q \in \mathbf{R}$ , then  $V(x)$  satisfies the relation  $(PT)V(x)(PT)^{-1} = V(x)$ . The positive energy eigenvalues are then given by

$$E_n = \frac{a^2}{4} \left( 2n+1 - \sqrt{1 - \frac{4V_2}{qa^2}} \right)^2 + \frac{V_3^2}{4} \left( 2n+1 - \sqrt{1 - \frac{4V_2}{qa^2}} \right)^{-2}, \quad n \geq 0, q \geq 1, \quad (43)$$

$$n < \frac{1}{2} \left[ \sqrt{\frac{2V_3}{a^2}} + \sqrt{1 - \frac{4V_2}{qa^2}} - 1 \right].$$

Which is consistent with [25]. Also for  $V_3 \rightarrow iV_3$ ,  $V_2, a, q \in \mathbf{R}$ , the potential (40) is  $PT$ -symmetric under the spatial reflection  $P \rightarrow \frac{\ln(q)}{a} - x$ , and in that case eigenvalues are

$$E_n = -\frac{a^2}{4} \left( 2n+1 - \sqrt{1 + \frac{4V_2}{qa^2}} \right)^2 + \frac{V_3^2}{4} \left( 2n+1 - \sqrt{1 + \frac{4V_2}{qa^2}} \right)^{-2}, \quad n \geq 0, q \geq 1, \quad (44)$$

Again for  $V_2, V_3, a, q \in \mathbf{IR}$ , the potential (40) is non- $PT$ -symmetric potential. The energy eigenvalues are given by

$$E_n = \frac{a^2}{4} \left( 2n+1 - \sqrt{1 - \frac{4V_2}{qa^2}} \right)^2 - \frac{V_3^2}{4} \left( 2n+1 - \sqrt{1 - \frac{4V_2}{qa^2}} \right)^{-2}, \quad n \geq 0, q \geq 1. \quad (45)$$

Setting  $V_2 = V_3 = 0$ , then Eq. (11) is known as Scarf potential [25].

## VIII. CONCLUSION

In this paper, the Schrödinger equation with Hulthén plus Manning-Rosen potential has been solved by using the Nikiforov-Uvarov method. Some interesting results including complex  $PT$ -symmetric and non- $PT$ -symmetric versions of the Hulthén potential and the Manning-Rosen potential have also been discussed. Energy eigenvalues for the Hulthén potential and the Manning-Rosen potential have been presented separately. It is shown that the results are in good agreement with the ones obtained by others. We have plotted the Hulthén plus Manning-Rosen potential, the Hulthén potentials, real and imaginary part of  $PT$ -symmetric Hulthén potentials and the Manning-Rosen potential for different parameter values. Figure 3 and 4 show that there is a periodic behavior of the  $PT$ -symmetric Hulthén potential and there is a real energy spectrum due to unbroken  $PT$ -symmetry. In figure 5, the Manning-Rosen potential is presented.

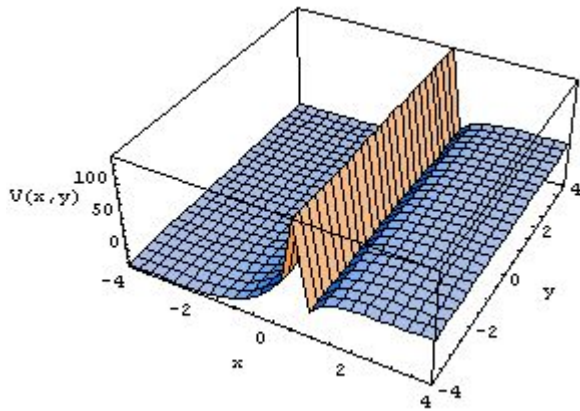


FIGURE 1. Hulthén plus Manning-Rosen potential for  $V_1 = 64, V_2 = 0.64, V_3 = 40, a = 1, q = 4$ .

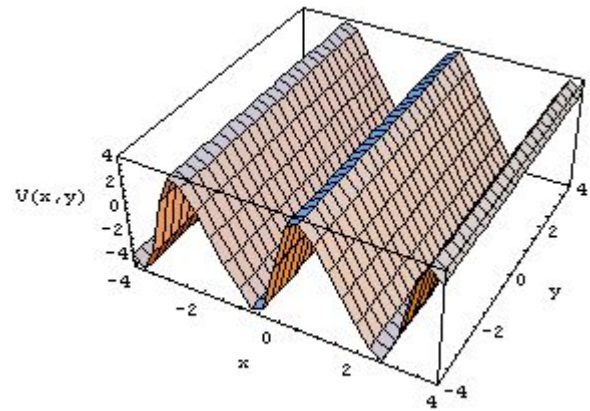


FIGURE 4. Imaginary part of  $PT$ -symmetric Hulthén potential for  $V_1 = 64, a = 1, q = 4$ .

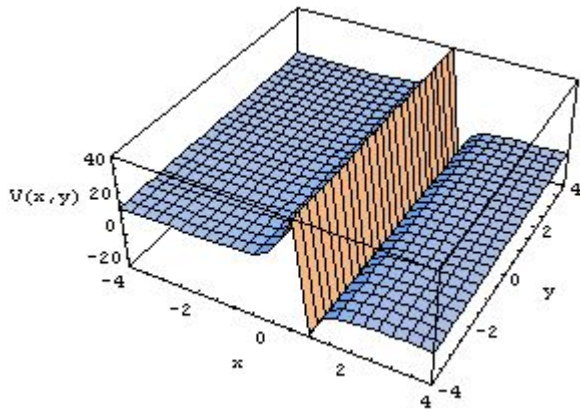


FIGURE 2. Hulthén potential for  $V_1 = 64, a = 1, q = 4$ .

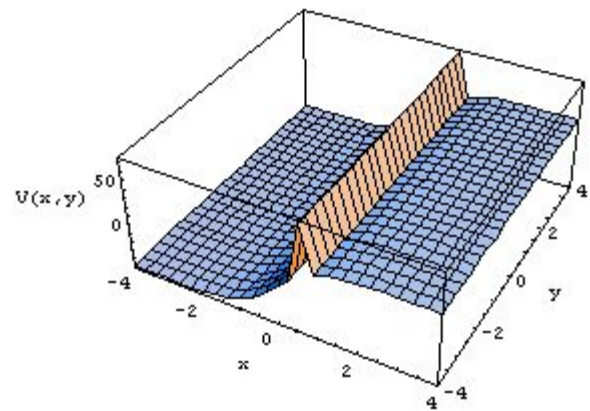


FIGURE 5. Manning-Rosen potential for  $V_2 = 0.64, V_3 = 40, a = 1, q = 4$ .

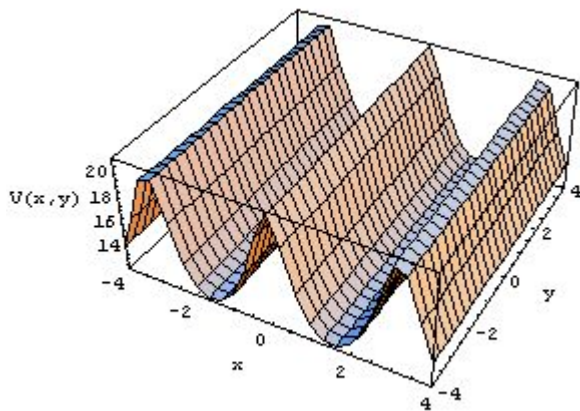


FIGURE 3. Real part of  $PT$ -symmetric Hulthén potential for  $V_1 = 64, a = 1, q = 4$ .

## REFERENCES

- [1] Bender, C. M. *et al*, *Variational Ansatz for  $PT$  - Symmetric Quantum Mechanics*, Phys. Lett. A **259**, 224-231 (1999).
- [2] Bender, C. M. *et al*, *Complex Periodic Potentials with Real Band Structure*, Phys. Lett. A **252**, 272-276 (1999).
- [3] Buslaev, V., Grecchi, V., *Equivalence of unstable anharmonic oscillators and double wells*, J. Phys. A **36**, 5541-5549 (1993).
- [4] Delabaere, E., Pham, F., *Eigenvalues of complex Hamiltonians with  $PT$  -symmetry*, Phys. Lett. A **250**, 25-28 (1998).
- [5] Bagchi, B., Quesne, C., *Non-Hermitian Hamiltonians with real and complex eigenvalues in a Lie-algebraic framework*, Phys. Lett. A **300**, 18-26 (2002).
- [6] Bagchi, B., Quesne, C.,  *$sl(2, C)$  as a complex Lie algebra and the associated Non-Hermitian Hamiltonians with real eigenvalues*, Phys. Lett. A **273**, 285-292 (2000).

- [7] Lévai, G. et al, *Algebraic and scattering aspects of a PT-symmetric solvable potential*, J. Phys. A **34**, 839-844 (2001).
- [8] Lévai, G. et al, *PT-symmetric potentials and the so(2,2) algebra*, J. Phys. A **35**, 5041-5057 (2002).
- [9] Simsek, M., Egrifes, H., *The Klein Gordon equation of generalized Hulthén potential in complex quantum mechanics*, J. Phys. A: Math. Gen. **37**, 4379-4393 (2004).
- [10] Egrifes, H., Sever, R., *Bound states of the Dirac equation for the PT-symmetric generalized Hulthén potential by the Nikiforov-Uvarov method*, Phys. Lett. A **344**, 117-126 (2005).
- [11] Grosche, C., *Path integral solutions for deformed Pöschl Teller-like and conditionally solvable potentials*, J. Phys. A: Math. Gen. **38**, 2947-2958 (2005).
- [12] Bender, C. M. and Boettcher, S., *Real Spectra in Non-Hermitian Hamiltonians Having PT-Symmetry*, Phys. Rev. Lett. **80**, 5243-5246 (1998).
- [13] Bender, C. M. and Boettcher, S., Meisinger, P. N., *PT-symmetric quantum mechanics*, J. Math. Phys. **40**, 2201-2209 (1999).
- [14] Bagchi, B., Quesne, C., Znojil, M., *Generalized Continuity Equation and Modified Normalization in PT-Symmetric Quantum Mechanics*. Mod. Phys. Lett. A **16**, 2047-2057 (2001).
- [15] Lévai, G. et al, *PT-symmetry breaking and explicit expressions for the pseudo-norm in the Scarf II potential*, Phys. Lett. A **300**, 271-281 (2002).
- [16] Meyur, S., Debnath, S., *Real Spectrum of Non-Hermitian Hamiltonians for Hulthén potential*, Modern Phys. Lett. A **23**, 2077-2084 (2008).
- [17] Japaridze, G. S., *Space of state vectors in PT-symmetrical quantum mechanics*, J. Phys. A **35**, 1709-1718 (2002).
- [18] Shin, K. C., *On the reality of the eigenvalues for a class of PT-symmetric oscillators*, Comm. Math. Phys. **229**, 543-564 (2002).
- [19] Cannata, F., Junker, G., Trost, J., *Schrödinger operators with complex potential but real spectrum*, Phys. Lett. A **246**, 219-226 (1998).
- [20] Caliceti, E., Cannata, F., Znojil, M., and Ventura, A., *Construction of PT-asymmetric non-Hermitian Hamiltonians with CPT-symmetry*, Phys. Lett. A **335**, 26-30 (2005).
- [21] Mostafazadeh, A., *Pseudo-Hermiticity versus PT symmetry. The necessary condition for the reality of the spectrum*, J. Math. Phys. **43**, 205-214 (2002).
- [22] Nikiforov, A.F., Uvarov, V.B., *Special Functions of Mathematical Physics*, Birkhäuser, Basel, 1988.
- [23] Abramowitz, M., Stegun, I., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1964.
- [24] Bayrak, O., Boztosun, I., *Analytical solutions to the Hulthén and the Morse potentials by using the asymptotic iteration method*, Journal of Molecular Structure: THEOCHEM **802**, 17-21 (2007).
- [25] Yeşiltaş, Ö., *PT/non-PT symmetric and non-Hermitian Pöschl Teller-like solvable potentials via Nikiforov Uvarov method*, Phys. Scr. **75**, 41-46 (2007).