Exactly solvable quantum mechanical systems generated from the anharmonic potentials

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Abstract

Exact analytic solution of the Schrödinger equation is reported for the newly constructed multiterm quantum mechanical potentials generated from the already solved sextic and dectic anharmonic potentials using the extended transformation method.

Keywords: Exactly solved potential, Schrödinger equation, extended transformation.

Resumen

Se reporta una solución analítica exacta de la ecuación de Schrödinger para la nueva construcción de potenciales multitérmino de la mecánica cuántica generados a partir de los ya resueltos potenciales sexticos y décticos anarmónicos utilizando el método de transformación extendida.

Palabras clave: Potencial exactamente resuelto, ecuación de Schrödinger, transformación extendida.

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I. INTRODUCTION

The exact solution of the non-relativistic Schrödinger equation for physically relevant potentials for different areas of physics and chemistry has attracted much attention of the physicist. Considerable effort [1, 2, 3, 4] has been made in order to obtain exact solution of the Schrödinger equation for the potential of physical interest. We obtain here a class of exactly solved potentials which are generated from the already exactly solved sextic and dectic anharmonic potentials using extended transformation [5, 6, 7, 8] method. The exactly solved potentials of the Schrödinger equation consist of bound state energy eigenfunctions and corresponding quantized energy eigenvalues. For each solution, an interrelation between the parameters of the potential and the orbital angular momentum quantum number has to be satisfied. The method of generation of exactly solved quantum systems is based on a transformation called the extended transformation (ET) that includes a coordinate transformation followed by a functional transformation and a set of plausible ansatze. In multiterm potential ET may be applied repeatedly by selecting the 'working potential' from the multiterm potential to generate a variety new quantum systems (QS) except for one which revert it back to the parent QS. A very useful property of the transformation method one should note is that the wavefunctions of the generated QSs are almost always normalizable. Normalizability of the eigenfunctions of the generated exactly solved potential (ESP) is essential as without this property the ESP cannot be used in quantum mechanical problem.

II. FORMALISM

A. Generation from sextic power potential

The radial part of the Schrödinger equation for a solved quantum bound state problem denoted by A-QS in three-dimensional spaces ($\hbar = 2m = 1$) is given by

$$\Psi_A''(r)+\frac{2}{r}\Psi_A'(r)+\left[E_A-V_A(r)-\frac{l_A(l_A+1)}{r^2}\right]\Psi_A(r)=0 . \qquad (1)$$

The potential of A-QS is doubly anharmonic sextic potential and is expressed as:

$$V_A(r)=ar^6+br^4+cr^2 . \qquad (2)$$

The constraint equation between the parameters of the A -QS potential and angular momentum quantum number l_A is given by [9]:

$$\frac{b^2}{4a}-\left(2_A+5\right)\sqrt{a}=c\,. \tag{3}$$

The normalized energy eigenfunctions of the A-QS is as given by [9]:

$$\Psi_A(r)=N_A r^{l_A}\exp\left[-\frac{1}{4}\sqrt{a}r^4-\frac{b}{4\sqrt{a}}r^2\right]. \tag{4}$$

The energy eigenvalues of A-QS are provided by [9]:

$$E_A=\frac{b}{\sqrt{a}}\left(l_A+\frac{3}{2}\right). \tag{5}$$

Applying ET to Eq. (1), this comprises of Coordinate transformation:

$$r\to g_B(r)\,. \tag{6}$$

Followed by a Functional transformation of the wavefunction:

$$\Psi_B(r)=f_B^{-1}(r)\Psi_A\left(g_B(r)\right), \tag{7}$$

where $\Psi_B(r)$ is the wavefunction of the transformed quantum system, henceforth called as B-QS.
This leads to following equation:

$$\Psi_B{}''(r)+\left(\frac{d}{dr}\ln\frac{f_B^2 g_B^2}{g_B{}'}\right)\Psi_B{}'(r)+\left[\left(\frac{d}{dr}\ln f_B\right)\left(\frac{d}{dr}\ln\frac{f_B{}' g_B^2}{g_B{}'}\right)+g_B{}'^2\left(E_A-V_A(g)-\frac{l_A(l_A+1)}{g_B^2}\right)\right]\Psi_B(r)=0. \tag{8}$$

The dimension of the Euclidean space of the transformed QS can be arbitrarily. Let it be denoted as D. This yield:

$$\frac{d}{dr}\ln\frac{f_B^2 g_B^2}{g_B{}'}=\frac{D-1}{r}=\frac{d}{dr}\ln r^{D-1}, \tag{9}$$

this fixes $f_{\mathbf{B}}(r)$.
Integrating

$$\ln\frac{f_B^2 g_B^2}{g'}=\ln r^{D-1}-2\ln N\,, \tag{10}$$

Where $-2\ln N$ is integration constant. This gives,

$$f_B(r)=Ng'^{\frac{1}{2}}g^{-1}r^{\frac{D-1}{2}}\,. \tag{11}$$

The transformed equation for B-QS changes to:

$$\Psi_B{}''(r)+\frac{D-1}{r}\Psi_B{}'(r)+\left[\frac{1}{2}\{g,r\}+\frac{D-1}{2}\frac{D-3}{2}\frac{1}{r^2}+g'^2\left(E_A-V_A(g)-\frac{\left(l_A+\frac{1}{2}\right)^2}{g^2}+\frac{1}{4g^2}\right)\right]\Psi_B(r)=0, \tag{12}$$

where

$$\{g,r\}=\frac{g'''}{g'}-\frac{3}{2}\left(\frac{g''}{g'}\right)^2. \tag{13}$$

To bring the second degree differential equation (12) to the standard Schrödinger equation form we make the following ansatze by selecting the working potential $V_A^W(g)=cg^2$:

$$g'^2\left(cg^2\right)=-E_B\,, \tag{14}$$

$$g'^2 E_A=-V_B^1(r)\,, \tag{15}$$

$$-g'^2\left(V_A(r)-cg^2\right)=-V_B^2(r)\,, \tag{16}$$

$$\frac{g'^2\left(l_A+\frac{1}{2}\right)^2}{g^2}=\frac{\left(l_B+\frac{D}{2}-1\right)^2}{r^2}\,. \tag{17}$$

Eq. (14) yields,

$$g(r)=\sqrt{2}\left(-\frac{E_B}{c}\right)^{\frac{1}{4}}r^{\frac{1}{2}}\,. \tag{18}$$

Which satisfies the local property $g(r)=0$, by putting the integration constant equal to zero.

Now Eqs. (15) and (16) lead to B-QS potential

$$V_B(r)=A_1 r^{-1}+B_1 r+C_1 r^2\,. \tag{19}$$

The parameters of the potential are:

$$A_1=C_B^2=\frac{\left(-E_A\right)}{2}\left(-\frac{E_B}{c}\right)^{\frac{1}{2}}, \tag{20}$$

Where C_B^2 is the characteristic constant of B-QS. It plays the same role as $-Ze^2$ in case of Coulomb and $\frac{1}{2}m\omega^2$ in case of harmonic Oscillator system and

$$B_1=2b\left(-\frac{E_B}{c}\right)^{\frac{3}{2}}\quad\text{and}\quad C_1=4a\left(-\frac{E_B}{c}\right)^2. \tag{21}$$

The constraint equation B-QS is found as:

$$A_1+\frac{B_1}{\sqrt{C_1}}\left(l_B+\frac{D-1}{2}\right)=0\,. \tag{22}$$

The energy eigenvalue of B-QS from Eq. (20) comes out to be:

$$E_B = (2l_B + D)\sqrt{C_1} - \frac{B_1^{\,2}}{4C_1}. \tag{23}$$

The corresponding energy eigenfunction is obtained from equation (7) as:

$$\Psi_B(r) = N_B r^{l_B} \exp\left[-\frac{1}{2}\sqrt{C_1}\, r^2 - \frac{B_1}{2\sqrt{C_1}}\, r\right]. \tag{24}$$

B. Generation from dectic power potential

To construct another new class of exactly solved quantum system we have applied our formalism on an already exactly solved central dectic power anharmonic potential and are given by [10]:

$$V_A(r) = ar^2 - br^4 + cr^6 - dr^8 + er^{10}. \tag{25}$$

The eigenfunction for the dimensionless Schrödinger equation for the above potential can be read as [10]:

$$\Psi_A(r) = N_A r^{l_A} \exp\left[\frac{1}{2}\alpha r^2 - \frac{1}{4}\beta r^4 + \frac{1}{6}\lambda r^6\right]. \tag{26}$$

The parameters a, b, c, d, e and angular momentum quantum number l_A are related to α, β and λ by the following set of equations:

$$\alpha^2 - 2\beta l_A - 3\beta = a, \tag{27}$$

$$5\lambda + 2\lambda l_A - 2\alpha\beta = -b, \tag{28}$$

$$\beta^2 + 2\alpha\lambda = c, \tag{29}$$

$$2\beta\lambda = d, \ \lambda^2 = e. \tag{30}$$

The corresponding energy eigenvalues of the dectic anharmonic QS is given by [10]:

$$E_A = \frac{(2l_A + 1)(d^2 - 4ce)}{8e\sqrt{e}}. \tag{31}$$

To implement ET, we have selected $V_A^w(g) = ag^2$ as the working potential and the transformation function is found as

$$g(r) = \sqrt{2}\left(-\frac{E_B}{a}\right)^{\frac{1}{4}} r^{\frac{1}{2}}. \tag{32}$$

The corresponding modified Eqs. (15) and (16) lead to

$$V_B(r) = \beta_1 r^{-1} + \beta_2 r^4 + \beta_3 r^3 + \beta_4 r^2 + \beta_5 r, \tag{33}$$

Where

$$\beta_1 = \frac{1}{2}\left(-\frac{E_B}{a}\right)^{\frac{1}{2}}(-E_A), \tag{34}$$

$$\beta_2 = 2^4 e\left(-\frac{E_B}{a}\right)^3, \tag{35}$$

$$\beta_3 = 8(-d)\left(-\frac{E_B}{a}\right)^{\frac{5}{2}}, \tag{36}$$

$$\beta_4 = 4c\left(-\frac{E_B}{a}\right)^2 \tag{37}$$

and

$$\beta_5 = 2(-b)\left(-\frac{E_B}{a}\right)^{\frac{3}{2}}. \tag{38}$$

The parameters of the potential are connected by the following two constraint equations:

$$(2l_B + D + 1)\sqrt{\beta_2} + \frac{\beta_3\beta_4}{2\beta_2} - \frac{\beta_3^{\,3}}{8\beta_2^{\,2}} = \beta_5, \tag{39}$$

$$(2l_B + D - 1)\left(\frac{\beta_4}{2\sqrt{\beta_2}} - \frac{\beta_3^{\,2}}{8\beta_2^{3/2}}\right) = \beta_1. \tag{40}$$

The energy eigenvalue of B-QS is found as:

$$E_B = -\left[\left(\frac{\beta_4}{2\sqrt{\beta_2}} - \frac{\beta_3^{\,2}}{8\beta_2^{3/2}}\right)^2 + \frac{(2l_B + D)\beta_3}{2\sqrt{\beta_2}}\right]. \tag{41}$$

The corresponding energy eigenfunction of B-QS can be obtained from equation (7) and is

$$\Psi_B(r) = N_B r^{l_B} \exp\left[\left(\frac{\beta_4}{2\sqrt{\beta_2}} - \frac{\beta_3^{\,2}}{8\beta_2^{\frac{3}{2}}}\right) r + \frac{\beta_3}{4\sqrt{\beta_2}} r^2 + \frac{\sqrt{\beta_2}}{3} r^3\right]. \tag{42}$$

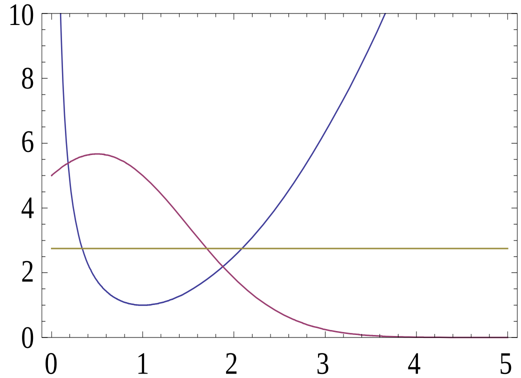

FIGURE 1. The curves are for the B-QS $\{V_{B0}(r), \Psi_{B0}(r), E_{B0}\}$ in generated from the sextic potential, where the parameter set is $A_1 = 1; B_1 = -1; C_1 = 1; l_B = 0; E_{B0} = 2.75$ **.**

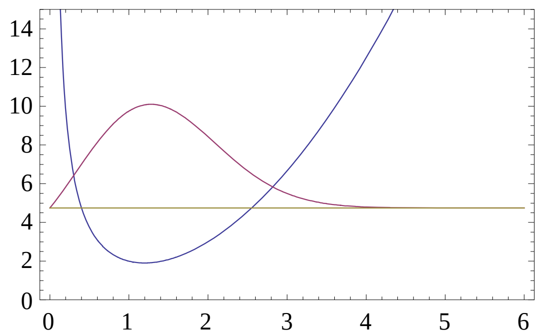

FIGURE 2. The curves are for the B-QS $\{V_{B1}(r), \Psi_{B1}(r), E_{B1}\}$ generated from the sextic potential, where the parameter set is $A_1 = 2; B_1 = -1; C_1 = 1; l_B = 1; E_{B1} = 4.75$.

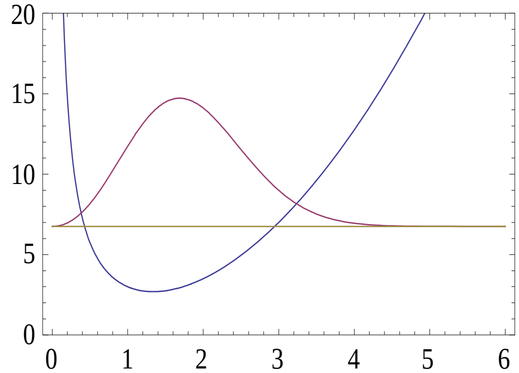

FIGURE 3. The curves are for the B-QS potential $\{V_{B2}(r), \Psi_{B2}(r), E_{B2}\}$ generated from the sextic potential, where the parameter set is $A_1 = 3; B_1 = -1; C_1 = 1; l_B = 2; E_{B2} = 6.75$ **.**

III. NORMALIZABILITY OF THE GENERATED QUANTUM SYSTEMS

The normalizability condition of the generated wavefunction of the bound state QS obtain by the extended transformation can be proved under fairly general condition, as it seems to preserve the normalizability property to a quite a good extent. Normalizability condition for $D-$dimensional B-QS eigenfunction is:

$$|N_B|^2 \int_0^\infty |\Psi_B|^2 r^{D-1} dr = finite\ .$$

This can be reduced to

$$|N_B|^2 \frac{\langle V_A(r)\rangle}{-E_B} = 1\,.$$

Since

$$g'^2(r) = \frac{-E_B}{V_A(g(r))}.$$

Hence the entire $\Psi_B(r)$ are normalizable for $E_B \neq 0$. For any physical QS, $\langle V_A(r)\rangle$ exists ensuring normalizability of daughter B-QS. As such, the wavefunction of the generated QS are always normalizable corresponding to a non-null eigenvalues, when the wavefunction of the parent QS are normalizable.

IV. CONCLUSIONS

We deal with the Schrödinger equation with some central anharmonic potential in arbitrary dimensional spaces for radial wavefunctions through a simple mapping procedure. We have obtained a class of exactly solved potentials for the Schrödinger equation which may find applications in different branches of Physics and Chemistry. The eigenfunctions are normalizable for must of the cases and bound state energy eigenvalues spectrum are found. The constraint equation relating the parameters of the potentials and angular momentum quantum numbers have been obtained for different potentials. In quantum multiterm potentials it is possible to generate a finite number of different exactly solved quantum systems by selecting the working potential, but we restrict ourselves to taking one term working potential for simplicity.

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