Determination of electromagnetic cavity modes using the Finite Difference Frequency-Domain Method

J. Manzanares-Martínez[1], D. Moctezuma-Enriquez[2], R. Archuleta-García[2]
[1]*Centro de Investigación en Física de la Universidad de Sonora, Apartado Postal 5-088, Hermosillo, Sonora CP 83190, México.*
[2]*Programa de Posgrado en Ciencias(Física), Centro de Investigación en Física de la Universidad de Sonora, Apartado Postal 5-088, Hermosillo, Sonora CP 83190, México.*

E-mail: jmanza@cifus.uson.mx

(Received 17 February 2010, accepted 16 May 2010)

Abstract
In this communication we propose a numerical determination of the electromagnetic modes in a cavity by using the Finite Difference Frequency-Domain Method. We first derive the analytical solution of the system and subsequently we introduce the numerical approximation. The cavity modes are obtained by solving an eigenvalue equation where the eigenvectors describe the eigenfunctions on the real space. It is found that this finite difference method can efficiently and accurately determine the resonance modes of the cavity with a small amount of numerical calculation.

Keywords: Numerical computation, Finite Differences, Frequency-Domain.

Resumen
En este trabajo proponemos una determinación numérica de los modos electromagnéticos en una cavidad por medio del uso del Método de Diferencias Finitas en el Dominio de la Frecuencia. Primero derivamos la solución analítica del sistema y subsecuentemente introducimos una aproximación numérica. Los modos de la cavidad son obtenidos al resolver la ecuación de eigenvalores donde los eigenvectores describen las eigenfunciones en el espacio real. Se puede observar que este método puede ser eficiente y preciso para determinar los modos de resonancia de la cavidad sin necesidad de cálculos numéricos excesivos.

Palabras clave: Computación numérica, Diferencias Finitas, Dominio de la Frecuencia.

PACS: 41.20.Jb,41.20.CV,42.25.Bs **ISSN 1870-9095**

I. INTRODUCTION

Since the introduction of Maxwell equations a great effort has been developed to the modeling of different practical situations. Analytical methods are limited to simple geometries. For complicate geometries it is necessary a numerical formulation. Different numerical methods have been applied to analyze electromagnetic problems. For example, for Photonic Crystals the Plane Wave Method (PWM) solves the wave equation using a Fourier expansion of the periodic functions [1, 2]. On the other hand, to determine the electromagnetic field distribution for non-periodic problems are used methods as the Finite Element Method (FEM), [3, 4, 5, 6, 7] Method of Moments (MoM) [8, 9, 10] or Finite Difference Time-Domain Methods (FDTD). [11, 12, 13] Usually PWM, FEM, MoM and FDTD allow a good approximation for problems with complex boundaries.

In this work we introduce the analysis of the wave equation using Finite Difference Frequency-Domain Method (FDFD). [14, 15] We consider an eigenvalue equation where the electromagnetic fields are described in the real space. We have found that our method is flexible and gives good convergence with a minimal numerical effort. Even if the physical system is simple, the formulation here described can be easily extended to more complicate problems.

This paper is organized as follows. Section 2 presents the problem and its analytical solution. Section 3 introduces the finite difference version of the wave equation. In order to illustrate our ideas, first we proceed by solving three analytical cases and then we consider the general problem. In section 4 we present an analysis to illustrate the accuracy of the method. Finally, conclusions are outlined in section 5.

II. ANALYTICAL ANALYSIS

We apply the FDFD to find the resonant modes in a one-dimensional metal cavity resonator. Several practical situations involve the propagation or excitation of electromagnetic waves in hollow metallic container or *cavities* [16].

The cavity is presented on Fig. 1. In panel (a) we illustrate an air segment *d* surrounded by metallic boundaries. The system is similar to the well-known infinite quantum well that we illustrate on panel (b).

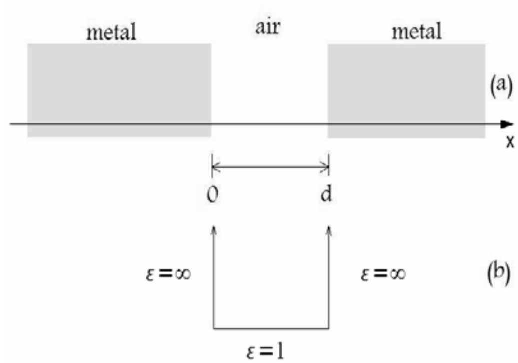

FIGURE 1. One dimensional cavity of width *d*. Panel (a) shows a segment of air limited by perfect metal. Panel (b) illustrates the similarity of the cavity with an infinite quantum well.

With sinusoidal time dependence $e^{-i\omega t}$ for the fields, the electromagnetic wave equation is

$$\frac{\partial^2}{\partial x^2}E(x) = -\left(\frac{\omega^2}{c^2}\right)\varepsilon(x)E(x). \tag{1}$$

The dielectric function is position dependent on the form

$$\varepsilon(x) = \begin{cases} \infty & x = 0, \\ 1 & 0 < x < d, \\ \infty & x = d. \end{cases} \tag{2}$$

The boundary conditions are

$$\begin{aligned} E(x=0) &= 0, \\ E(x=d) &= 0. \end{aligned} \tag{3}$$

The analytical solution of the wave equation is

$$E(x) = E_0 \sin\left(\frac{m\pi x}{d}\right) \tag{4}$$

In the Figure 2 we illustrate the first four eigenfunctions. The allowed frequencies of the resonant cavity are

$$\left(\frac{m\pi}{d}\right)^2 = \left(\frac{\omega}{c}\right)^2. \tag{5}$$

It is convenient to write the solutions in terms of a reduced frequency

$$\hat{\omega}_m^A = \pm m, \tag{6}$$

where we have introduced

$$\hat{\omega}_m^A = \frac{\omega d}{\pi c}. \tag{7}$$

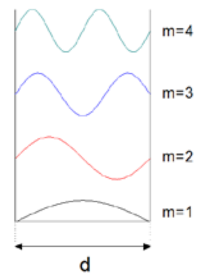

FIGURE 2. Illustration of the first eigenfunctions of the resonant cavity.

III. THE FINITE DIFFERENCE SOLUTION

We consider a finite difference approximation to the derivative using central finite differences [11]

$$\frac{\partial}{\partial x}E(x) = \frac{E(x+\Delta x/2) - E(x-\Delta x/2)}{\Delta x}. \tag{8}$$

Where $\Delta x = d/N$ and N is the number of partitions of the cavity. The second derivative is

$$\frac{\partial^2}{\partial x^2}E(x) = \frac{E(x+\Delta x) - 2E(x) + E(x-\Delta x)}{(\Delta x)^2}. \tag{9}$$

We substitute this equation in eq. (1) to obtain the finite difference version of the wave equation

$$-E(x_i+\Delta x) + 2E(x_i) - E(x_i+\Delta x) = \lambda E(x_i), \tag{10}$$

where we have introduced

$$\lambda = \left(\frac{\omega d}{cN}\right)^2. \qquad 11)$$

It is convenient to write the reduced frequency as

$$\hat{\omega} = \frac{N}{\pi}\sqrt{\lambda}. \tag{12}$$

To illustrate the solutions of Eq. (10) we first consider an analytical treatment for the cases of N =2, 3, 4. Then we consider the case of arbitrary number of partitions N.

A. The case of N=2

The case of a partition N=2 for the cavity is illustrated in Fig. 3, panel (a). The finite difference wave equation for the point $x_1 = d/2$ is

$$-E(d)+2E(d/2)-E(0)=\lambda E(d/2). \tag{13}$$

Considering the boundary conditions of Eq. (3) we find that the solution is

$$\lambda = 2. \tag{14}$$

The reduced frequency is

$$\hat{\omega} = \frac{2\sqrt{2}}{\pi} = 0.9003. \tag{15}$$

This value is our first approximation to the analytical solution $\hat{\omega}_1^A = 1$. In order to obtain a better convergence it is necessary to take more partitions *N*.

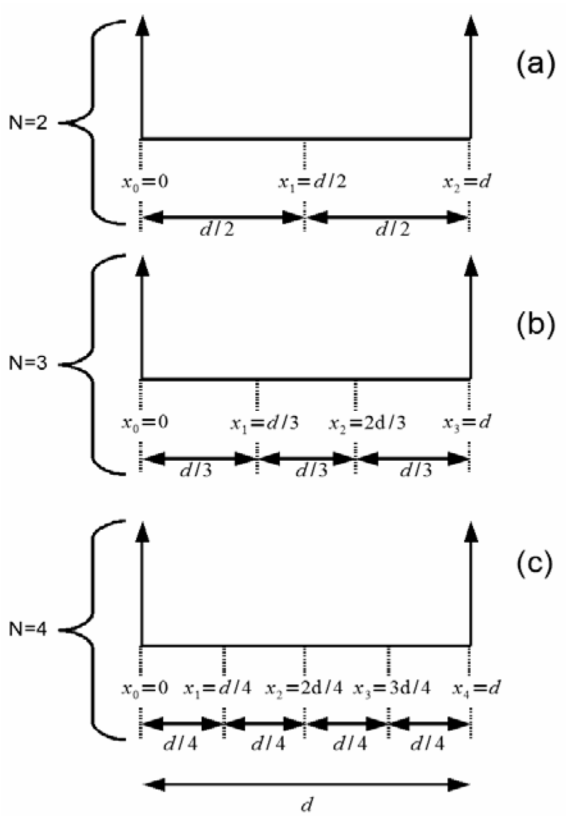

FIGURE 3. Discrete Partitions of the cavity of width *d*. Panel (a), (b) and (c) show the cases for N=2, 3 and 4, respectively.

B. The case of *N*=3

In Fig. 3, panel (b) is show the partition of the cavity for the case *N*=3. The finite difference version of the wave equation for the points $x_1 = (1/3)d$, $x_2 = (2/4)d$ is

$$-E(2d/3)+2E(d/3)-E(0)=\lambda E(d/3), \tag{16}$$

and

$$-E(d)+2E(2d/3)-E(d/3)=\lambda E(2d/3). \tag{17}$$

Taking account of the boundary conditions [Eq. (3)], these equations can be written as an eigenvalue problem

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} E(d/3) \\ E(2d/3) \end{bmatrix} = \lambda \begin{bmatrix} E(d/3) \\ E(2d/3) \end{bmatrix}. \quad 18)$$

The solutions can be found by solving the determinant

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0. \tag{19}$$

The characteristic polynomial is

$$\lambda^2 - 4\lambda + 3 = 0. \tag{20}$$

The solutions are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= 3. \end{aligned} \tag{21}$$

Using Eq. (12) we found the reduced frequencies

$$\hat{\omega}_1 = \frac{3}{\pi} = 0.9549, \tag{22}$$

and

$$\hat{\omega} = \pm\frac{3\sqrt{3}}{\pi} = 1.6539. \tag{23}$$

We have found that the first eigenvalue is closer to the analytical. On the other hand, $\hat{\omega}_2 = 1.6539$ is far from the analytical solution $\hat{\omega}_2 = 2$.

C. The case of *N*=4

The case of four partitions is illustrated in Fig. 3, panel (c). The wave equation for the discrete points $x_1 = (1/4)d$, $x_2 = (2/4)d$, $x_3 = (3/4)d$ is

$$-E(2d/4)+2E(d/4)-E(0)=\lambda E(d/4), \tag{24}$$

$$-E(3d/4)+2E(2d/4)-E(d/4)=\lambda E(2d/4). \tag{25}$$

and

$$-E(d)+2E(3d/4)-E(2d/4)=\lambda E(3d/4). \tag{26}$$

This system can be written as a an eigenvalue problem in the form

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} E(d/4) \\ E(2d/4) \\ E(3d/4) \end{bmatrix} = \lambda \begin{bmatrix} E(d/4) \\ E(2d/4) \\ E(3d/4) \end{bmatrix}. \tag{27}$$

The solutions are found by solving the determinant

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0. \tag{28}$$

The characteristic polynomial is

$$(2-\lambda)\left(\lambda^2 - 4\lambda + 2\right) = 0. \tag{29}$$

The solutions are

$$\begin{aligned} \lambda_1 &= 2-\sqrt{2}, \\ \lambda_2 &= 2, \\ \lambda_3 &= 2+\sqrt{2}. \end{aligned} \tag{30}$$

The corresponding reduced frequency solutions are

$$\hat{\omega}_1 = \frac{4}{\pi}\sqrt{2-\sqrt{2}} = 0.9795, \tag{31}$$

$$\hat{\omega}_2 = \frac{4}{\pi}\sqrt{2} = 1.8006, \tag{32}$$

$$\hat{\omega}_3 = \frac{4}{\pi}\sqrt{2+\sqrt{2}} = 2.3526. \tag{33}$$

The first reduced frequency has now a good approximation to the analytical solution. Nevertheless, for the second and third value we have not acceptable values.

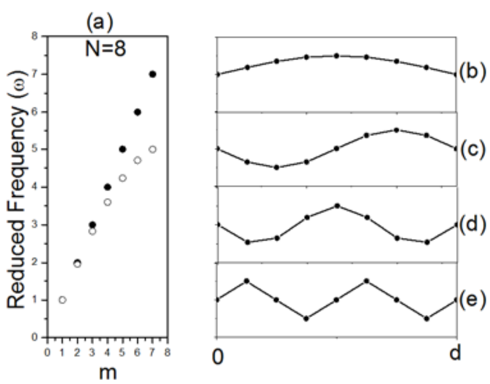

FIGURE 4. Numerical solution for N=8. Panel (a) show the comparison between analytical solution (filled circles) with the numerical solution (open circles). Panels (b)-(e) present the first four eigenfrequencies of the cavity (m=1, 2, 3 and 4).

D. The general Case

For the general case of any further number of partitions N we can write the system of equations

This system is of the form $A\mathbf{X} = \lambda\mathbf{X}$ and we can solve with standard numerical techniques to find the eigenvalues (λ) and the eigenvectors ($\mathbf{X}$).

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} E(d/N) \\ E(2d/N) \\ E(3d/N) \\ \vdots \\ E[(N-3)d/N] \\ E[(N-2)d/N] \\ E[(N-1)d/N] \end{bmatrix} = \lambda \begin{bmatrix} E(d/N) \\ E(2d/N) \\ E(3d/N) \\ \vdots \\ E[(N-3)d/N] \\ E[(N-2)d/N] \\ E[(N-1)d/N] \end{bmatrix}. \tag{34}$$

TABLE I. Comparative table for the numerical approximations of the eigenvalues. In the first Column we have the analytical solution. On the other columns we have the numerical approximations for the several values of N.

$\hat{\omega}_m^A$	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	0.9003	0.9549	0.9795	0.9836	0.9886	0.9916	0.9936
2		1.6539	1.8006	1.8710	1.9099	1.9338	1.9490
3			2.3526	2.5752	2.7009	2.7768	2.8295
4				3.0273	3.3080	3.4641	3.6013
5					3.6892	4.0150	4.2346
6						4.3445	4.7053
7							4.9951

IV. NUMERICAL EXAMPLES

In table I we present a comparison for the approximation of the eigenvalues for the first N = 8 partitions values. The first column presents the analytical eigenvalue. The others columns present numerical eigenvalues. In Figure 4 we present the solutions that are obtained for N=8. In panel (a) we present with solid circles the analytical solutions and with open circle the numerical solution. We observe that the convergence is very good for the first eigenfrequency, m=1. For the second and third eigenfrequencies (m=2, 3), the convergence is acceptable. For the fourth and greater frequencies ($m \geq 4$), we have not an acceptable convergence of the numerical approximations. In panels (b) - (e) we present the first four eigenfunctions.

We have as the number of partitions increases. In Fig. 5, panels (a) and (b) we present the comparison of the analytical (solid circle) and numerical (open circle) for partitions of N=20 and N=50, respectively. We find a good convergence for $m \leq 5$and $m \leq 12$, respectively.

V. CONCLUSIONS

We have presented a numerical procedure to determine the resonant modes for a cavity limited by metallic boundaries. We have introduced a finite difference version of the wave equation and then we propose an eigenvalue problem where the eigenfunctions describe the electromagnetic field in the real space. We have found that our procedure obtains good accuracy for the first eigenvalues. Our future work will be devoted to apply this method to more complex geometries in periodic systems, in particular we are interested in the application of this method to the case of photonic crystals.

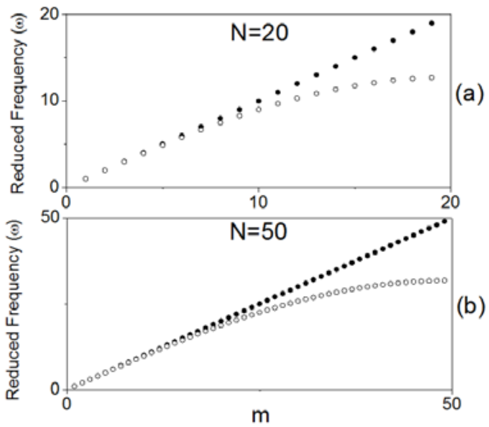

FIGURE 5. Comparison of the analytical (filled circles) and numerical (open circles) determination of the eigenvalues. Panel (a) and (b) show the cases for N=20 and N=50, respectively.

ACKNOWLEDGEMENTS

JMM thanks the financial support provided by *Universidad de Sonora* and CONACYT (Mexico) for a sabbatical Year at *Instituto de Optica-CSIC* (Spain). This work was partially supported by 90211-CONACYT project. We also thank the special support of “Programa para el mejorameinto del Profesorado” PROMEP 2009, Mexico.

REFERENCES

[1] Archuleta-García, R., Manzanares-Martínez, M. B., Manzanares-Martines, J., *Una descripción del método de ondas planas para el cálculo de bandas fotónicas,* Rev. Bol. de Fís. **13**, 79-85 (2007).

[2] Joannopoulous, J. D., Jhonson, S., Winn, J. N., *Photonic Crystals: Molding the flow of light* , (Princeton University Press, Princeton New-Jersey, 1995).

[3] Jianming, J.,*The Finite Element Method in Electromagnetic* (Wiley-Interscience, New York, 1993).

[4] Ozgun, O. and Kuzuoglu, *Finite Element Analysis of Electromagnetic Wave Problems via Iterative Leap-Field Domain Decomposition Method* M., Journal of Electromagnetic Waves and Applications JEMWA 2**2**, 251-266 (2008).

[5] Zhou, X. and Pan, G. W., *Application of Physical Spline Finite Element Method (PSFEM) to Fullwave Analysis of Waveguides Progress in Electromagnetics Research*, Progress in Electromagnetics Research PIER **60**, 19-41 (2006).

[6] Wei, X. C., Li, E. P., Zhang, Y. J., *Application of the Improved Finite Element-Fast Multipole Method on Large Scattering Problems*, Progress in Electromagnetics Research PIER **47**, 49-60 (2004).

[7] Zhou, X, *Physical Spline Finite Element (PSFEM) Solutions to One Dimensional Electromagnetic Problems* , Progress in Electromagnetics Research PIER **40**, 271-294 (2003).

[8] Khalaj-Amirhosseini, M., *Analysis of Longitudinally Inhomogeneous Waveguides Using the Method of Moments*, Progress in Electromagnetics Research PIER **74**, 57-67, (2007).

[9] Wang, C. F. and Gan, Y. B., *2D Cavity Modeling Using Method of Moments and Iterative Solvers*, Progress in Electromagnetics Research PIER **43**, 123-142, (2003).

[10] Li, L. W., Zhao, T. X., Leong, M. S., Yeo, T., *A Spatial-Domain Method of Moments Analysis of a Cylindrical-Rectangular Chirostrip* , Progress in Electromagnetics Research PIER **35**, 165-182, (2002).

[11] Taflove, A., *Computational Electrodynamics:The finite difference Time-Domain Method*, (Artech House, Boston, 1995).

[12] López-Esquer, M. A., Gaspar-Armenta, J., Manzanares-Martínez, J., *Aplicación del método de diferencias finitas en el dominio del tiempo a la simulacion del campo electromagnetico usando Matlab*, Revista Mexicana de Física- E **52**, 58, (2006).

[13] Manzanares-Martinez, J. y Gaspar-Armenta, J., *Direct integration of the constitutive relations for modeling dispersive metamaterials using the finite difference time-domain technique*, Journal of Electromagnetics Waves and Applications JEMWA **21**, 2297-2310 (2007).

[14] Kuzu, L., Demir, V.,Elsherbeni, A. Z., Arvas, E., *Electromagnetic Scattering from Arbitrarily Shaped Chiral Objects Using the Finite Difference Frequency Domain Method*, Progress in Electromagnetics Research PIER **67**, 1-24 (2007).

[15] Al-Sharkawy, M. H., Demir, V., and Elsherbeni, A. Z., *The Iterative Multi-Region Algorithm Using a Hybrid Finite Difference Frequency Domain and Method of Moment Techniques* Progress in Electromagnetics Research PIER **57**, 19-32, (2006).

[16] Geyi, W., *Time-Domain Theory of Metal Cavity Resonator*, Progress in Electromagnetics Research, PIER **78**, 219-253, (2008).