# Wave packet evolution of damped oscillator with a time dependent linear potential

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**Abstract**

We discuss the general solution of a time-dependent Schrödinger wave equation (SWE) with time-dependent linear potential within the framework of invariant operators. We obtain the Gaussian wave packet evolution by choosing the ansatz for the weight factor of the form $g(\lambda) = Ae^{-a\lambda^2+\frac{b}{\lambda^2}}$ which is the eigen function of this operators.

**Keywords:** Quantum Mechanics, time-dependent Harmonic Oscillator.

**Resumen**

Se discute la solución general de una ecuación de onda de Schrödinger dependiente del tiempo con un potencial lineal dependiente del tiempo dentro del marco de referencia de operadores invariantes. Se obtuvo la evolución del paquete Gaussiano de onda escogiendo el ansatz para el factor de peso de la forma $g(\lambda) = Ae^{-a\lambda^2+\frac{b}{\lambda^2}}$, la cual es la eigen función de este operador.

**Palabras clave:** Mecánica Cuántica, Oscilador Armónico dependiente del tiempo.

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## I. INTRODUCCIÓN

In recent time the time-dependent Harmonic Oscillator (TDHO) continues to have widespread applications in various branches of Physics [1, 2, 3, 4, 5, 6, 7]. A great attention has been paid to solving exactly time-dependent quantum mechanical problems. Among such methods of solving this problem is the famous quantum invariant operator by Lewis [8]. Other interesting methods of solving (TDHO) includes the propagator [7, 9, 10], and time-space transformation approaches [11]. The study of TDHO apart from their mathematical interest has many applications in the fields of quantum transport [12, 13], quantum optics [14, 15] and quantum information [16]. The analytical solutions of the one dimensional TDHO with a time-dependent linear potential has attracted considerable interest recently [17, 18]. Guedes [17] solved the TDHO with a time-dependent linear potential using the invariant operator of Lewis and Riesenfeld [19]. Following the trend, Feng [18] use the space -- time transformation approach to find the plane-wave type and the Airy -- packet solutions. Bekkar et al however commented that the Airy packet solutions are just a superposition of the plane-wave type solution [20].

Schuch [21] has solved the TDHO using the Riccati and Ermaker methods for harmonic and the free motion. In terms of the test-function methods, G. Lu *et al.* [22] has constructed an exact wave -- packet train (GWPT) solution, whose centre moves accelerating along the corresponding classical trajectory. In addition, the wave packet solutions in terms of the Wigner Distribution Functions (WDF) had been constructed by Schuch [23].

Luan and Tang [24] in their paper re-examined the solution of the TDHO in the presence of a time dependent linear potential using the invariant operator raised by Guedes [17].Following the approach raised by Luan and Tang [24] and Maamache and Sadi [26], we choose a generalized ansatz for the weight factor and solve the Schrödinger equation to obtain the general wave function solution for the TDHO with linear damping potential .The organization of the paper is as follows. In section 2, we examine the invariant operator of time dependent Hamiltonian. In section 3, we evaluate the exact solution of the SWE and Wave packet evolution of the system. A brief conclusion is given in section 4.

## II. INVARIANT OF TIME-DEPENDENT HAMILTONIAN

The time-dependent Schrödinger equation describing the motion of a damped system is given by

$$i\hbar\frac{d}{dt}\psi(q,t)=\hat{H}\psi(q,t), \tag{1}$$

where $\hat{H}$ is the Hamiltonian Operator being induced by the external time dependent driving force $F(t)$, is define as

$$\hat{H}(t)=e^{-\gamma t}\frac{\hat{p}^2}{2m}-F(t)\hat{q}e^{\gamma t}, \tag{2}$$

where $\hat{p}$, $\hat{q}$ are the canonical momenta and co-ordinate and $\gamma$ is the damping coefficient.

The solution of Eq. (1) is possible if a non-trivial Hermitian operator $I(t)$ exists and satisfies the invariant equation

$$\frac{dI(t)}{dt}=\frac{\partial \mathrm{I}(t)}{\partial t}+(i\hbar)^{-1}[I,H]=0. \tag{3}$$

The invariant operator $I(t)$ obeys the eigen value equation

$$\hat{I}(t)\varphi_\lambda(q,t)=\lambda\varphi_\lambda(q,t), \tag{4}$$

where $\varphi_\lambda(q,t)$ is the eigen function and $\lambda$ is the corresponding eigen values. The solution of Eq. (4) takes the form,

$$\varphi_\lambda(q,t)=exp\left\{\frac{i}{\hbar}\left[\frac{2\left(\lambda-C(t)q-B_oq^2\right)}{2A(t)}\right]\right\}, \tag{5}$$

where the constants $A(t), B_o(t)$ and $C(t)$ are to be determined later. The general wave function $\Psi(q,t)$ of Eq. (1) is related to $\psi_\lambda(q,t)$ by

$$\psi(q,t)=\int d\lambda\; g(\lambda)e^{i\alpha_\lambda(t)}\varphi_\lambda(q,t), \tag{6}$$

where $\alpha_\lambda(t)$ satisfies the eigen value equation for the Schrödinger equation as

$$\hbar\varphi_\lambda(q,t)\dot{\alpha}_\lambda(t)=i\left(\hbar\frac{\partial}{\partial t}-H\right)\varphi_\lambda(q,t)\ . \tag{7}$$

The time-dependent invariant operator $I(t)$ is of the form [24]

$$\hat{I}(t)=A(t)\hat{p}+B(t)\hat{q}+C(t), \tag{8}$$

and taking the time-derivative of Eq. (8) yields

$$\frac{d\hat{I}(t)}{dt}=\dot{A}(t)\hat{P}+A(t)\dot{\hat{P}}+\dot{B}(t)\hat{q}+B(t)\dot{\hat{q}}+\dot{C}(t), \tag{9}$$

where the dot denotes the derivative with respect to time. The commutation of the invariant operator $I(t)$ and the Hamiltonian, using Eqs. (2) and (8) becomes

$$\left[I(t),\hat{H}\right]=i\hbar A(t)F(t)e^{-\gamma t}+\frac{i\hbar}{m}\hat{P}B(t)e^{\gamma t}, \tag{10}$$

and putting Eqs. (9-10) into Eq. (3), we obtain the time-dependent coefficients as:

$$A(t)=A_o-\frac{B_o}{2}\left(t-\frac{2}{\gamma}\right)e^{\gamma t}, \tag{11}$$

$$B(t)=B_o\ , \tag{12}$$

$$C(t)=C_o-A_oF(t)\left(1+\frac{1}{\gamma}\right)e^{\gamma t}+\frac{B_o}{m}\left(\frac{t^2}{4}+\frac{t}{2}+\frac{t}{\gamma}\right)e^{\gamma t}-\frac{e^{\gamma t}}{\gamma^2}. \tag{13}$$

Furthermore, substituting Eqs. (5), (11-13) into Eq. (7), we get

$$\alpha_\lambda(t)=\alpha_\lambda(0)-\int_0^t\left\{\left[\frac{\lambda-\left(C_o-A_oF(t)\left(1+\frac{1}{\gamma}\right)e^{\gamma t}+\frac{B_o}{m}\left(\frac{t^2}{4}+\frac{t}{2}+\frac{t}{\gamma}\right)e^{\gamma t}-\frac{e^{\gamma t}}{\gamma^2}\right)}{2m\hbar\left[A_o-\frac{B_o}{2}\left(t-\frac{2}{\gamma}\right)e^{\gamma t}\right]^2}\right]^2 dt\right\}+\int_0^t\frac{i}{2m\gamma t}\frac{B_O}{A_O}\ln\left[\frac{B_O}{2A_O}\left(t-\frac{2}{\gamma}\right)\right]dt\ , \tag{14}$$

Now using Eq. (6), we write the wave function $\psi_\lambda(q,t)$ as

$$\psi_\lambda(q,t)=e^{i\alpha_\lambda(t)}\varphi_\lambda(q,t), \tag{15}$$

and on substituting Eqs. (5) and (14) into Eq. (15) yields:

$$\psi_\lambda(q,t)=\sqrt{\frac{B_o}{2A_o}\left(t-\frac{2}{\gamma}\right)}\,e^{-i\left(\frac{B_o}{A_o}\right)\gamma t+i\alpha_\lambda(0)}\;X$$

$$exp\left\{-i\int_0^t\left[\frac{\left[\lambda-\left(\sqrt{C_o-A_oF(t)\left(1+\frac{1}{\gamma}\right)e^{\gamma t}-\frac{B_o}{m}\left(\frac{t^2}{4}+\frac{t}{2}-\frac{t}{\gamma}\right)e^{\gamma t}-\frac{e^{\gamma t}}{\gamma^2}}\right)\right]^2}{2m\hbar\left[A_o-\frac{B_o}{2}\left(t-\frac{2}{\gamma}\right)e^{\gamma t}\right]^2}\right]dt\right\}$$

$$X\ \exp\left\{\frac{i}{\hbar}\left[\frac{2\left(\lambda-\left(C_o-A_oF(t)\left(1+\frac{1}{\gamma}\right)e^{\gamma t}-\frac{B_o}{m}\left(\frac{t^2}{4}+\frac{t}{2}-\frac{t}{\gamma}\right)e^{\gamma t}-\frac{e^{\gamma t}}{\gamma^2}\right)\right)q-B_oq^2}{2\left[A_o-\frac{B_o}{2}\left(t-\frac{2}{\gamma}\right)e^{\gamma t}\right]}\right]\right\} \tag{16}$$

Here $\frac{B_o}{A_o}\equiv\mathcal{F}_o$ as defined in Ref [24] and the factor $\mathcal{F}_o$ must satisfy

$$Im\left(\mathcal{F}_o\right)=0, \tag{17}$$

to ensure the physical acceptable solution of Eq. (4).

## IIII. WAVE PACKET EVOLUTION

The general solution of the Schrödinger equation is obtained via Eq. (6) by choosing the appropriate weight factor. The weight factor chosen in [20] leads to Airy function solutions while that used by Maamache and Sadi [26] gives a general wave-packet solution.

For our discussion, we use the ansatz for the weight factor in the form;

$$g(\lambda)=\left(\frac{\sqrt{a}}{\sqrt{2\hbar A_o\pi\sqrt{2\pi}}}\right)^{\frac{1}{2}}e^{-\left(a\lambda^2+\frac{b}{\lambda^2}\right)}, \tag{18}$$

where a and b are real constant. We present in figure (1) and Figure (2) the plot of $g(\lambda)$ with $\lambda$ for Eq. (18) and that of Maamache and Sadi [26].

Substituting, Eqs. (18) and (16) into Eq. (6) yields

$$\psi(q,t)=\sqrt{\frac{\sqrt{a}}{\hbar A(t)\sqrt{2\pi}\left(a+i\int_0^t\frac{1}{2\hbar mA^2(t)}dt'\right)}}\;e^{-\frac{i}{\hbar}\left[\frac{C(t)q+B_oq^2}{2A(t)}\right]}$$

$$exp\left\{-\left[\frac{\frac{q}{\hbar A(t)}-\left(\int_o^t\frac{C(t)}{2\hbar mA^2(t)}+\sqrt{\left(a+i\int_0^t\frac{1}{2\hbar mA^2(t)}dt'\right)b}\right)}{4\left(a+i\int_0^t\frac{1}{2\hbar mA^2(t)}dt'\right)}\right]\right\} \tag{19}$$

$$X\ exp\left\{-\frac{1}{i}\alpha_\lambda(o)+iF_o\gamma t\right\}$$

After some simplification, we find

$$\left|\psi(q,t)\right|^2=\frac{1}{\sqrt{2\pi A_o\left(1-\frac{B_o}{2}\left(t-\frac{2}{\gamma}\right)e^{\gamma t}\left[a^2+\frac{1}{4\hbar^2m^2}\left(\int_0^t\frac{dt'}{A^2(t')}\right)^2\right]\right)}}$$

$$Xexp\left\{\frac{-\left[q-\left\{\hbar A(t)\left(\int_o^t\frac{C(t)}{2\hbar mA^2(t)}+\sqrt{\left(a^2+\left(\int_0^t\frac{dt'}{2m\hbar A^2(t)}\right)^2\right)b}\right)\right\}\right]}{2\hbar A(t)\left[a^2+\frac{1}{4\hbar^2m^2}\left(\int_0^t\frac{dt'}{A^2(t')}\right)^2\right]}\right\} \tag{20}$$

From Eq. (20), it can be observed that; at any time $t$ the wave packet is peaked at

$$q=\hbar A(t)\left[\int_0^t\frac{C(t)}{2\hbar mA^2(t)}+\sqrt{\left\{\left(a^2+\frac{1}{4m^2\hbar^2}\left(\int_0^t\frac{dt'}{A^2(t')}\right)^2\right)\right\}b}\right]. \tag{21}$$

and this result coincides with the expectation value of $q$. Denoting the following parameters in Eq.(19) as

$$N=\sqrt{\frac{\sqrt{a}}{\sqrt{\hbar A(t)\sqrt{2\pi}\left(a+i\int_0^t\frac{dt'}{2\hbar mA^2(t')}\right)}}},$$

$$\beta=A(t)\hbar\left(\int_o^t\frac{C(t)}{2\hbar mA^2(t')}+\sqrt{\left(a+i\int_0^t\frac{dt'}{2\hbar mA^2(t')}\right)b}\right) \tag{22}$$

$$\alpha=\left[4\left(a+i\int_0^t\frac{dt'}{2\hbar mA^2(t')}\right)\right]^{-1},$$

simplifies it as follows:

$$\psi(q,t)=Nexp\left\{-\left(\alpha\left[q-\beta\right]^2+\frac{1}{2\hbar A(t)}\left[C(t)q+B_oq^2\right]+\frac{1}{i}\alpha_\lambda(o)-i\mathcal{F}_o\gamma t\right)\right\}. \tag{23}$$

Now using Eq. (23), we obtain the expectation value in momentum as

$$\left\langle\hat{P}\right\rangle=\left\langle\psi(q,t)\middle|\hat{P}\middle|\psi(q,t)\right\rangle=$$
$$=\frac{N^2}{4\sqrt{\alpha}}-2i\hbar N^2\sqrt{\alpha}-\sqrt{\frac{\pi}{32}}N^2\frac{C(t)}{A(t)}-\sqrt{\frac{\pi}{8}}N^2\beta\frac{B_o}{A(t)}. \tag{24}$$

Similarly, the expectation value in coordinate is

$$\left\langle \hat{q}^2 \right\rangle = \left\langle \psi\left(q,t\right) \middle| \hat{q}^2 \middle| \psi\left(q,t\right) \right\rangle,$$

$$=\hbar A\left(t\right)\left(a+i\int_0^t \frac{dt}{2m\hbar A(t)^2}\right)$$

$$+\hbar A(t)^2\left[\int_0^t \frac{C(t)}{2m\hbar A(t)^2}+\sqrt{\left(a+i\int_0^t \frac{dt}{2m\hbar A(t)^2}\right)b}\,\right]^2, \tag{25}$$

and this leads to the uncertainty relation in position as

$$\Delta q = \hbar A(t)\left(a+i\int_0^t \frac{dt}{2m\hbar A(t)^2}\right), \tag{26}$$

and its momentum uncertainty relation counterpart becomes

$$\Delta p = \frac{1}{\Delta q}\left[\frac{\hbar^2}{4}+\left\{\hbar^2\left(4\beta^2+2\beta\right)+i\left(\frac{B_0}{A(t)^2}-\frac{C(t)}{A(t)^2}+\frac{B_0 C(t)}{\alpha A(t)}-\alpha^2\right)\right\}\right]^{\frac{1}{2}}. \tag{27}$$

Thus the uncertainty product is expressed from Eq.(27) as

$$\Delta p\Delta q = \frac{\hbar}{2}\left[1+\left\{\left(\beta^2+\frac{\beta}{2}\right)+\frac{4i}{\hbar^2}\left(\frac{B_0}{A(t)^2}-\frac{C(t)}{A(t)^2}+\frac{B_0 C(t)}{\alpha A(t)}-\alpha^2\right)\right\}\right]^{\frac{1}{2}}. \tag{28}$$

This in general does not attain the minimum uncertainty value. However, for a time dependent oscillator; we cannot expect to find strictly coherent state $(\Delta p\Delta q) = \frac{\hbar}{2}$ for all time $t$ [25]. In addition, the full quantum behavior of the system is manifested by the change of the width in Eq. (20):

$$\sigma\left(t\right) = \left\{2\pi\hbar A_0\,(1-\frac{B_0}{2}\left(t-\frac{\gamma}{2}\right)\exp(\gamma t)\left[a^2+\frac{1}{4\hbar^2 a^2}\left(\int_0^t \frac{dt}{A(t)^2}\right)^2\right]^2\right\}^{\frac{1}{2}}. \tag{29}$$

Here the width of the damped wave packet at $\gamma \to 0$ leads to exactly the same result obtained by [26] using a Gaussian like weight function and this width determines the shape of the spreading wave packets.

## IV. CONCLUSION

In this paper we studied the Schrödinger equation with a time dependent damped linear potential via invariant operator .With the choice of our ansatz as Eq. (18),we obtain the exact solution of the time dependent SWE with linear damped potential. We also evaluated the wave packet evolution of the system and we have shown the uncertainties and their relations to the TDHO.

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