doi: 10.5007/1808-1711.2011v15n3p453

GÖDEL'S INCOMPLETENESS THEOREMS AND PHYSICS

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Abstract. This paper is a summary of a lecture in which I presented some remarks on Gödel's incompleteness theorems and their meaning for the foundations of physics. The entire lecture will appear elsewhere.

Keywords: Infinitary logic; Gödel and infinitary methods; incompleteness in physics.

I

The usual version of a mathematical formalism requires that it is recursive. In particular, the axioms, logical and specific, have to compose a recursive set, and the rules of inference must be constructive or, better, recursive.

A formalism constitutes essentially an algorithm or a Turing machine. It involves logic and specific axioms, being, so to say, an inference device; its proofs and syntactic procedures are all recursively decidable, in the sense that we are able to verify if they satisfy the conditions of their (in general) inductive definitions.

Gödel's fundamental idea was to view a (mathematical) formalism as part of arithmetic via the process of arithmetization, and then to take advantage of this move. His first incompleteness theorem says, in outline, that a consistent formalism containing elementary arithmetic is always incomplete: there are sentences S such that neither S nor $\neg S$ (the negation of S) are provable in the formalism. His second theorem, that can be proved as a corollary to the first, asserts that a consistent formalism, $\mathscr{F}$, encompassing arithmetic, cannot prove certain arithmetic sentence that constructively expresses the consistency of $\mathscr{F}$.

In consequence, all classical, strong and consistent mathematical theories are incomplete. As a Gödel says,

> The human mind is incapable of formulating (or mechanizing) all its mathematical intuitions, *i.e.*, if it has succeeded in formulating some of them, this very fact yields new intuitive knowledge, *e.g.*, the consistency of this formalism. This fact may be called the "incompletability" of mathematics. On the other hand, on the basis of what has been proved so far, it remains possible that there may exist (and even be empirically discoverable) a theorem-proving machine which in fact is equivalent to mathematical intuition, but cannot be proved to be so, nor even be *proved* to yield only correct theorems of finitary number theory. (1954, p. 324.)

Principia **15**(3): 453–459 (2011).
Published by NEL — Epistemology and Logic Research Group, Federal University of Santa Catarina (UFSC), Brazil.

Anyhow, for Gödel we are able to discover new mathematical truths and intuitively establish them. We are able, in reality, to get to know new reasonable axioms and to continue more or less systematically the search for truth. He wrote that,

> Despite their remoteness from sense experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, *i.e.*, in mathematical intuition, than in sense perception... The set theoretical paradoxes are hardly any more troublesome for mathematics than deception of the senses are for physics ... Evidently, the 'given' underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather, they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (1964, p. 272)

So, one could argue that, in the spirit of Gödel's stance, it would be acceptable to extend the notion of mathematical formalism, eliminating some of its (constructive) restrictions. For example, non-constructive rules of inference that, if intuitively valid, could be acceptable.

Let us consider some examples of the use of non-constructive rules of inference.

If we add to Peano arithmetic (see Church 1956, Kleene 1952, Shoenfield 1967) the ω-rule

$$\frac{A(0), A(1), A(2), \ldots}{\forall x A(x)}$$

where $A(x)$ is a formula and x a variable, then the new formal system PA^* is syntactically complete: any sentence (closed formula) S of PA or its negation, $\neg S$, is provable in PA^*. This extended formalism has, as its theorems, all true sentences of PA, that is, sentences that are true of the standard model of PA and only them.

Both incompleteness theorems do not apply to PA^*. Although PA^* has an intuitive appeal, its notion of proof is not decidable (proofs may be of infinite length). Since the first incompleteness theorem is not valid for PA^*, its concept of proof cannot be constructive (recursive).

Notwithstanding, PA^* possesses an intuitive interpretation and deserves to be included in the class of (extended) mathematical formalisms.

The preceding situation can be generalized (cf. da Costa 1974).

$\mathscr{L}$ is a first-order language containing a family of distinct terms $(t_i)_{i\in\alpha}$), where α is an ordinal greater than 0, and Γ denotes a set of sentences of $\mathscr{L}$ (in particular, it may be a theory). An α-model of Γ is a model in the ordinary meaning, such that every element of it is denoted by at least one term t_i, $i \in \alpha$. Given a sentence F of $\mathscr{L}$, F is said to be a semantic α-consequence of Γ if and only if it is true in any α-model of Γ.

C_α ($C_\alpha^=$) designates a first-order predicate calculus (with equality) having the following primitive symbols: 1) Connectives: $\vee$ (or) and $\neg$ (not); 2) The universal quantifier: $\forall$ (for all); 3) Individual variables: a denumerably infinite set of individual variables; 4) Individual constants: a family $(c_i)_{i\in\alpha}$ of (distinct) constants, where α is an ordinal greater than 0; 5) Predicate symbols: a family $(R_j)_{j\in\beta}$ of (distinct) predicate symbols, where β is an ordinal greater than 0 (for every $j \in \beta$, R_j has a finite rank; in the case of $C_\alpha^=$, one of the predicate symbols is the symbol of equality); 6) Auxiliary symbols: parentheses and comma. The concepts of formula, of free variable, sentence, etc., are introduced as usual. The connectives $\rightarrow$ (implication), $\wedge$ (and) and $\leftrightarrow$ (equivalence), as well as the existential quantifier ($\exists$) are defined in the standard way. The common metalinguistic conventions and notations are not made explicit.

The postulates (axiom schemes and rules of inference) of C_α ($C_\alpha^=$) are the ordinary ones, and the notions of deduction, theorem ... are those of da Costa 1974 with clear adaptations. If Γ is a set of formulas and F is a formula, F is said to be a syntactic consequence of Γ if $\Gamma \vdash F$ in C_α ($C_\alpha^=$); as usual, $\vdash F$ means $\emptyset \vdash F$.

If the family $(c_i)_{i\in\alpha}$ above is replaced by $(t_i)_{i\in\alpha}$, then the concepts of an α-model, semantic α-consequence, etc., are immediately defined. When F is a formula and Γ is a set of formulas, we write $\Gamma \models_\alpha F$ if every α-model that satisfies the formulas of Γ also satisfies F.

We denote by C_α^* ($C_\alpha^{=*}$) the calculus C_α ($C_\alpha^=$) to which rule α is added. Then, it is easy to define deduction, proof and other familiar concepts for the new calculus obtained.

One has, for example (see da Costa 1974):

Theorem 1. *In C_α^* ($C_\alpha^{=*}$), whenever $\Sigma \vdash_\alpha H$, we also have $\Sigma \models_\alpha H$.*

Theorem 2. *Suppose $\alpha \leq \omega$. Then, $\vdash_\alpha H$ in C_α^* ($C_\alpha^{=*}$) $\Longleftrightarrow \vdash H$ in $C_\alpha^=$ ($C_\alpha^=$).*

Theorem 3. *Suppose that either $\overline{\overline{\alpha}} = \aleph_0$ and $\overline{\overline{\beta}} = \aleph_0$ or $o < \alpha < \omega$; if $\Gamma \cup \{F\}$ is a set of sentences of C_α^* ($C_\alpha^{=*}$) and $\Gamma \models_\alpha F$, then $\Gamma \vdash_\alpha F$ in C_α^* ($C_\alpha^{=*}$), that is to say, C_α^* ($C_\alpha^{=*}$) is α-complete.*

The second theorem expresses the α-completeness of the logics C_α^* and $C_\alpha^{=*}$.

(The preceding ideas and results lead us to a generalization of the so-called ω-logic, *i.e.*, to the construction of α-logic, where α is an ordinal greater than ω (see da Costa & Pinter 1976).

It also known that rule α, adjoined to second-order logic or to higher-order logics, does not guarantee α-completeness (see Shoenfield 1967).

Anyhow, with the help of very strong rules of inference, α-completeness is secured (see da Costa 1974).

Another kind of extended formalism, in fact a very abstract variety of formalism, is concerned with languages with formulas of infinite length. Among the large collection of results in this domain, I shall present only one. In order to motivate it, I quote a passage of Hodges (1997, p. 93):

> The twenty-three year old Felix Klein in his famous Erlanger Programm (1872) proposed to classify geometries by their automorphisms. He hit on something fundamental here, in a sense, *structure is whatever is preserved by automorphisms*. One consequence — if slogans can have consequences — is that a model-theoretic structure implicity carries with it all the features which are set-theoretically definable in terms of it, since these features are preserved under all automorphisms of the structure.
>
> There is a trivial model-theoretic slogan: *structure is whatever is definable*. Surprisingly, this slogan points in the same direction as the previous one. For example, if we have a field K, we can define the projective plane over K. But precisely because the projective plane is definable from K, any automorphism of K will induce an automorphism of the plane too. Either way, the plane comes with the field; in some abstract sense it is the field, but looked at from an unusual point of view.

In infinitary languages, formulas may have infinite length. Conjunction, disjunction and blocks of consecutive quantifiers may be infinite. There exists infinitary languages with special particularities, for example, the lengths of formulas are strictly less than a fixed cardinal and the sequence of consecutive quantifiers in any formula are finite. All these infinitary languages possess semantics analogous to the set-theoretic semantics of the classical first-order logic. It is possible to define concepts like truth of a sentence, semantic consequence, etc.

Given a mathematical structure of any order whatever (group, ring, topological space, differentiable manifold, etc.), it is associated with its group of automorphisms, that is, the bijections of its basic domain which, in a certain sense, leave its primitive relations invariant.

Then, the connection between definability and invariance, referred to in the above passage of Hodges (1997), takes the following formulation:

Theorem 4. *A relation (0-adic, monadic, binary, ternary, ...) is invariant under the group of automorphisms of a structure S if, and only if, the relation is definable on the basis of the primitive terms of S in a convenient infinitary language (S is a structure of any order whatever).*

The topic of definability and invariance is studied in da Costa & Rodrigues 2007; the preceding theorem constitutes a reformulation of a result of the Portuguese mathematician José Sebastião e Silva.

The common languages, for example first-order languages, are normally fragments of infinitary languages. Moreover, there exist infinitary logics based on such infinitary languages, so that there are also theories or "abstract formalisms" whose underlying logics are infinitary. Clearly, Gödel's incompleteness theorems are not applicable to such theories or abstract formalisms.

An interesting point is that all results mentioned can be treated inside set theory (for example, in ZF with choice). In particular, this is clearly true of the traditional (recursive) formalisms. However, the last formalisms have an informal version, via constructions, similar to those of the intuitionism *à la* Brouwer, what does not happen with the extended, abstract ones. Traditional formalisms so conceived deserve to be called "concrete" formalisms.

It seems that an extension of the traditional conception of a mathematical formalism, as it was sketched above, fits in very well with the philosophical views of Gödel. Moreover, Gödel's proof of the consistency of Peano arithmetic by means of abstract functionals, his proof of the semantic completeness of the first-order predicate calculus and his investigations of intuitionistic logic and arithmetic not only corroborate the deepness of his ideas, but show that they are wide and cope with most of the fundamental questions of mathematics.

For him, the traditional, strict formalisms are effective tools to systematize parts of the mathematical knowledge. However, extended formalisms are acceptable, since they are, in fact, kinds of structures. The engineering of building concrete formalisms or axiomatic systems does not encompass all mathematics; this discipline involves more than that, comprising an abstract level. Maybe, even if this assertion is opposed to Gödel's stance, the abstract level reduces to inter-connections between systems of abstract relations and extended formalisms. This way, the intuition underlying abstract mathematics would be formal, not material.

II

Extant physics does not exist without mathematics. So, what are the consequences of Gödel's incompleteness theorems for the foundations of physics?

Clearly, Peano arithmetic can be viewed as a physical theory; it copes with whole numbers, conceived as concrete physical objects, and afterwards considers abstract objects, extending its initial goal via idealization and generalization. Since Peano arithmetic is not complete (supposed consistent), it follows that there exists an incomplete physical theory. Moreover, since Peano arithmetic is contained in practically every mathematically strong physical theory, no such theory is complete if consistent. Therefore, physics is incomplete or inconsistent (and, hence, in the last hypothesis, logically trivial). Worse, any physical, strong theory is incompletable in principle. In particular, some ambitions of string theory are impossible.

In addition, there is no way to prove the consistency of a mathematically strong physical theory using constructive methods in the sense of Gödel. As André Weil would say, God does exist because physics seems to be consistent, but the Devil also exists, because we are unable to prove it.

Anyhow, we way believe, mathematics based on non-constructive axiomatization (formalisms) is not subject to Gödel's restrictions. Perhaps, there could be complete physical theories established with the help of infinitary methods. As we mentioned above, PA^* is a theory of this kind. But the main point is to know if with strong infinitary rules one would be able to obtain a reasonable, consistent and complete, physical theory. This constitutes an open problem. Moreover, in this case a proof of consistency would lose any strong and intuitive meaning.

III

The gist of Gödel's researches on the foundations of general relativity is well described by S. W. Hawking, who wrote the following (Hawking 1990, pp. 189–90):

> Gödel showed that it was possible to have solutions of the Einstein field equations in which the galaxies were rotating with respect to the local inertial frame. He therefore demonstrated that general relativity does not incorporate Mach's principle. Whether or not this is an argument against general relativity depends on your philosophical viewpoint, but most physicists nowadays would not accept Mach's principle, because they feel that it makes an untenable distinction between the geometry of space-time, which represents the gravitational and inertial field, and other forms of fields and matter.
>
> [In one paper (1949)] Gödel presented a rotating solution that was not expanding but was the same at all points of space and time. This solution was the first to be discovered that had the curious property that in it was possible to travel into the past. This leads to paradoxes such as "What happens is you go back and kill your father when he was a baby"? It is generally agreed that this cannot happen in a solution that represents our universe, but Gödel was the first to show that it was not forbidden by the Einstein equations. His solution generated a lot of discussion of the relation between general relativity and the concept of causality.
>
> [In another paper (1952)] describes more reasonable rotating cosmological models that are expanding and that do not have the possibility of travel into the past. These models could well be a reasonable description of the universe that we observe, although observations of the isotropy of the microwave background indicate that the rate of rotation must be very low.

Perhaps an open minded person like Gödel, who elaborated models of general relativity as those described by Hawking, wouldn't, *a priori*, condemn the introduction of infinitary logical tools in physics ...

References

Church, A. 1956. *Introduction to Mathematical Logic*. Vol. 1. Princeton: Princeton University Press.

da Costa, N. C. A. 1974. α-models and the systems T and T^*. *Notre Dame J. Formal Logic* **15**: 443–54.

da Costa, N. C. A. & Pinter, C. 1976. α-logic and infinitary languages. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **22**: 105–12.

da Costa, N. C. A. & Rodrigues, A. A. M. 2007. Definibility and invariance. *Studia Logica* **86**: 1–30.

Gödel, K., 1954. J. W. Gibbes Lecture. Quoted by H. Wang, in *From Mathematics to Philosophy*. Humanities Press, 1974.

———. 1964. What is Cantor continnum problem? Reprinted in P. Benacerraf and H. Putnam (eds.) *Philosophy of Mathematics*. Prentice-Hall, pp. 258–73.

Hawking, S. W. 1990. *Gödel 1949*: Introductory Note to *1949* and *1952*. In: *Kurt Gödel's Collected Works*, vol. II. Oxford: Oxford University Press, pp. 189–90.

Hodges, W. 1997. *A Shorter Model Theory*. Cambridge: Cambridge University Press.

Kleene, S. C. 1952. *Introduction to Metamathematics*. New York: D. van Nostrand.

Shoenfield, J. 1967. *Mathematical Logic*. Reading, Mass.: Addison-Wesley.

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Resumo. Este artigo é o resumo de uma conferência na qual apresentei algumas observações sobre os teoremas de incompletude de Gödel e seu significado para os fundamentos da física. A conferência inteira será publicada em outro lugar.

Palavras-chave: Lógica infinitária, Gödel e métodos infinitários, incompletude em física.