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The "Great Confusion" concerning MPT

Markowitz, Harry

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Abstract

The "Great Confusion" is between necessary and sufficient conditions for the use of mean-variance analysis in practice. Normal (Gaussian) return distributions is a sufficient condition but not a necessary one. For those (such as the author) who accept the expected utility maxim for rational decision making, the necessary and sufficient condition is that a careful choice from the mean-variance frontier will almost maximize expected utility for a wide variety of concave (risk-averse) utility functions. Over fifty years of extensive (but remarkably little-known) research shows that certain functions of mean and variance do a quite good job of estimating expected utility. Recent research indicates that they actually do a *better* job than functions of mean and the leading *alternate* measures of risk.

Keywords:

MPT, Mean-variance, Semivariance, MAD, VaR, CVaR, Geometric Mean.

JEL classification:

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Markowitz, H. ✉ *UC San Diego, Rady School of Management. Otterson Hall, Room 4S134. 9500 Gilman Drive #0553. La Jolla, CA 92093-0553.* ✆ *858.534.3741. Fax: 858.534.0745. Email: hmarkowitz@ucsd.edu*

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La "Gran Confusión" **en relación a la MPT**

Markowitz, Harry

Resumen

La "Gran Confusión" radica en las condiciones necesaria y suficiente para la utilización del análisis media-varianza en la práctica. Las distribuciones Normales (Gaussianas) de rendimientos constituyen una condición suficiente, pero no necesaria. Para aquellos que (como el autor) aceptan la maximización de la utilidad esperada en la toma de decisiones racionales, la condición necesaria y suficiente es que una cuidadosa selección a partir de la frontera media-varianza prácticamente maximice la utilidad esperada para un amplio abanico de funciones de utilidad cóncavas (aversión al riesgo). Más de cincuenta años de extensa (pero ciertamente poco conocida) investigación muestran que determinadas funciones de la media y la varianza han funcionado bastante bien en lo que a la estimación de la utilidad esperada se refiere. La investigación reciente indica que realmente funcionan mejor que las funciones de la media y medidas líderes *alternativas* de riesgo.

Palabras clave:

MPT, Media-varianza, Semivarianza, MAD, VaR, CVaR, Media geométrica.

1. Introduction

Our field is plagued by a Great Confusion, namely the confusion between necessary and sufficient conditions for the use of mean-variance analysis in practice. Normal (Gaussian) return distributions are sufficient to justify the use of mean-variance analysis: But they are not necessary. If you believe (as many do, including me) that rational decision making should be consistent with expected utility maximization, then the necessary and sufficient condition for the use of mean-variance analysis is that a carefully selected portfolio from the mean-variance efficient set will approximately maximize expected utility, for a great variety of concave (risk-averting) utility functions. This was the argument for mean-variance analysis that I presented in Markowitz (1959). A large number of subsequent research papers, by me and others, following up along the same lines, have generally been supportive of mean-variance analysis—subject to certain caveats. In this paper I will briefly summarize some highlights of this literature with emphasis on its practical significance. See Markowitz (2012b) for a more complete review of the literature.

The first section below reviews the fundamental assumptions of Markowitz (1959), of which the maximization of single-period expected utility is a part. Subsequent sections review mean-variance approximations to expected utility, including recent work comparing such approximations to ones using other risk-measures.

2. Markowitz's fundamental assumptions

Markowitz (1959) justifies mean-variance analysis by relating it to the theory of rational decision making over time and under uncertainty, as developed by von Neumann and Morgenstern (1944), Savage (1954) and Bellman (1957). The fundamental assumptions of the book appear in Part 4, Chapters 10 through 13. Specifically, Chapter 10 deals with single-period decision-making with known odds. It echoes the view that, in this case, the rational decision maker (RDM) may be assumed to follow certain axioms, from which follows the expected utility maxim. Below I assume the reader is familiar with the expected utility maxim and justifications for it. This is covered in many modern texts on decision making, including the aforesaid Chapter 10 of Markowitz (1959).

Chapter 11 of my 1959 book considers many-period games, still with known odds. It shows that essentially the same set of axioms as in Chapter 10 implies that an RDM would maximize expected utility for the game as a whole which, in turn, implies that the RDM would maximize the expected values of a sequence of *single-period* utility functions, each using a Bellman "derived" utility function. Research on the relationship between single-period mean-variance analysis and the many-period game, beyond the observations in Markowitz (1959), is reported in Markowitz and van Dijk (2003). The application of the Markowitz-van Dijk approach to the rebalancing of portfolios at State Street Bank is described in Kritzman *et al*. (2008).

Chapter 12 of Markowitz (1959) considers single or multiple-period decision-making with unknown odds. Taking off from Savage's work, it adds a "sure thing" principle to the axioms of Chapter 10 and 11 and concludes that, when odds are unknown, the RDM maximizes expected utility using "probability beliefs" where objective probability are not known. These probability beliefs shift according to Bayes rule as evidence accumulates.

Chapter 13 applies the conclusions of Chapters 10 through 12 to the portfolio selection problem. In particular, it extends an observation made in Chapter 6 for the logarithmic utility function, that if a probability distribution of a portfolio's returns is not "too spread out," a function of its mean and variance closely approximates its expected utility. I review this argument in the next section.

The reason the fundamental assumptions of Markowitz (1959) are presented at the back rather than at the front of the book was that I feared that if I started with an axiomatic treatment of the theory of rational decision-making under uncertainty, no one involved with managing money would read the book. This may have been a wise strategy at the time, but its side-effect is that a very small percent of our industry understand the conditions for the applicability of mean-variance analysis.

3. Quadratic approximations to expected utility

Suppose an investor wished to maximize the expected value of a logarithmetic utility function

$$U = Ln(1+R) \tag{1}$$

where R is return on the investor's portfolio. Perhaps this is the investor's goal because of the reasons Daniel Bernoulli (1954) gave in favor of this function when he first proposed maximizing expected utility rather maximizing expected income; or perhaps because of its connection with the growth rate G (i.e., the "geometric mean" return) of the portfolio, namely

$$Ln(1+G) = ELn(1+R) \tag{2}$$

where E is the expected value operator. How bad would it be for such an investor if he or she had to be satisfied with a portfolio from a mean-variance efficient frontier?

Consider Table 1 here, which is Table 2 of Chapter 6 on Page 121 of Markowitz (1959). The first column lists return R, the second $Ln(1+R)$ and the third $R-\frac{1}{2}R^2$ There is little difference between $Ln(1+R)$ and this quadratic approximation to it for returns between a 30% loss and a 40% gain *on the portfolio-as-a-whole*. For example, at $R=-0.30$ (a thirty percent loss) $Ln(1+R)=-0.36$ whereas the quadratic is -0.35. At $R=0.40$ (a forty percent gain) $Ln(1+R)=0.34$ whereas the quadratic is 0.32. Between these two values, i.e., for $R=-0.20, -0.10, ..., +0.30$, the approximation equals the log utility function to the two-places shown. Even at a forty percent loss or a fifty percent gain, the difference is noticeable but not great: –0.51 vs –0.48 in the one case; 0.41 vs. 0.38 in the other. As the range of possible returns increases further, however, the approximation deteriorates at an increasing rate. In particular, $Ln(1+R)$ goes towards minus infinity as R approaches –1.0, a hundred percent loss, whereas the quadratic goes to –1.5. Conversely, as R increases $Ln(1+R)$ increases without bounds whereas the quadratic reaches a maximum at $R=1$ and then declines.

Table 1. Comparison of $Ln(1+R)$ with $R-½R^2$

R	$Ln(1+R)$	$R-½R^2$
-.50	-.69	-.63
-.40	-.51	-.48
-.30	-.36	-.35
-.20	-.22	-.22
-.10	-.11	-.11
.00	.00	-.00
.10	.10	.10
.20	.18	.18
.30	.26	.26
.40	.34	.32
.50	.41	.38

As long as the returns on a portfolio are within the range in which $Ln(1+R)$ and $R-\frac{1}{2}R^2$ are close, the expected value of the one must be close to the expected value of the other. But the expected value of the quadratic depends only on portfolio mean and variance. Thus Markowitz (1959) concludes that for choice among return distributions which are mostly within the range of a thirty or forty percent loss to a forty or fifty percent gain on the portfolio-as-a whole, and do not fall outside this range "too far, too often," the $E[Ln(1+R)]$ maximizer will almost maximize expected utility by an appropriate choice from the mean-variance efficient frontier.

Note that this argument does not depend on the shape of the return distribution: It can be skewed to the left, skewed to the right, bimodal—whatever! Just as long as it is not spread out "too much" in the sense illustrated by Table 1.

In general Markowitz (1959) suggested two types of approximations to any utility function, $U(R)$:

$$Q_Z(R)=U(0)+U'(0)R+0.5U''(0)R^2 \quad (3)$$

$$Q_E(R)=U(E)+U'(E)(R-E)+0.5U''(E)(R-E)^2 \quad (4)$$

where a prime denotes differentiation. For example, for the natural logarithm utility function, $U=Ln(1+R)$, approximations (3) and (4) are, respectively,

$$q_Z(R)=R-\tfrac{1}{2}R^2 \quad (5)$$

as shown in Table 1, and

$$q_E(R)=Ln(1+E) + (R-E)/(1+E) - (R-E)^2/[2(1+E)^2]. \quad (6)$$

The expected values of Equations 5 and 6 are the following functions of mean and variance,

$$f_Z(E,V)=E - (E^2+V)/2 \quad (7)$$

$$f_E(E,V)=Ln(1+E)-\tfrac{1}{2}V/(1+E)^2 \quad (8)$$

Q_Z in Equation 3 is the Taylor approximation to $U(R)$ centered at $R=0$; Q_E in Equation 4 is that centered at $R=E$. Markowitz (1959) observed that Q_E was superior to Q_Z. This was confirmed by subsequent research. For example, Markowitz (2012a) presents historical comparisons between geometric mean and six different mean-variance approximations to it, for two databases. One database consists of the historical returns on asset classes widely used in asset allocation decisions. The second database contains the real returns during the 20th Century of the equity markets of sixteen countries. The six approximations included f_Z and f_E in Equations 7 and 8. Of the six, three did well for a wide range of distributions, including ones with observations well beyond the 30 to 40 percent loss and 40 to 50 percent gain within which f_Z would be expected to do well. As it turned out, f_Z did poorly whereas f_E was one of the three that did quite well.

4. Why not just maximize expected utility?

If one believes that action should be in accord with the maximization of expected utility, i.e., the max EU rule, why seek to *approximately* maximize *EU* via a mean-variance analysis? Why not just maximize expected utility? In considering this question, distinguish three types of expected utility maximization:

- explicit
- MV-approximate
- implicit

I refer to it as "*explicit*" EU maximization when a utility function is given and analytic or numerical methods are used to find the portfolio that maximizes the expected value of this function. In contrast, I refer to it as "*MV-approximate*" when a mean-variance approximation to expected utility is maximized. An example would be to approximately maximize $ELn(1+R)$ by generating an MV efficient frontier, and choosing from it the portfolio that maximizes the approximation in Equation 7 or 8.

As reviewed below, Levy and Markowitz (1979) find that mean-variance approximations are usually quite accurate. From this they conclude, for some hypothetical investor Mr. X, that "If Mr. X can carefully pick the MV efficient portfolio which is best for him, then Mr. X, who still does not know his current utility function, has nevertheless selected a portfolio with maximum or almost maximum expected utility." I refer here to such a process as "*implicit*" expected utility maximization.

Typically it is much more convenient and economical to determine the set of mean-variance efficient portfolios than it is to find the portfolio which maximizes expected utility. Historically, one source of inconvenience and added expense for the latter was computational. One typically had to wait longer (perhaps hours longer) and pay a higher computer bill to find an expected-utility-maximizing portfolio than to trace out a mean-variance frontier. This computational problem is now trivial thanks to faster, cheaper computers. It still takes many times as long to compute the expected value of most concave functions as it does to trace out a mean-variance efficient frontier. But neither calculation takes long enough to be a practical limitation.

There are, however, other expenses and inconveniences that remain for explicitly maximizing expected utility as compared to using an MV or implicit approximation to it. The first of the remaining economically significant differences in cost and convenience concerns parameter estimation. The only inputs required for a mean-variance analysis are the means, variances and covariances of the securities or asset classes of the analysis. (A factor model can serve in place of individual variances and covariances). Typically, more than this is required to explicitly maximize the expected value of a utility function. The formulas relating the expected return and variance of a portfolio to the expected values, variances and covariances of return of securities do not depend on the form of the probability distribution. For example, letting E_p be the expected return on the portfolio; X_i, the fraction of the portfolio invested in the ith security and E_i the expected return on the ith security, the relationship

$$E_p=\sum_{i=1}^{n} X_i E_i \tag{9}$$

holds whether or not returns are normally distributed. More generally, Equation 9 is true whether or not distributions are symmetrically distributed, and whether or not the return distributions have "fat tails," as long as the E_i exist and are finite. Similarly, letting V_p be the variance of the portfolio, and σ_{ij} be the covariance between security returns r_i and r_j, the formula for portfolio variance

$$V_p = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j \sigma_{ij} \qquad (10)$$

is true whether or not the return distributions are normal, or symmetric or have fat tails as long as the $V_i = \sigma_{ii}$ are finite. The case is different when one explicitly maximizes expected utility. Then one needs to determine what type of joint probability distribution generates return combinations, $(r_1, r_2 ..., r_n)$, and must estimate the parameters for such a joint distribution. Accomplishing this can be a substantial research project.

A second difficulty with using explicit expected utility maximization, as opposed to implicit EU maximization, is that someone must determine the investor's utility function. As von Neumann and Morgenstern explain, theoretically this should be done by answering a series of questions as to what probabilities p_a of returns R_a versus $(1-p_a)$ of R_c the investor considers just as desirable as return R_b with certainty. This would be challenging enough for an institutional investor, such as an endowment or pension fund with a single large portfolio, but seems hardly possible in any thorough way on behalf of the many clients of a financial advisor. This step is not necessary when implicit EU maximization is used.

Finally, another advantage of using implicit EU maximization is that no one has to explain the expected utility concept to the individual investor, or to the supervisory board of an institutional investor, or to the typical financial advisor. Instead, portfolio choice can be couched in the familiar terms of risk versus return.

5. Levy and Markowitz (1979)

The Levy-Markowitz study had two principal objectives:

(1) to see how good mean-variance approximations are for various utility functions and portfolio return distributions; and

(2) to test an alternate way of estimating expected utility from a distribution's mean and variance.

The Levy-Markowitz "alternate way" was to fit a quadratic approximation to U at three values of R:

$$(E - k\sigma_p), (E), (E + k\sigma_p) \qquad (11)$$

where σ_p is the portfolio's standard deviation. They tried their approach for

$$k = 0.01,\ 0.1,\ 0.6,\ 1.0 \text{ and } 2.0$$

Of these, $k=0.01$ did best in almost every case. This is essentially the same as the approximation in Equation 4. I will therefore relate their results for $k=0.01$ and subsequently treat these as if they were results for the Equation 4 approximation.

Table 2 shows the Levy-Markowitz results for four data sets. The first column of the table lists various utility functions. The next shows results based on the annual returns for 149 mutual funds for the years 1958 through 1967. (These were all the funds whose returns Wiesenberger 1941 reported at the time for the full period.) Levy and Markowitz considered these 149 return series as 149 real-world return distributions. This second column of the table shows correlations between average utility

$$EU = \sum_{t=1}^{T} U(r_t)/T \tag{12}$$

and the mean-variance approximation $f_{.01}(E,V)$ based on the quadratic fit through the three points in Specification 11 with $k=0.01$.

Table 2. Correlation between EU and $f_{.01}$ (E,V)
For four historical return series. 1958 -1967

Utility function		Annual returns of 149 mutual funds	Annual returns on 97 stocks	Monthly returns on 97 stocks	Random portfolios of 5 or 6 stocks
$\text{Log}(1+R)$		0.997	0.880	0.995	0.998
$(1+R)^a$	$a=0.1$	0.998	0.895	0.996	0.998
	$a=0.3$	0.999	0.932	0.998	0.999
	$a=0.5$	0.999	0.968	0.999	0.999
	$a=0.7$	0.999	0.991	0.999	0.999
	$a=0.9$	0.999	0.999	0.999	0.999
$-e^{-b(1+R)}$	$b=0.1$	0.999	0.999	0.999	0.999
	$b=0.5$	0.999	0.961	0.999	0.999
	$b=1.0$	0.997	0.850	0.997	0.998
	$b=3.0$	0.949	0.850	0.976	0.958
	$b=5.0$	0.855	0.863	0.961	0.919
	$b=10.$	0.447	0.659	0.899	0.768

The utility functions used were the logarithmic, and the power and exponential functions for the values of a and b shown in the table. For the logarithmic utility function, and for all the power utility functions considered, the correlation (over the 149 return distributions) between average utility and the mean-variance approximation to it was at least 0.997. Since this is more precision than one may expect from forward looking estimates of means, variances and covariances for a mean-variance analysis—or from estimates of joint distributions for an explicit expected utility maximization—Levy and Markowitz concluded that, for such utility functions and return distributions, for all practical purposes EU and its mean-variance approximation are indistinguishable.

On the other hand, MV approximation was much less successful for exponential utility

$$U = -\exp\{-b(1+R)\}$$

for $b=5$ and, especially, for $b=10$. This raises serious questions about the applicability of mean-variance analysis to certain kinds of investors: In particular, what are the characteristics of such investors? And what needs to be done for them? I return below to these questions.

The other columns of Table 2 show the correlation between EU and $f_{.01}$ for three more sets of historical distributions reported by Levy and Markowitz. The second data set reported in the table shows correlations for annual returns on 97 randomly chosen U.S. common stocks during the years 1948-1968. It is understood, of course, that mean-variance analysis is to be applied to the portfolio-as-a-whole rather than individual investments. Annual returns on individual stocks were used, however, as examples of return distributions with greater variability than that found in the portfolios reported in the prior column. As expected, correlations for individual stocks are poorer than for the mutual fund portfolios. For $U=Ln(1+R)$, for example, the correlation is 0.880 for the annual returns on stocks as compared to 0.997 for the annual returns on the mutual funds.

Since monthly returns tend to be less variable than annual returns, we would expect the correlations between EU and $f_{.01}$ to be higher for the former than the latter. The correlations for monthly returns on the same 97 stocks are shown in the fourth column of Table 2. For the logarithmic utility function, for example, the correlation is 0.995 for the monthly returns on individual stocks as compared to 0.880 for annual returns on the stocks, and 0.997 for annual returns on the mutual funds. On the whole, the correlations for monthly returns on individual stocks are comparable to those of the annual returns on mutual funds.

The central limit theorem implies that compounded returns would tend to a log normal distribution if successive returns were independent. This suggests that annual returns should be closer to log normal than monthly returns. However, the point that Markowitz (1959) makes in connection with our Table 1 is that even though monthly returns may have a less Gaussian-like or lognormal-like shape than do annual returns, one should expect f_Z or f_E to provide a better approximation to EU for monthly return distributions than for annual return distributions because they are less spread out. This is amply confirmed by the Levy-Markowitz data.

As noted above, annual returns on individual stocks—i.e., on completely undiversified portfolios—had perceptibly smaller correlation, between EU and $f_{.01}$, than do the annual returns on the well diversified portfolios of mutual funds. The final column in Table 2 presents such correlations for "slightly diversified" portfolios consisting of a few stocks. Specifically, it shows the correlations between EU and $f_{.01}$ on the annual returns for 19 portfolios of 5 or 6 stocks randomly drawn (without replacement) from the 97 U.S. stocks. We see that for the logarithmic utility function, correlation is 0.998 for the random portfolios of 5 and 6, up from 0.880 for individual stocks. Generally, the correlations for the annual returns on the portfolios of 5 and 6 were comparable to those for the annual returns on the mutual funds. These results were among the most surprising of the entire analysis. They indicate that, as far as the applicability of mean-variance analysis is concerned, at least for joint distributions like the historical returns on stocks for the period analyzed, a little diversification goes a long way.

6. Highly risk-averse investors

The Levy-Markowitz results for the exponential with $b=10$ differ markedly from those of the other utility functions reported in Table 2. In this section we explore the reasons for this. In particular, why do mean-variance approximations have difficulty with such utility functions and what characterizes such investors? A later section, reviewing the work of Simaan (1993) addresses the question of what to do about it.

For $E=0.1$ and $\sigma=0.15$, Table 3 compares the exponential utility function with the quadratic Q_E of Equation 4. The utility function is rescaled as follows

$$U = 1000e^{-10(1+R)}$$

(As von Neumann and Morgenstern explain, such multiplication of a utility function by a positive constant does not affect its choices among probability distributions) With this scaling the difference between $U(0.5)$ and $U(-0.3)$ is of the same order of magnitude as that for $Ln(1+R)$ in Table 1, namely, $0.41-(-0.36) = 0.77$ in the latter case versus about 0.91 in the former. Table 3 is presented to four places, rather than two as in Table 1, since $U(R)$ rounds to 0.00 to two places for $R \geq 0.3$ for the exponential.

Table 3. Comparison of exponential utility with the Q_E quadratic approximation

For $U = 1000e^{-10(1+R)}$; $E = .1$; and $\sigma = .15$

R	$U(R)$	$Q_E(R)$	U–Q_E
-.30	-.9119	-.2171	-.6948
-.20	-.3355	-.1420	-.1935
-.10	-.1234	-.0835	-.0399
.00	-.0454	-.0418	-.0036
.10	-.0167	-.0167	.0000
.20	-.0061	-.0084	.0022
.30	-.0023	-.0167	.0144
.40	-.0008	-.0418	.0409
.50	-.0003	-.0835	.0832

The first column of Table 3 lists R; the second, $U(R)$; the third, the quadratic approximation Q_E; the fourth column presents the difference between utility U and the quadratic Q_E, namely $d_E(R)=U(R)–Q_E(R)$. The table sheds light on why a quadratic approximation does much better for $Ln(1+R)$ than for $–\exp\{–10(1+R)\}$. As Table 1 showed, for R between –0.30 and +0.40 the maximum difference between $Ln(1+R)$ and $Q_Z(R)$, is 0.02. Table 5 shows that, with U scaled for comparability with Table 1, the absolute value of the difference, $|d_E|$, is 0.69 at R=–0.3 —over thirty times as great. (The approximation Q_1 fit to the three Levy-Markowitz points in Specification 11 with $k = 1$, did a little better, but not much better, than Q_E or $f_{.01}$ in its correlation with EU and its fit to $U(R)$.)

The reason that a quadratic has trouble approximating the utility function in Table 3 is that this $U(R)$ turns too quickly in the neighborhood of $R=E$. Between R=–.30 and R=.10 utility increases by 0.74 from $U(–.3)$=–.912 to $U(.1)$=–.017. But since $U \le 0.0$ everywhere, it becomes comparatively flat as R increases further. Specifically, it rises less than 0.2 between R=.10 and "$R=\infty$." Essentially $U(R)$ has a knee at $R = E$.

Levy and Markowitz observe that an investor who had $-e^{-10(1+R)}$ as his or her utility function would have some strange preferences among probability distributions of return. Since $U(R)<0$ for all R, it follows that

$$\tfrac{1}{2}U(0.0)+\tfrac{1}{2}U(R)<-0.0227 < U(.1) \quad \text{for all } R.$$

Therefore, the investor would prefer
(A) a 10 percent return with certainty, to
(B) a 50-50 chance of zero return (no gain, no loss) *versus* a gain of 10^9 percent or more.

Put another way, such an investor would prefer 10 percent with certainty to a 50-50 chance of either breaking-even or a "blank check." Markowitz, Reid and Tew (1994) find that real investors do not assign such a low "value of a blank check," (VBC). In their survey of brokerage customers, the median value of VBC was 404 percent as a fraction of the investor's portfolio, or 143 percent as a fraction of the investor's total wealth, well above the less-than-ten percent of the $U(R)$ in Table 3. They conclude that few if any real investors have utility functions for which Levy-Markowitz found that Q_E provides a poor approximation to expected utility.

7. Highly risk-averse investors and a risk-free asset

Simaan explores the efficacy of MV-approximate maximization for investors with an exponential utility function when a risk-free asset is available versus when such a risk-free asset is not available. He finds that, for investors with exponential utility functions with large values of b, MV-approximate EU maximization is highly efficacious when a risk-free asset is available, and much less so when it is not.

In deriving these results, Simaan assumes that security returns follow a factor model,

$$1+r_i = a_i + b_i F + u_i \qquad i=1,\ldots,n \tag{13}$$

where the u_i are normally distributed, not necessarily independently, and F is a (skewed) random variable with a Pearson Type Three distribution. Simaan also assumes that the only constraint on portfolio choice is

$$\sum X_i = 1 \tag{14}$$

without regard to the sign of the X_i. Given these assumptions, Simaan is able to solve for the optimum portfolio.

Simaan illustrates his solution in terms of monthly returns for ten randomly selected securities. The measure of efficacy used by Simaan is what he calls the "optimization premium," namely, the return θ which would have to be added to the MV-approximate maximum portfolio in order to make it as desirable for the investor as the explicit optimum.

Table 4 presents the Simaan results. The first column shows the coefficient b in the exponential utility function; the second column shows the optimization premium when a risk-free asset is *not* available; the third column shows it when a risk-free asset *is* available. For example, for b=10, if there is no risk-free asset, one would have to add 0.00323, i.e., roughly 3/10 of 1% percent of the value of the portfolio, each month to make it just as good as the explicitly maximized portfolio; whereas if a risk-free asset is available 0.00001, i.e., 1/1000th of one percent per month (roughly a basis point per annum) need be added. Simaan concludes that, given his assumptions and sample, as long as a risk-free asset is available the MV-approximation delivers essentially the same expected utility as explicit EU maximization. In other words, if you are going to cater to "pathologically risk-averse" investors, among others, be sure to include a risk-free asset in your universe of securities.

Table 4. **Simaan's optimization premiums**

Exponential coefficient	Without a risk-free asset	With a risk-free asset
2	0.00023	0.00050
4	0.00073	0.00025
6	0.00144	0.00017
8	0.00229	0.00012
10	0.00323	0.00010
15	0.00581	0.00007
20	0.00859	0.00005
25	0.01147	0.00004
50	0.02646	0.00002
100	0.05719	0.00001

8. Recent research

Markowitz (2012a) reports on the ability of six different functions of mean and variance to approximate the geometric mean or, equivalently $ELn(1+R)$ as in Equation 2 for two different databases. The first database was that of the frequently used asset classes listed in Table 5a with data from Morningstar's EnCorr back to 1926 where available. The second database was the Dimson, Marsh and Staunton (2002) database of real returns of the equity markets of the 16 countries listed in Table 5b for the 101 years, 1900-2000. Of the mean-variance approximations considered, f_Z in Equation 7 was eliminated early as the worst of the lot. This is perhaps not surprising since both databases include series with returns that fell well outside the interval (30 or 40 percent loss to 40 or 50 percent gain) for which f_Z was expected to do well. The three approximations that did best were f_E of Equation 8; an approximation, f_{LN}, that is exactly right if portfolio returns are log normal, and another f_{HL}, due to Henry Latané, which is exactly right if the return distribution has only two outcomes $E_p-\sigma_p$ and $E_p+\sigma_p$. f_{HL} did best for the asset class database, but the other two did fairly well also. For the DMS database f_{LN} did best but, again, the other two did fairly well. Markowitz (2012a) shows that necessarily

$$f_{LN} \geq f_E \geq f_{HL}$$

As to how well is "fairly well," below I report such numbers from recent research comparing the efficacy of f_E in approximating $ELn(1+R)$ versus that for other risk-measures.

Table 5a. Frequently used asset classes used in Markowitz (2012b and 2013)

Large Cap Stocks	U.S. Treasury bills
Small Cap Stocks	Inflation
Long-term corporate bonds	EAFE (Developed non-U.S. markets)
Long-term government bonds	Energy Markets
Intermediate-term government bonds	

Table 5b. Sixteen countries whose real equity returns, 1900-2000, are used in Markowitz (2012b and 2013)

Australia	Japan
Belgium	Netherlands
Canada	South Africa
Denmark	Spain
France	Sweden
Germany	Switzerland
Ireland	U.K.
Italy	U.S

9. Related research

I have summarized only a small portion of the literature on mean-variance approximations to expected utility. A more complete survey is presented in Markowitz (2012b). Perhaps the most interesting article omitted here but reviewed there is that of Hlawitschka (1994). It is often said that mean-variance analysis is inapplicable if a portfolio includes derivative securities, since these have quite non-Gaussian return distributions and are not linearly related to underlying risk-factors. Hlawitschka demonstrates that this view is wrong. While an MV approximation to EU for a single put or call would do quite poorly, Hlawitschka found that MV approximations to EU did quite well for portfolios of ten calls each. For randomly drawn stocks Hlawitschka assumed the calls to be 5 percent out-of-the-money and were priced according to Black-Scholes. He also assumed that all portfolios had 10 percent invested in T-Bills, to eliminate the possibility of a 100 percent loss. For such portfolios Hlawitschka concluded that "empirically, two-moment approximations to the utility functions studied here perform well for the task of portfolio selection."

Other measures of risk

The results of Markowitz (2012a) raise the question: Could an approximation based on a different risk-measure have done better? This is the topic of Markowitz (2013) which considers the following risk-measures

Variance (V)
Mean Absolute Deviation (MAD)
Semivariance (SV)
Value at Risk (VaR)
Conditional Value at Risk (CVaR)

These are defined as follows:

$$\text{MAD} = E|R-E(R)|$$

$$\text{SV} = E(Min(0,R-E(R))^2$$

VaR is the largest number such that

$$\text{Prob}(R \leq -VaR) = p$$

$$CVaR = E(R|R \leq -VaR)$$

Markowitz (2013) used $p = 0.05$.

Konno and Yamazaki (1991) is the principal proponent of MAD for portfolio selection; Sortino & Satchell (2001) argue for semivariance, a.k.a. downside risk; see Jorion (2006) regarding VaR; and Kaplan (2012) who recommends CVaR.

Rearranging the terms in Equation 8, we see that

$$Ln(1+E) - L(1+g_{QE}) = \tfrac{1}{2}V/(1+E)^2 \qquad (15)$$

where g_{QE} is the f_E–based estimate of the geometric mean G. Thus the expression on the right hand side of Equation 15 characterizes the f_E method for approximating the difference on the left. Let

$$\Delta L \equiv Ln(1+E) - Ln(1+G) \qquad (16)$$

Markowitz (2013) tests approximations to ΔL of the form

$$\Delta L = \beta \cdot f(RM) \tag{17}$$

where $f(RM)$ represents a function of some risk-measure. The $f(RM)$ considered by Markowitz (2013) are listed in the first column of Table 6. RawVaR is computed as if the returns in each data series were equally likely and were the only possible returns in the population. Thus -RawVaR at the five percent level is the largest loss such that this loss—plus all returns which are worse than it—constitute *at least* five percent of the population. For a small data series there may be a considerable gap between -RawVaR and the next lower return. Interpolated VaR assumes, instead, that the return distribution has a step-function probability-density with returns uniformly distributed between -RawVar and the next lower return. Thus interpolated VaR is a linear interpolation between these two values. Since each series in the DMS database has 101 observations, the fifth from the worst return was used to define VaR. This makes VaR be at the 5/1.01 percent level, and RawVar precisely equal to VaR. For both databases CVaR was computed as the average return given that return equaled -RawVaR or worse. It was deemed unnecessary to compute CVaR using both RawVaR and interpolated VaR where these differed, since there is a large overlap in the range of the two computations.

In a series with no variability, $E = G$ and thus $\Delta L = 0$. Therefore the beta coefficients were fit by regressions in which the intercept was forced to be zero. This was done separately for each of the two databases.

Table 6 shows the root-mean-squared (RMSQ) error made by each tested $f(RM)$ for each of the two databases. The first column of the table lists the $f(RM)$ considered; the second column lists the RMSQs in the asset class database; and the third shows the same for the DMS database. RMSQ is expressed as a percent. For example, using a confidence interval equal to the estimate plus or minus two RMSQ, in the asset class database if adjusted variance estimated a geometric mean of 10.0%, this estimate would be subject to an error of probably no more than plus or minus $2 \cdot (0.05) = 0.1$ percent, i.e., 10 bps (where a basis point, bp, is $1/100^{th}$ of 1%). MAD, on the other hand, has an RMSQ of 0.51, therefore is subject to an estimated error of probably no more than plus or minus 102 bps, slightly over one percentage point. Adjusted MAD-squared does much better, with an RMSQ 0.20, therefore a confidence interval of ± 40 bps, about twice that of variance and four times that of adjusted variance.

Viewing the entire second column of Table 6, we see that the best fit in the asset class database is provided by adjusted variance with an RMSQ of 0.05, followed by (unadjusted) variance, semivariance, adjusted semivariance and $CVaR^2$ with RMSQs of 0.10, 0.11, 0.12 and 0.15 respectively.

All $f(RM)$ did worse in the DMS database than they did in the asset class database. Specifically, in this database unadjusted variance does slightly better than adjusted variance, with RMSQs equal to 0.17 and 0.18. The next closest risk measure is semivariance, adjusted and not, with an RMSQ of 0.35, about twice that of variance. MAD-squared and adjusted MAD-squared are a bit behind the semivariance measures. The worst performers are unadjusted MAD and the various functions of VaR and CVaR with RMSQs ranging from 0.46 to 0.70.

Thus in the DMS database with its large losses, functions of VaR and CVaR—which are promoted as the measures-to-use in case of large deviations—have substantially larger errors of approximation than do functions of variance.

Table 6. **Root mean-squared errors (percent)**

$f(RM)$	Frequently used asset classes	Real returns for 16 countries 1900-2000
Variance	0.10	0.17
Variance/$(1+E)^2$	0.05	0.18
MAD	0.51	0.70
MAD^2	0.29	0.40
MAD$^2/(1+E)^2$	0.20	0.42
Semivariance	0.11	0.35
Semivariance/$(1+2)^2$	0.12	0.35
RawVaR	0.65	0.68
$((\text{RawVaR}+E)/K)^2$	0.38	0.61
$((RV+K)/K)^2/(1+E)^2$	0.46	0.60
InterpVaR	0.32	—
$((\text{IntVaR}+E)/K)^2$	0.38	—
$((IV+E)/K)^2/(1+E)^2$	0.30	—
CVaR	0.48	0.55
CVaR^2	0.15	0.49
CVaR$^2/(1+E)^2$	0.17	0.46

10. Postscript

It is now over a half-century since Markowitz (1959) justified mean-variance by its ability to approximate expected utility. In light of repeated confirmation of this ability, the persistence of the Great Confusion is as if cartographers of 1550 still thought the world was flat.

Endnote

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