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A Note on Natural Number

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RESUMEN

Este artículo discute los modos en los que puede precisarse la caracterización por parte de Kant en la *Crítica de la razón pura* de la noción de número natural. La propuesta es que los números naturales son entidades funcionales cuya mejor manera de ser expresados está en la lógica de primer orden complementada con el cálculo lambda. A la vez que se alcanza este punto de vista, se proporciona un análisis de los puntos de vista de Frege sobre el número natural en *Die Grundlagen der Arithmetik*.

ABSTRACT

This paper discusses ways in which Kant's characterisation of natural number in the *Critique of Pure Reason* could be made precise. The proposal is that natural numbers are functional entities which are best expressed in first order logic augmented by the lambda calculus. In reaching this view, an analysis of Frege's views on natural number in *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*) is provided.

This note is intended to provide a formal definition of natural number that satisfies the following classic characterisation of number in Kant:

> [N]umber […] comprises the successive addition of homogenous units. Number … is simply the unity of the synthesis of the manifold of a homogeneous intuition in general […] [Kant (1781 & 1787/1933), A142-3 & B182).

What Kant seems to be saying is that a natural number is the name given to the unity that results when a finite set of objects that share a common property (i. e. are homogeneous with respect to that property and therefore are units with respect to the property) are synthesised into a single object[1] by counting. In many ways, Kant's definition is not an especially good definition by modern standards. Terms like "unity", "finite" and "counted" seem to be as difficult to characterise as natural number itself. On the other hand, the definition does make clear that number is dependent on a prior homogenisation (abstraction over properties) and that number has a functional character ("plurality considered as unity" in short[2]). The definition below attempts to capture these characteristics in a precise manner[3].

$0 := (\lambda P)(\forall a)(\neg Pa)$

$$s(n):=(\lambda P)(\lambda b)[Pb\wedge n(\lambda y(Py\wedge y\neq b))]$$

"λ" is the abstraction operator of the lamdba calculus and s(n) indicates the successor of n in the natural number sequence. This definition can be seen to be correct if the substitution rule of the lambda calculus is recollected: if term s is substituted for x in term t ($=\lambda y.t'$), denoted $t[s/x]$, if $x=y$ then $t[s/x]=t$, while if $x\neq y$ $t[s/x]=(\lambda y)t'[s/x]$ if y is not free in s or x is not free in t' or $t[s/x]=(\lambda z)t'[z/y][s/x]$ otherwise (i. e. y is free in s and x is free in t')[4]. Thus, for example:

$1:= s(0)$	Definition
$s(0):= (\lambda P)(\lambda b)[Pb\wedge 0(\lambda x(Py\wedge y\neq b))]$	Definition
$s(0):= (\lambda P)(\lambda b)[Pb\wedge(\forall x)(\neg(Px\wedge x\neq b))]$	λ application with x for y
$s(0):= (\lambda P)(\lambda b)[Pb\wedge(\forall x)(Px\Rightarrow x=b)]$	Propositional logic

Hence 1 is the concept of being a property satisfied by a unique individual.

Similarly,

$2:=s(1)$	Definition
$s(1):=(\lambda P)(\lambda b)[Pb \wedge 1(\lambda y(Py\wedge y\neq b))]$	Definition
$s(1):= (\lambda P)(\lambda b)[Pb \wedge (\lambda b)[Pb\wedge (\forall x)(Px\Rightarrow x=b)][\lambda y(Py\wedge y\neq b)/P]]$	Substitution from s(0)
$s(1):= (\lambda P)(\lambda b)[Pb\wedge(\lambda c)[Pc\wedge c\neq b\wedge(\forall x) (Px\wedge x\neq b \Rightarrow x=c)]$	Substitution as b is free in $\lambda y(Py\wedge y\neq b)$ and P is free in $Pb\wedge(\forall x)(Px\Rightarrow x=b)$
$s(1):= (\lambda P)(\lambda b)(\lambda c)[Pb\wedge Pc\wedge c\neq b\wedge(\forall x) (Px\Rightarrow x=b\vee x=c)]$	Change of λ scope and propositional logic

As can be seen, 2 is the concept of being a property satisfied by a pair of individuals; and in general n is the concept of being a property satisfied by n individuals. This may seem circular, but it is only an informal characterisation. In fact, as the definition shows, the construction and understanding of natural number is inductive in that a number can only be constructed and understood once its predecessor has been constructed. If we wanted to bring out the functional aspect of the definition of natural number, we could say that a natural number is the characteristic function that decides whether an arbitrary property holds for exactly n individuals.

The above definition of natural number, and indeed most of the formal framework within which it sits, is due ultimately to Frege. However, Frege did not define natural numbers as concepts but as objects identified with sets of concepts. To be precise, the number belonging to a property P is the set of all properties the extensions of which can be put in one-to-one correspondence with P. n is a number if there is a property P to which n belongs. To show that for each natural number there is a property to which it belongs, Frege took $P=\lambda x(x\neq x)$ for 0 and $P=\lambda x(0\leq x\leq n)$ for s(n)[5]. Thus, 0 is the set u such that $(\forall F)(F\in u \Leftrightarrow Feqx\neq x)$ and s(n) is the set u such that $(\forall F)(F\in u \Leftrightarrow Feq0\leq x\leq n)$, where FeqG if $(\exists H)[\forall x(Fx\Rightarrow(\exists!y)(Hxy\wedge Gy)) \wedge\forall y(Gy\Rightarrow(\exists!x)(Hxy\wedge Fx))]$. The relation "eq" stands for "is equinumerous with".

Although the genius of its conception is not to be denied, there are a number of problems with Frege's account of number. The most serious problem is that the definition is circular. For example, 1 belongs to the property $\lambda x(x=1)$ as well as to $=\lambda x(x=0)$ $(=\lambda x(0\leq x\leq 0))$. Thus 1 is included in properties which are included in the definition of 1. This circularity holds equally of other numbers, and can be called the *problem of numbering numbers*. This name is generally appropriate because $\lambda x(1\leq x\leq n)$ has number n and will thus appear in the definition of n.

Slightly less serious, although against the inductive nature of the definition of the natural numbers, is the fact that larger numbers will appear within properties that define smaller numbers. For example, 2 appears in the property $\lambda x(x=2)$ which is a property included in the definition of 1.

Another kind of problem is that Frege treats numbers as abstract *objects*. While there are problems in the literature[6] concerning the possibility of epistemological access to abstract objects based on Tarski's truth definition and the causal theory of knowledge, if abstract objects are allowed to have concrete instances, the belief in abstract objects seems consistent because concrete instances of abstract objects can be chosen to represent the objects provided that the instances chosen form an ω-sequence[7]. However, there is a concern that objects should not have instances and, moreover, that natural numbers as abstract objects do not obviously relate to the functional characteristic of number that is apparent when determining the size of a finite set of individuals by counting, and which Kant's definition does capture.

These problems are not necessarily fatal to Frege's account of number, but they do require some major surgery. Circularity seems essential to Frege's account, for the existence of the natural number sequence is demonstrated inductively by reference to previous numbers. In order to avoid the circularity we can introduce numbers as a separate abstract type above a domain of objects that does not include numbers. The most natural way to do this is to take the domain of concrete (spatio-temporal) objects. Natural numbers are definable as sets of properties of concrete objects, with the proviso that there are only as many natural numbers as there are concrete objects. In order to ensure that all natural numbers are definable, the domain of concrete objects needs to be infinite. This can be achieved by constructing objects in a suitable open space (e. g. a Euclidean space) in the way envisaged by Hilbert [Hilbert (1925/1967)][8] or by successively subdividing an object into homogeneous parts[9]. If there is a hierarchy of objects, properties of objects, properties of properties of objects, etc., natural numbers will then appear at each level in the hierarchy above the second. To be precise, numbers of objects appear at the third level (as abstraction over properties of objects), numbers of properties at the fourth level, etc. Each of these numbers will differ only in the type of objects numbered, so a natural number could be defined as the equivalence class of equinumerous properties at each level in the hierarchy greater than the second.

Formally, we have:

> 0 is the class z such that $(\forall F)[F\in z \Leftrightarrow F eq(\lambda x)(x\neq x)]$ where x is a concrete object,

and

> s(n) the set s' such that $(\forall F)(\exists L)[F\in s' \Leftrightarrow (\exists b)(Fb\wedge \lambda y(Fy\wedge y\neq b) eqn(L))]$, where L is a property of concrete objects and FeqG if $(\exists H)[\forall x(Fx\Rightarrow (\exists !y)(Hxy\wedge Gy))\wedge \forall y(Gy\Rightarrow (\exists !x)(Hxy\wedge Fx))]$ where F and G, x and y may be at different levels in the hierarchy.

There is something counterintuitive about this definition. The definition of a natural (finite!) number is at the very least an infinite set (allowing properties of each object), while if the hierarchy of levels is extended indefinitely into the transfinite, natural numbers will be proper classes rather than sets. To attempt to resolve this difficulty we could replace a set with the property that define the set. This gives rise to the following definition:

> 0 is the property Z such that $(\forall F)[Z(F)\Leftrightarrow F eq(\lambda x)(x\neq x)]$ where x is a concrete object,

and

> s(n) the property S' such that $(\forall F)(\exists L)[S'(F) \Leftrightarrow (\exists b)(Fb\wedge \lambda y(Fy\wedge y\neq b) eqn(L))]$, where L is a property of concrete objects and FeqG if $(\exists H)$

$[\forall x(Fx \Rightarrow (\exists!y)(Hxy \wedge Gy)) \wedge \forall y(Gy \Rightarrow (\exists!x)(Hxy \wedge Fx))]$ where F and G, x and y may be at different levels in the hierarchy.

It should be noted that Z and S will be at a level in the hierarchy higher than the level of each property in the hierarchy. If, for example, the property hierarchy extends to all finite levels above concrete objects, then Z and S will be defined at the first transfinite level, ω, of the hierarchy. If the hierarchy is extended indefinitely into the transfinite, Z and S will not be defined in resulting transfinite type theory.

The revised definition of natural number above has much to commend it. By replacing sets with properties, the definition no longer has the problems associated with abstract objects. The definition also has a functional character if the property is identified with its characteristic function. The definition is even inductive, as the hierarchy is defined inductively from the basis of the domain of concrete objects. There are two problems that remain with the definition: viz, the definition does not capture the *intrinsic*, *structural* nature that enables numbers at different levels in the hierarchy to be identified, and the definition is not especially obvious.

If we return to our original definition, viz.:

$0:=(\lambda P)(\forall a)(\neg Pa)$
$s(n):=(\lambda P)(\lambda b)[Pb \wedge n(\lambda y(Py \wedge y \neq b))]$

we see that if circularity is to be avoided natural numbers need to form the same hierarchy as under the revised Fregean definition. Thus the definition of n is specific to a particular level in the hierarchy. To define the general notion of natural number, note that the only difference between the definitions of natural number at different levels in the hierarchy is the types of the abstracted variables. Thus if it is possible to transform the types of the variables to make two definitions of a number identical, the two definitions will define the same number. For want of a better term, two such definitions will be called isomorphic. A number n is then the property of being isomorphic to a number n of concrete objects. To put this more formally:

$P \approx n(G)$ where P is a property at any level in the hierarchy if P(G) is identical to n(G) and where the types of other variables are appropriate,

and

$n:= (\lambda P)(\lambda L)(P \approx n(L))$, where P is a property at any level in the hierarchy and L is a property of concrete objects.

This definition seems to capture the nature of numbers. Numbers are abstract functions that are formal (ie relating to a form or structure) in that they apply equally to properties of different types derived from a basis of concrete objects. Numbers are functions, which must be grasped inductively. Numbering numbers is possible within the hierarchy of properties by taking numbers at certain level in the hierarchy as individuals in the numbering process. It is not possible to number the general notion of natural number on pain of circularity. This is not surprising as one would not expect to be able to apply a structural notion to itself.

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NOTES

[1] "Intuition" or "singular representation" would be more accurate, but "object" has the advantage of being a fairly neutral term.

[2] See Kant (1781&1787/1933), B111 for Kant's view that number belongs to the category of totality, which is plurality considered as unity.

[3] This definition is derived from Frege [Frege (1884/1950)], but see also Hilbert & Ackermann (1938/1950) p137.

[4] See Turner (1990), Chapter 2 for further details of the untyped lambda calculus.

[5] This summary of Frege (1984/1950) follows Gillies (1982), Chapter 7.

[6] See Benacerraf (1973).

[7] See Thiel (1995) for example, for accounts of ω-sequences.

[8] This type of construction has been developed by Lorenzen (1951), (1965/1971) and more recently by Thiel (1995) by abstracting over concrete representations (see Powell (1997) for a discussion). The possibility of performing the constructions can be motivated by Lorenzen's foundation of Euclidean geometry in terms of openness and homogeneity (see Lorenzen (1987) and Powell (1997). Hellman (1989) contains an interesting modal nominalist account of the construction of natural numbers as concrete objects.

[9] This line of thought can be traced to Kant (see Parsons (1984)) if the notion that a continuum has an indeterminate number of parts is merged with the view of a real number as represented by a nested sequence of intervals on a line (a space form that can be traversed in two directions).

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