ANALYTIC STUDY OF A PREY-PREDATOR MODEL WITH ALLEE EFFECT

Estudio analítico de un modelo predador-presa con efecto Allee

ABSTRACT
The Allee effect is introduced into intraspecific and two-species competitive models. We use phase plane analysis and simulation to investigate the dynamics of these models. Combining simple modeling and simulations, we demonstrate that the Allee effect alone could lead to alternative stable states in two-species competitive systems. In such systems, if interspecific competition is intense, both species may go extinct even if their population densities are high.

KEYWORDS: *Alee effect, prey-predator model***.**

RESUMEN
El efecto Allee se introduce en modelos de competencia intraespecífica y dos especies en competencia. Se utiliza análisis de plano de fase y simulación para investigar la dinámica de estos modelos. Combinando modelación simple y simulación, se demuestra que el efecto Allee por sí solo podría llevar sistemas de dos especies en competencia a estados estables alternativos. En tales sistemas, si la competencia interespecífica es intensa, las dos especies podrían extinguirse aunque su densidad de población sea alta.

PALABRAS CLAVES: *Efecto Allee, Modelo predador-presa.*

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1. INTRODUCTION

The Allee effect, also called as `negative competition effect' (Begon and Mortimer, 1981), refers to a decrease in per-capita growth rate at low population densities (Allee, 1931, 1938; Fowler and Baker, 1991; Burgman et al., 1993; Begon et al., 1996). The logistic model assumes that percapita growth rate declines monotonically with density. But for populations subject to an Allee effect, per-capita growth rate shows a humped curve increasing (from negative to positive) at low density, up to a maximum at intermediate density and then declining.

2. CONTENT

2.1 THE MODEL

Let us consider following prey--predator model

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1-\dfrac{x}{k}\right) - \dfrac{mx}{x+b} - \dfrac{qxy}{x^2+a}. \\ \\ \dfrac{dy}{dt} = sy\left(1-\dfrac{y}{nx}\right). \end{cases} \tag{1}$$

In this model $x = x(t)$ represents the number of prays and $y = y(t)$ is the number of predators at a given moment t ($t \geq 0$). System (1) is a dynamical system in the plane $x-y$. This is a typical Lotka-Volterra model. Furthermore, the parameters have the following meanings:

(1) r and s are the intrinsic growth rates or biotic potential of the prey and predators, respectively.

(2) q is the maximal predator per capita consumption rate, i.e., the maximum number of prey that can be eaten by a predator in each time unit.

(3) a is the number of prey necessary to achieve one-half of the maximum rate q .

(4) n is a measure of the food quality that the prey provides for conversion into predator births.

(5) k is the prey environment carrying capacity [8].

(6) m and b are constants that indicate the severity of the Allee effect that has been modeled.

This model considers three aspects for describing the interaction:

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(i) A logistic type growth function for predator;

(ii) A functional response of predators of nonmonotonic type; and

(iii) The Allee effect [5,9,11,24], which is considered in the prey growth function and occurs whenever fitness of an individual in a small or sparse population decreases as the population size or density also declines [5].

We shall study model (1) from the point of view of the general theory autonomous systems. First, we find equilibrium points of this system. These points are obtained by solving the algebraic system

$$\begin{cases} rx\left(1-\dfrac{x}{k}\right)-\dfrac{mx}{x+b}-\dfrac{qxy}{x^2+a}=0. \\ \\ sy\left(1-\dfrac{y}{nx}\right)=0. \end{cases} \quad (2)$$

Second equation of system (2) gives

$$y=0 \text{ or } y=nx.$$

2.2 COEXISTENCE OF PREYS AND PREDATORS

Let us suppose that $y>0$ in (2). Then

$$y=nx,$$

i.e., the number of predators y increases proportionally to the number of preys x.
Now, we substitute equation $y=nx$ into the first equation of system (2) and after some algebraic calculations we arrive at the following fourth degree polynomial equation in the variable x:

$$\begin{aligned} &rx^4+(b-k)rx^3+(ar+k(m+nq-br))x^2+ \\ &(bknq+a(b-k)r)x+ak(m-br)=0. \end{aligned} \quad (3)$$

Let

$$L=\frac{br-m}{s}. \quad (4)$$

First Case: $L=0.$

We will assume that m and b are subject to

$$\begin{cases} bknq+a(b-k)r=0. \\ m-br=0. \end{cases}, \quad (5)$$

i.e.,

$$m=\frac{akr^2}{ar+knq} \text{ and } b=\frac{akr}{ar+knq}. \quad (6)$$

This assumption gives equation (8) in the form

$$rx^2+(b-k)rx+(ar+k(m+nq-br))=0. \quad (7)$$

Taking into account equations (9a), equation (10) has the solution

$$x=\frac{k^2nqr}{2r(ar+knq)}\left(1\pm\sqrt{1-\frac{4(ar+knq)^3}{k^4n^2q^2r}}\right). \quad (8)$$

provided that

$$k^4n^2q^2r\geq 4(ar+knq)^3. \quad (9)$$

Observe that restrictions (6) imply that

$$0<m<kr \text{ and } 0<b<k. \quad (10)$$

For example, our assumptions hold if we set

$$a=3,\ k=10,\ n=2,\ q=2 \text{ and } r=3. \quad (11)$$

This choice gives

$$b=1.83673,\ m=5.5102.$$

Thus, from (6) and (11) the corresponding equilibrium points are

$$(8.16327,0),\ \ (4.65294,9.30588) \text{ and } (3.51033,7.02065).$$

The phase portrait near these points is shown in Figure 1. This graphic was made with the aid of *Maple* 13.

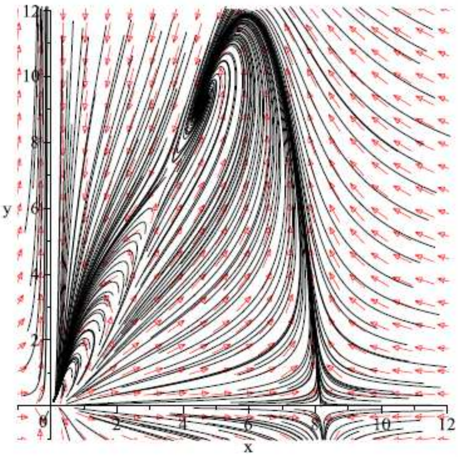

Figure 1

Another possible choice for the parameters is

$$b=\frac{k}{2}, a=\frac{k^2nq}{2m}, r=\frac{2m}{k}, m>16nq. \qquad (12)$$

Second Case: $L<0$

We choose

$$r=\frac{\left(2a^2+a+1\right)nq}{a(a-1)},$$
$$m=\frac{(2a+1)^2\left(a^2+1\right)nq}{2a^2(a-1)}, \qquad (13)$$
$$k=\frac{2a^2+a+1}{a},$$
$$b=\frac{2a^2+a+1}{2a},$$

where a is subject to the condition

$$a>\frac{1}{4}\left(7+\sqrt{41}\right)\approx 3.35078.$$

Observe that

$$L=\frac{br-m}{s}=-\frac{2am}{(2a+1)^2\left(a^2+1\right)s}<0.$$

The advantage here is that our selection gives equation (7) in the form

$$(x-1)^2\left(2ax^2-\left(2a^2-3a+1\right)x+2a\right)=0. \qquad (14)$$

Solving equation (14) gives

$$x=1$$
$$x=\frac{1}{4a}\left(2a^2-3a+1\pm\sqrt{\left(a+2a^2+1\right)\left(2a^2-7a+1\right)}\right).$$

These data allow simulating the model.

3. CONCLUSIONS

We studied a prey-predator model with Allee effect. We also showed the way we may chose some of the parameters in a wide range in order to simulate the model. This helps in the understanding the way in which prey and predators may coexist.

4. REFERENCES

[1] Allee, W.C., 1931. Animal Aggregations: A Study in General Sociology. University of Chicago Press, Chicago, IL.
[2] Allee, W.C., 1938. The Social Life of Animals. Heinemann, London.
[3] Allison, T.D., 1990. Pollen production and plant density affect pollination and seed production in Taxus canadensis. Ecology 71, 516--522.
[4] Amarasekare, P., 1998. Allee effects in metapopulation dynamics. Am. Nat. 152, 298--302.
[5] Asmussen, M.A., 1979. Density-dependent selection II. The Allee effect. Am. Nat. 114, 796--809.
[6] Begon, M., Mortimer, M., 1981. Population Ecology: A Unified Study of Animals and Plants. Blackwell Scientific, Oxford.
[7] Begon, M., Harper, J.L., Townsend, C.R., 1996. Ecology. Individuals, Populations and Communities, Third edition. Blackwell Science, Oxford.
[8] Beddington, J.R., 1974. Age distribution and the stability of simple discrete time population models. J. Theor. Biol. 47, 65--74.
[9] Birkhead, T.R., 1977. The effect of habitat and density on breeding success in the common guillemot (Uria aalge). J. Anim. Ecol. 46, 751--764.
[10] Burgman, M.A., Ferson, S., Akcakaya, H.R., 1993. Risk Assessment in Conservation Biology. Chapman and Hall, London.
[11] Dennis, B., 1989. Allee-effect: population growth, critical density, and chance of extinction. Nat. Resource Model. 3, 481--538.
[12] Dobson, A.P., Lyles, A.M., 1989. The population dynamics and conservation of primate populations. Conserv. Biol. 3, 362--380.
[13] Feinsinger, P., Tiebout III, H.M., Young, B.E., 1991. Do tropical bird-pollinated plants exhibit density-dependent interactions? Ecology 72, 1953--1963 field experiments.

[13] Fowler, C.W., Baker, J.D., 1991. A review of animal populations dynamics at extremely reduced population levels. Rep. Int. Whal. Comm. 41, 545--554.
[14] G. Wang et al. : Ecological Modelling 124 (1999) 192 183--192 Gilpin, M.E., Case, T.J., 1976. Multiple domains of attraction in competition communities. Nature 261, 40-- 42.
[15] Hanski, I., 1991. Single-species metapopulation dynamics: concepts, models and observations. Biol. J. Linnean Soc. 42, 17--38.
[16] Hopf, F.A., Hopf, F.W., 1985. The role of the Allee effect in species packing. Theor. Popul. Biol. 27, 27--50. Horn, H.S., 1981. Succession. In: May, R.M (Ed.), Theoretical Ecology: Principles and Applications. Blackwell Scientific, Oxford.
[17] Kunin, W.E., 1992. Density and reproductive success in wild populations of Diplotaxis erucoides (Brassicaceae). Oecologia 91, 129--133.
[18] Kunin, W.E., 1993. Sex and single mustard: population density and pollinator behavior effects on seed-set. Ecology 74, 2145--2160.
[19] Kuussaari, M., Saccheri, I., Camara, M., Hanski, I., 1998. Allee effect and population dynamics in the Glanville fritillary butterfly. Oikos 82, 384--392.
[20] Lamont, B.B., Klinkhamer, P.G.L., Witkowski, E.T.F., 1993. Population fragmentation may reduce fertility to zero in Banksia goodii: a demonstration of the Allee Effect. Oecologia 94, 446--450.
[21] Lande, R., 1987. Extinction thresholds in demographic models of territorial populations. Am. Nat. 130, 624--635.
[22] Lande, R., 1988. Genetics and demography in biological conservation. Science 241, 1455--1460.
[23] Lande, R., Engen, S., Saether, B.E., 1994. Optimal harvesting, economic discounting and extinction risk in fluctuating populations. Nature 372, 88--90.
[24] Lewis, M.A., Kareiva, P., 1993. Allee dynamics and the spread of invading organisms. Theor. Popul. Biol. 43, 141--158.
[25] Lotka, A.J., 1932. The growth of mixed populations: two species competing for a common food supply. J. Washington Acad. Sci. 22, 461--469.
[26] May, R.M., 1977. Thresholds and breakpoints in ecosystems with a multiplicity of stable states. Nature 269, 471--477.
[27] McCarthy, M.A., 1997. The Allee effect, finding mates and theoretical models. Ecol. Model. 103, 99--102.
[28] Stephan, T., Wissel, C., 1994. Stochastic extinction models discrete in time. Ecol. Model. 75-76, 183--192.
[29] Veit, R.R., Lewis, M.A., 1996. Dispersal, population growth, and the Allee effect: dynamics of the house finch invasion of eastern North America. Am. Nat. 148, 255--274.
[30] Volterra, V., 1926. Variations and fluctuations of the numbers of individuals in animal species living together. In: Chapman, R.N. (Ed.), Animal Ecology. McGraw Hill, New York Reprinted in 1931.