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A Scientific Revolution in Real Time

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RESUMEN

Frege no se dio cuenta de que el trabajo semántico de los cuantificadores incluye expresar, debido a las propias dependencias formales entre ellos, dependencias reales entre variables. Este error no corregido condujo a una lógica de primer orden deficiente, a paradojas en la teoría de conjuntos y a una compresión inadecuada de la lógica de orden superior. Un error que comenzó a corregirse tan solo por medio de la introducción de la lógica IF. Esta reforma conduce a una lógica de primer orden más rica, a una reducción de la lógica de orden superior a un nivel de primer orden, a un rechazo de las habituales axiomatizaciones de primer orden de la teoría de conjuntos, a un nuevo concepto de probabilidad y a una reevaluación de los resultados de incompletitud, indefinibilidad e indemostrabilidad.

PALABRAS CLAVE: *lógica IF, paradojas de teoría de conjuntos, dependencia de cuantificadores, probabilidad*.

ABSTRACT

Frege did not realize that the semantical job of quantifiers includes expressing, by their own formal dependencies on each other, actual dependencies between variables. This uncorrected mistake led to flawed first-order logic, set-theoretical paradoxes and inadequate grasp of higher-order logic. It began to be corrected only through the introduction of IF logic. This reform leads to a richer first-order logic, to a reduction of higher-order logic to first-order level, to a rejection of the usual first-order axiomatizations of set theory, to a new concept of probability, and to a re-evaluation of the incompleteness, indefinability, and unprovability results.

KEYWORDS: *Independence Friendly Logic, Paradoxes of Set Theory, Quantifier dependence, Probability*.

I. THE SETTING OF A SCIENTIFIC REVOLUTION

The purpose of this report is to bring to my readers' attention the extraordinary developments that are going on in the foundations of mathematical sciences and of scientific philosophy. Ever since Thomas Kuhn's 1962 book, *The Structure of Scientific Revolutions*, the notion of a scientific revolution has been used and discussed. Many thinkers, the present writer included, have been skeptical and have found the concepts Kuhn uses too vague to illuminate the nature of major shifts in the developments of actual science. Kuhn himself gave up the key term "paradigm" that he originally used.

No attempt is made in this report to discuss Kuhn's ideas except by pointing out that they have turned out to be uncannily timely. Believers and skeptics alike are now finding themselves in the midst of an unmistakable revolution. The scientific community in philosophy and in the mathematical sciences is facing an unexpected situation. Most mathematicians and most philosophers are nevertheless unaware of what is going on.

This erupting crisis is best viewed from a historical perspective. In 1800, mathematics consisted, broadly speaking, of the theory of numbers, the theory of numerical functions (analysis), and of geometry. It has since become an abstract science of structures of any kind. The main steps in this development include prominently the introduction of symbolic logic and of set theory as well as the use of the axiomatic method. Modern symbolic logic goes back to Gottlob Frege (1848-1925). Set theory was founded by Georg Cantor (1845-1918) and the most important defender of axiomatization in foundational studies was the mathematician David Hilbert (1862-1943). The revolution discussed here springs from the realization that all these three paradigmatic ideas have been used and developed in an inadequate way, with disastrous results.

This certainly is especially dramatic in the case of the most fundamental of the three, our basic logic. The central part of logic is the theory of the quantifiers "every" and "some". This theory was first systematized and formalized by Frege in the form of the first-order part of his overall logic. Later logicians have mostly followed the main features of Frege's approach. Its most common form is the well-known received first-order logic (RFOL). This logic comprehends much, but not all, of the modes of reasoning used by practicing mathematicians. The rest Frege proposed to capture by means of the higher-order part of his logic. Others have tried to capture this part of the reasoning in mathematics in terms of set theory.

Independently of Frege, quantificational reasoning played a crucial role in the nineteenth-century development of a rigorous calculus and in its arithmetization. Mathematicians are familiar with this informal logic of quantifiers in the form of the "epsilon-delta" technique of calculus texts. It has been assumed by practically everybody that this quantificational reasoning can be captured in Frege's and his followers' symbolic logic.

II. Which Mathematical Logic is the Logic of Mathematics?

This belief has been shown to be mistaken. In fact, Frege failed to understand adequately the unformalized logic of quantifiers that his fellow mathematicians like Karl Weierstrass (1815-1897) were using. This is spelled out in my 2012 paper, “Which Mathematical Logic is the Logic of Mathematics?” [Hintikka (2012a)]. As a consequence, Frege’s own logic of quantifiers is not incorrect, but it is unnecessarily weak. For a historically significant example, most of the so-called uniformity concepts (uniform convergence, uniform differentiability, etc.) that are important in analysis cannot be expressed in Frege’s *Begriffsschrift* [Frege (1879)] or in the RFOL. Amazingly, there does not seem to be any account of the meaning of uniformity concepts in the literature of logic, and even the best mathematicians have made mistakes when trying to spell out their meaning in logical terms.

As a consequence, the vast majority of logicians and philosophers (as well as those mathematicians who use explicitly formulated logic) have for well over a century been using unnecessarily weak logic, meaning weak in representative power, not necessarily weak deductively.

Of course, using too weak a logic is not per se a mistake. However, you are liable to philosophical mistakes if you think that your weak logic is the whole story. Moreover, Frege’s real mistake is a far deeper one than a failure to locate fully the principles of logical inference. The deeper mistake is a failure to understand the semantical nature of quantifiers. Frege assumed that the semantical job description of quantifiers is exhausted by their ranging over a class of values. For him, quantifiers were higher-order predicates expressing the non-emptiness or the exceptionlessness of lower-order predicates. For other thinkers, the same fallacious idea has taken other forms, such as constructing quantified sentences as long disjunctions or conjunctions or the theory of rigid designation.

In reality, quantifiers have another, hugely important semantical function. By their formal (syntactical) dependencies on each other, they express the dependence relations between their respective variables.

In mathematicians’ pre-symbolic jargon, such dependencies are handled by constructing the relevant variables, not as being “bound” to a quantifier, but as slots for a choice term, as in mathematical textbooks’ “epsilon-delta” technique for defining the basic concepts of calculus, say the differentiability of $f\,x$ at x. “For any ε, however small, one can choose δ such that…” What separates uniform concepts from pointwise ones is that in their case “choice” is independent of x (for some specified interval $x_1 < x < x_2$). Semantically, this means that the variable for δ is independent of x in the intended sense.

Such independencies cannot in general be expressed by means of RFOL, and remained uninvestigated in the mainstream logic tradition. At the same time, mathematicians, even though they handled uniformity relations in practice, never recognized their logical nature.

In the light of hindsight we can now see that the logic that mathematicians had already been using before symbolic logic was explicitly formulated only around 1990 largely by Jaakko Hintikka and Gabriel Sandu under the title "independence-friendly logic" (IF logic). Strictly speaking, what is known and studied under this title is only a partial liberalization of dependence relations, crucially the possibility of freeing existential quantifiers from a dependence on universal quantifiers. Various further modes of liberalization are nevertheless possible.

This shows vividly the systematic and historical status of IF logic. It is neither a further development of RFO logic nor an "alternative logic". It replaces RFO logic by eliminating some of its limitations. It should not carry any particular epithet. It would be more accurate to rename instead RFOL and call it "dependence handicapped" or "independence challenged" logic.

III. The Inconclusiveness of Incompleteness Results

This thumbnail history suffices to show that the current conceptions concerning the philosophical and other general theoretical significance of contemporary symbolic logic are inaccurate. Its great achievement in the 20^{th} century are generally taken to include the famous incompleteness and other impossibility results by Kurt Gödel (1906-1978), Alfred Tarski (1901-1983), and Paul Cohen (1934-2007). On a closer scrutiny, these famous results turn out to be as much symptoms of the weakness of the logic these ingenious gentlemen were using as indications of the limits of logic, mathematics, axiomatic method or human reason in general. For instance, Tarski proved that the concept of truth cannot be defined for a first-order language in the same language. This impossibility holds only if one is using RFO logic. In contrast, such truth definitions are possible for an IF first-order language.

Gödel's so-called second incompleteness theory says that the consistency of elementary arithmetic cannot be proved by means of the resources of elementary arithmetic itself. Again, such a proof is possible if IF logic is used, as is shown in [Hintikka and Karakadilar (2006)].

Gödel's first incompleteness theorem will be discussed later in this report, and so will be the reasons why Gödel's and Cohen's results concerning set theory have no relevance to the question whether the continuum hypothesis or any other set theoretical theorem holds. Whatever general theoretical significance symbolic logic has, it does not lie in such impossibility results, except perhaps for their role as warning signs about mathematical and logical misconceptions.

In light of this overall development of logic, the entire twentieth-century analytic philosophy is beginning to look different. One major aspect of this development was the gradual decline of logical positivism. There does not seem to be any particular mystery about it, even if we restrict ourselves to the internal rather than external reasons. For what did the logical positivists promise? In effect, to solve all the problems in the foundations of science and of mathematics by studying the logical syntax of the language of science and of mathematics. In spite of all the solid work they did, they cannot be said to have succeeded. But as I have tried to dramatize history, imagine that Carnap and his friends had definitively solved the foundational problems of science and mathematics. Where would we be then? I dare say, we would all be logical empiricists.

The reason for the failure of the logical positivists is usually taken to be the inadequacy of their logical and formal apparatus. Yes, we can now see that the logic of logical positivists was inadequate, but not because of any general limitations of logical analysis, but because the particular logic they were using was too weak. If this is the case, what philosophers should have done was not just to give up the Vienna Circle's logical methodology, but to build a better one.

IV. Logic of what Order?

But the implications of Frege's mistake were not restricted to general philosophical issues. It affected the prospects of his projects in logic and the foundation of the mathematics. The expressive poverty of Frege's quantification theory was not architectonic, but affected his theory and practice of logic in a variety of ways. For his theory of number, Frege needed the concept of the equinumerosity (equicardinality) of the extensions of two predicates $A(x)$ and $B(y)$. By means of IF logic, this equicardinality can be expressed as follows:

$$(1)\ (\forall x)\ (\forall z)\ (\exists y / \forall z)\ (\exists u / \forall x)\ ((A(x) \supset B(y))\ \&\ (B(z) \supset A(u))\&\ ((y = z) \leftrightarrow (x = u))).$$

In a Skolem form, it equals

$$(2)\ (\exists f)\ (\exists g)\ ((\forall x)\ (\forall z)\ (A(x) \supset B(f(x))\ \&\ (B(z) \supset A(g(z)))\ \&\ ((f(x) = z) \leftrightarrow (x = g(z))).$$

The last equivalence in (2) shows that f and g are inverse functions, and (1) says therefore that the sets $\{x : A(x)\}$ and $\{y : B(y)\}$ can be mapped on each other one-one.

The independence-indicating slash / is not eliminable from (1). Hence equicardinality is not expressible in RFOL languages, including the language of the usual first order set theories.

Frege's failure to recognize the dependence indicating task of the quantifiers encouraged another oversight, which may very well have been the most widespread one. If all that matters in a quantifier was ranging over a class of values, first-order and higher-order quantification could be expected to behave in the same way. For on this view of the logic of quantifiers, all quantifiers do ultimately the same job of ranging over some domain.

But when quantifiers are thought of as proxies of choice functions, a conceptual difference opens between first-order ones, for to choose a set or other second-order entity is conceptually different from choosing a particular object. Hence, the logical laws of higher-order quantifiers are not exhausted by the laws of first-order logic. You simply cannot apply traditional first-order logic in the study of sets or other kinds of higher-order entities and merely stipulate that one's (first-order) quantifiers range over such entities.

Yet this is precisely what logicians have repeatedly done. The main assumption of Frege's higher-order version of set theory, Frege's Basic Law V says essentially only that one can deal with sets (extension of concepts) by means of first-order quantifiers. It is this assumption that led him into paradoxes.

Likewise, the great majority of work in set theory has used as its logic RFO logic. This work is therefore of dubious value (see sec. VI below).

On Frege's construal of quantifiers, the difference between first-order and second-order quantifiers can be in the metaphysical and logical status of their respective values: particulars versus sets or other higher-order entities. The quantifiers themselves would presumably have to behave in the same way.

In reality, as soon as the independencies examined in IF logic are recognized, it can be seen that the logical laws governing first- and second-order logic are intertwined. Hence to use first-order quantifiers in the role of second-order ones smuggles in a wealth of assumptions and accordingly cannot produce a presuppositionless analysis of higher-order quantification.

A striking example of this entanglement is the so-called axiom of choice. It is typically considered as one of the modes of reasoning that are vital in classical mathematics but not captured by RFO logic. It is typically formulated as a set-theoretical axiom, consequently a higher-order assumption. This is again a self-inflicted impossibility, due to logicians' custom of formulating rules of logical inference by inference to the governing ("outmost") quantifier or other logical operator. As soon as we allow one of the most fundamental rules, the rule of existential instantiation, to operate on sub-formulas of larger formulas, the "axiom" of choice becomes a purely logical and even first-order principle.

The same conceptual entanglement of the two kinds of quantifiers is exemplified by the following pair of sentences, one first-order and the other second-order:

(3) $(\exists f)\, S\, [\text{—}\, A\, (f(t)\, \text{—}]$

(4) $(\forall x)\,(\forall z)\,(\exists y\,/(\forall z))\,(\exists u\,/(\forall x))\,(((x = z) \leftrightarrow (y = u))\ \&\ S\,[-\,(\forall w)\,((x = w) \supset (A\,(y))\,-])$.

In (3) and (4), S is a formula in which f occurs only in atomic subformulas $A(f(t))$ with identities of the form $A_i\,(f(t_j))$. Clearly in (4) y is a function of x and u a function of z. The equivalence clause of (4) shows that they are the same function. This enables us to see that (3) and (4) are logically equivalent. In other words, we have in (4) a translation of a second-order formula (3) into IF first-order language.

This translation can be extended easily to all sigma one-one formulas of a second-order language. The translation depends essentially on the use of IF logic. By means of quantifier independencies that go beyond the usual basic IF logic, we can express also contradictory negation of any RFO or IF formula. These two results put together enable us to translate second-order logic into a sufficiently enriched IF logic. In principle, the entire second-order logic can for foundational purposes be reduced to an enriched first-order IF logic.

The details of this reduction are presented in [Hintikka and Symons (forthcoming)]. A tentative formulation of the reduction is found in my paper, "A Proof of Nominalism" [Hintikka (2009)]. It needs a further explanation, however. The relation obviously works if we can translate into the IF first-order notation both sigma one-one formulas and also contradictory negations. The possibility of the former was just explained informally. However, it might seem impossible to define contradictory negation $\neg S$ in terms of IF logic. Such a negation says game-theoretically that for any strategy by the "verifier" in a semantical game, there is a counterstrategy that prevents the verifier from winning. This involves in effect quantification over strategies, and hence second-order quantifiers.

The surprising way out is the insight that to allow the verifier to use an arbitrary ("any") strategy, amounts to making such player's moves completely free from restrictions in the sense of absence of all slashed independence requirements.

Such a liberalization is possible in an extended form of IF logic in which further independencies between quantifiers are allowed. Thus overcoming Frege's mistake opens wide new theoretical vistas. It turns out that at least for foundational purposes, higher-order logics as well as set theory are dispensable. For instance, we need not ask what the existence of higher-order entities means and how it can be established as a problem separate from existence *simpliciter*. A purely nominalistic theory of mathematics is in principle possible in the sense of Hilbert's dream that all mathematical reasoning can be understood in terms of structures of particular objects.

V. Revolution in Probability

The revolution described here is not restricted to pure logic and pure mathematics, but affects their applied uses as well. A fascinating perspective into those uses is offered by no lesser a figure than John von Neumann (1903-1957). Among his many feats, he largely created the widely used Hilbert space formalism for quantum theory, but he did not think that it was the right mathematics for quantum phenomena. This was not a casual opinion, either. von Neumann made a tremendous effort in trying to find a better one, among other things developing his extensive theory of operator algebras for the purpose. He even formulated guidelines as to what the better mathematics would have to be like. It would have to be so deeply different as to include a new "quantum logic". Its centerpiece seems to have been a new concept of probability, based directly on logic and facilitating among other things a general logical definition of entropy. Alas, von Neumann never found what he was seeking, and even the real significance of his search escaped most scientists. This story is told in the volume edited by [Rédei and Stöltzner (2001)].

Around 2005, Sandu and Hintikka discovered that IF logic comes with a completely explicit new concept of probability, actually in a most natural way. The crucial idea is to allow in the by this time well-known game-theoretical truth definitions the players to use mixed strategies. [See here Hintikka (2008) and Mann et al. (2011)]. The so-defined probability behaves rather like the "classical" one, but with apparently inconspicuous exceptions due to the failure of *tertium non datur* in IF logic.

Inconspicuous or not, this new IF probability calculus and its applications remain largely uninvestigated. It is nevertheless becoming patent that it opens extremely important doors for conceptualization and theorizing not only in logic and mathematics but in science. It is hard not to see it as an answer to von Neumann's question.

The way the connection was discovered is a telling illustration of the connection. von Neumann points out that in order for a logic space to match the structure of a physical space (a state space), it must have a counterpart to the geometrical notion of orthogonality. It is here that logicians became alerted of something significant when they noticed that the strong negation characteristic of IF logic behaves just like orthogonality.

Even though most detailed research remains to be done, there are convincing indications that the new logic and the new conception of probability can be as useful in applications of mathematics as in physical sciences. IF probability enables us to define a more general notion of entropy than the current ones. It is also suggestive that the notorious Bell's inequality does not hold in IF probabilities that are used instead of classical ones. It seems therefore that the problem of entanglement can be dealt with the help of IF probability.

VI. THE CRASH OF FIRST-ODER AXIOMATIC SET THEORY

Rightly understood, the way quantifiers operate logically includes representation of material dependencies between variables through the formal (syntactical) dependence relations between the quantifiers to which they are bound. By so doing, quantifiers can be involved in other kinds of dependence and independence relations. An important case is found in the logical theory of definitions. There it must be required that the definiens must be independent of the definiendum.

What does this requirement imply? In a case study, consider the axiom schema of comprehension in set theory. In a completely unrestricted version it has the form

$$(5)\ (\exists\, y)\ (\forall\, x)\ ((x \in y) \leftrightarrow (D\,[x])).$$

This can be looked upon as a definition of a set [y] combined with the assertion of its existence. In (5), the definiens $D\,[x]$ is a formula with x as its only free variable. In order to avoid circularity, y must not occur in $D\,[x]$. There can of course be quantifiers and connectives in $D\,[x]$.

In the usual accounts of definitions this is the whole story. But it should not be. For the definiens $D\,[x]$ to be independent of the definiendum, the quantifiers in it must obviously be independent of $(\exists\, y)$. A moment's thought shows that they must also be independent of $(\forall\, x)$ and that $(\forall\, x)$ must likewise be independent of $(\exists\, y)$.Yet in the usual discussion of definitions and definability, no attention has been paid to these independencies. The reason for this neglect is implicit in what has been said. Those independence relations are not expressible in RFO logic.

What horrible things can happen if these independencies are not heeded? A partial answer is that, among other mistakes, set-theoretical paradoxes can arise. And historically speaking they did arise. This is spelled out in my 2012 paper, "IF Logic, Definitions and the Vicious Circle Principle" [Hintikka (2012b)]. The paradoxes of set theory are consequences of the same mistake about quantifiers and dependence as Frege committed. This mistake was therefore the source of the *Grundlagenkrisis* of mathematics. In a bird's eye view on the history, logicians never found a way for set theorists to escape the problems and paradoxes of set-theoretical reasoning. What happened was that in the absence of a correct diagnosis of the problem logicians and mathematicians were forced to weaken their other presuppositions in set theory, in the first place to restrict the admissible choices for $D\,[x]$ in the comprehension schema (5). Unfortunately, the restrictions involved in the currently used axiomatizations of set theory do not eliminate the real source of trouble. One can literally say that for instance the Zermelo-Fraenkel system of set theory violates the rightly understood Vicious Circle Principle. In other words, set theorists could escape outright contradictions only by weakening their reasoning methods. This undermined the purpose of set theory. For instance, one of Zermelo's aims in his first axiomatizations of set theory was to vindicate the reasoning principle misleadingly known as the "axiom" of choice (AC), by incorporating it into an axiom system. (Zermelo did use AC in a crucial way in his proof of the well-ordering theorem.) It turns out, however, that the unrestricted form of the AC is incompatible with the other axioms of the Zermelo-Fraenkel set theory. The fact that AC is arguable a purely logical and even first-order principle makes the prospects of Zermelo-Fraenkel type set theories even dimmer.

Together with other reasons this shows that the current way of approaching set theory as a first-order axiom system is a dead end street. In my old slogan, axiomatic set theory has become a "Fraenkelstein's Monster" that destroys its own original purpose. This is argued in greater detail in my forthcoming paper, "Axiomatic Set Theory *in memoriam*" [Hintikka (forthcoming)]. And if the giving up of the current axiomatic set theories is not a scientific revolution, it is hard to imagine what could be.

VII. Axiom Systems are not "Logical Systems"

One reason for the wrong turn that set theory has taken can be seen in a confusion as to the nature of the axiomatic method. One common defense of set theory as it is currently practiced is that it can at the very least be cultivated as one possible mathematical theory in its own right. The conception of mathematical theorizing underlying this defense construes mathematical theories as logical systems. A logical system in its simplest form is supposed to consist of a finite number of axioms plus a finite set of rules of inference. In keeping with the idea (ideal) of symbolic logic, these rules of inference are purely formal. A mathematician's task is mainly to prove theorems from axioms by means of the given rules of inference.

In other words, mathematical theories are taken to be examples of logical systems. It is widely assumed, albeit in most cases tacitly, that applications of logic to mathematical or scientific theorizing should take the form of such a "logical system" or "formal system". This is among other things in keeping with the current dogma that the alpha and omega of mathematical practice are proofs of theorems. For another instance, different set theories are formulated as so many logical systems.

This conception of a formal system might look like a contemporary form of the idea of an axiomatic system. In reality, it is a distinctly different notion, and a dangerously misleading one.

According to the idea of a logical system the purpose of axiomatizations is interpreted as deductive systematization. Axioms are the ultimate first premises of all deductions of theorems. Theorems are simply different logical consequences of the axioms. By proving such theorems we extend our knowledge of the subject matter in question.

A more realistic view is found in that great theorist and practitioner of axiomatic method, David Hilbert. It is no accident that Hilbert paid special attention to the axiomatizations of scientific theories, even listing among his famous list of important open systematical problems number 6: Axiomatize physical (*sic*) theories.

Such an axiomatization can be characterized as studying a class of structures, for example electromagnetic systems, as models of a number of axioms. Then a scientist can study those structures by studying the axiom system purely logically and mathematically without having to acquire further empirical information. For instance, all electromagnetic systems are solutions of Maxwell's equation, in other words, models of the system of those equations.

This conception might at first seem to be essentially the same as the unfortunate idea of a "logical system". For instance, the requirement that the rules of inference of a logical system be purely formal seems nothing but an implementation of the obvious supposition of the success of an axiom system and presupposes that the logic used in the investigation of the models of an axiom system do not smuggle in new information.

The purely formal character of the derivation of theorems was indeed vigorously maintained by Hilbert. But what does "purely formal" mean? It means that the validity of a rule of inference depends only on its logical form. But the formal character of rules in this sense does not presuppose that in a given system we can get along with only a finite number of formal rules of inference. From results like Gödel's first incompleteness theorem it follows that such finitism is possible on only very poor mathematical theories. The cold truth is that "logical systems" do not cut much ice theoretically or by way of applications.

Rightly understood, what an axiom system is calculated to help us to master are not deduction relative between propositions, but certain structures, the models of the axioms. And they can be studied by means of logic even when the deductive relationships of propositions about them cannot be handled by a finite number of rules of inference. This shows clearly the real import of Gödel's first incompleteness theorem – and the limits of its theoretical significance.

Thus the characteristic activity of an axiomatizer is not to deduce theorems from axioms (and earlier theorems). Such deductions are in any case among the lesser accomplishments of an axiomatic theory in that they can only reveal what all the structures under scrutiny have in common. Among the uses of logical arguments in an axiomatic theory is to show what structures are possible according to the given theory, not just to show what all the models of the theory must be like.

It follows that the task of finding a proof for a theorem is not a matter of combining applications of a fixed number of rules in different formal ways with each other as in computation. In nontrivial cases, it involves in effect finding a new rule (or new rules) of inference.

A failure to see this point has quietly created a dangerously biased idea of what causes the differences in the difficulty of different mathematical and logical problems. To put the main point in simple terms, the mistake is to think of theoretical problems in mathematics and logic in the same way as computational problems. For instance, finding a proof for a hypothesis is according to this mistaken view like constructing an algorithm that produces in order the different steps of a formal proof.

As pretty much everybody knows by this time, from our collective experience of computing including the computer industry, to develop such algorithms ("software") requires in a typical nontrivial case major collective effort by a team of computer scientists working over long periods of time. If problems of theoretical mathematics are essentially computational, the same will hold of them. If so, there is little realistic hope that an individual mathematician could solve any major problem on her or his own, except perhaps by decade-long effort. The history of such famous problems as the four-color problem seems to feed such a pessimistic view.

Concrete evidence is hard to come by, but anecdotal evidence suggests that this defeatist view has taken over to an extraordinary degree. A famous mathematical logician who is also the president of the mathematical society of his important metropolitan area was asked recently what would happen if his society announced a meeting where a guest lecturer would present a solution to the P vs. NP problem, one of the Clay Foundation's millennium problems. The unhesitating answer was, "Nobody would come." Nobody would as much as entertain the possibility that the visitor might have a solution enough to be curious.

As a consequence, logicians and mathematicians have largely lost the sense of important common enterprise when it comes to mathematical theorizing and problem solving. This threatens to undermine the work ethic in mathematical sciences. While acknowledging the complexity and elusiveness of human motivation, it seems to me unmistakable that this lack of commitment on the part of members of the mathematical community is what drove Grigori Perelman (who solved the Poincaré problem, the first millennium problem to be solved) to his self-imposed intellectual exile. [See here Gessen (2009)].

This loss of courage on the part of logicians and mathematicians is shown to be premature by the scientific revolution here discussed. Admittedly, some famous classical problems have been studied so much that the solutions to those remaining open are likely to be hugely labor-intensive. But we can now see that in some cases at least difficulties have been partly self-inflicted. For one instance, the continuum hypothesis has been approached axiomatically as a question of whether the hypothesis is provable in the Zermelo-Fraenkel set theory with the help of suitable further axioms. What was seen in sec. VI above shows that such a proof would not even be relevant to the real continuum problem. And even the deductive problem has been approached in a hapless manner. How can anyone realistically expect success when (as in Zermelo-Fraenkel system) you do not even have the full force of the "axiom" of choice available to you?

Thus the long-term impact of the scientific revolution discussed here is not only negative. On the contrary, it opens tremendous opportunities for progress. There already exists a model-theoretical proof of the continuum hypothesis on the net, although not in print.

A crucial step in unblocking these constructive possibilities would be to abandon the disastrous idea of a "logical system" as the standard format of a mathematical or logical theory.

VIII. Logic and Computability

The promise of progress applies also to the areas where logic and computer science overlap. An overview is currently difficult to reach, but some of the main features of the situation are reasonable clear [see Hintikka (2011)]. At first sight, a logical theory and a theory of computability should merge with each other seamlessly, via a formalized logical theory of arithmetical computation. In order to calculate the value of $f(x)$ for $(x = b)$, it should be necessary and sufficient to prove formally an equation of the form f (b) = a. The computation proceeds largely from equations involving auxiliary functions while conventional logical deduction uses formulas with quantifiers.

The two are not quite equivalent, however. The auxiliary functions are in effect the choice functions of the corresponding quantificational formulas. Skolem functions are the truthmakers of quantified sentences. They provide the "witness individuals" which would show its truth. For instance a Skolem function of $(\forall x)(\exists y) F[x, y]$ satisfies $(\forall x) F[x, f(x)]$. They form the underlying structure of the logic of quantification, so much so that a quantificational sentence is true if and only if there exists a full set of "truthmaking" (i.e. Skolem) functions for it. These choice functions were mentioned in sec. II above. They are the Skolem functions (named after the Norwegian mathematician Thoralf Skolem, 1887-1963) mentioned earlier. This example shows how Skolem functions and existential quantifiers do the same job in their two different ways.

Now the fact is that if RFO logic is used, there are not always sets of Skolem functions available for the purpose. This is because a Skolem function comes from an existential quantifier in the labeled tree that a formula is while its arguments come from terms occurring earlier in the same branch. This means that the argument sets of Skolem functions must have a related tree structure too. Not all sets of auxiliary functions for an algorithm nevertheless have such a structure. Hence arbitrary algorithms cannot be captured as deductions in the sense of RFO logic.

However, in IF logic this restriction is removed. By using IF logic, we can therefore build a general theory of numerical computability in parallel with deductive first-order IF logic. This among other things allows us to use all the resources of logical proof theory in the theory of computability.

The opportunities in this direction largely remain unexploited [see nevertheless my paper "Function Logic and the Theory of Computability", Hintikka (2013a)]. In a slightly different direction, attention to the independence relations computable in IF logic helps to correct important mistakes in the current literature. An example is offered by the interesting theorem of Paris and Harrington (1977). When Gödel published his first incompleteness result, mathematicians, philosophers, logicians and questioning minds in general were puzzled. What are Gödel's strange creatures like, arithmetical propositions that are true, but not provable? Gödel's own examples seemed artificial and uninformative. It created therefore a great deal of interest when Paris and Harrington published a modified version of a well-known finite Ramsey theorem in combinatorial mathematics, a theorem that could only be proved by an infinitistic argument.

This theorem has been taken to exemplify (and hopefully to illuminate) the incompleteness phenomenon that Gödel was exploring. In my paper, "The modified Ramsey theorem is not a Gödel sentence" [Hintikka (2013b)], I show that the Paris-Harrington theorem is not representable in the language Gödel used and hence not an instance of a Gödel sentence.

Indirectly, this mistake nevertheless illustrates the nature of Gödelian incompleteness. What Gödel brings out is that the set of true arithmetical sentences is not recursively enumerable. No computer can be programmed to list all and only arithmetical truths. Now an instance of Gödel sentences would be relative to a particular axiomatization, in other words, to one particular attempt to enumerate recursively truths and only truths. As such, it would help to show only why it is that that particular enumeration fails. It is unlikely to yield any insight into why any such attempted enumeration must fail.

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