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An iterative algorithm for fuzzy mixed production planning based on the cumulative membership function

Algoritmo iterativo para la planeación de la producción mixta basado en la función acumulativa de pertenencia

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Abstract

This paper shows an application of a novel algorithm for Fuzzy Linear Programming (FLP) problems with both fuzzy technological coefficients and constraints, which deals with any kind of fuzzy membership functions for technological parameters and fuzzy linear constraints.

The presented approach uses an iterative algorithm which finds stable solutions to problems with fuzzy parameter sinboth sides of an FLP problem. The algorithm is based on the soft constraints method proposed by Zimmermann combined with an iterative procedure which gets a single optimal solution.

Key words: Fuzzy linear programming, Cumulative membership function, Production planning.

Resumen

Este artículo presenta una aplicación de un algoritmo nuevo para problemas de programación lineal difusa (FLP) con restricciones y coeficientes difusos, con restricciones difusas lineales y coeficientes tecnológicos difusos con funciones de pertenencia no-lineales.

El modelo propuesto usa un método iterativo que encuentra una solución estable aproblems con parámetros difusos en ambos lados de las restricciones de un problema de programación lineal. El algoritmo se basa en el método de restricciones suaves propuesto por Zimmermann, combinado con una rutina iterativa que llega a soluciones óptmas únicas.

Palabras claves: Programación Lineal Difusa, Función Acumulativa de Pertenencia, Planeación de la Producción

1. Introduction and motivation

As fuzzy sets were defined by Zadehin middle 60s, Fuzzy Linear Programming (FLP) models were considered as a natural application of fuzzy-based decision making methods. Different issues can be addressed with FLP, but its resolution methods vary depending on the

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complexity of the model, the membership functions used for their parameters, the quality of solutions desired by the analyst, etc.

There is no single method to find an optimal crisp solution for FLP problems. Therefore, the use of alternative methods arises as an option for decision makers who want to get optimal solutions to uncertain problems. The recent use of the Cumulative Membership Function (CMF) in optimization allows us to find solutions to Linear Programming (LP) problems with fuzzy righth and side parameters to gether with fuzzy left hand side parameters.

Similar problems have been addres sed by Ghodousiana and Khorram [10], Sy-Min and Yan-Kuen [21], Tanaka and Asai [22], Tanaka, Okudaand Asai [23], Inuiguchi [12], [11] and [13]. All these methods propose solutions for several linear fuzzy membership functions. Some conceptual papers about fuzzy and possibilistic optimization are given by Lodwick and Bachman [16], Mahdavi-Amiri and Nasseri [17]. Kearfott and Kreinovich [14] proposed the use of possibilistic measuresinlinear optimization. All these approaches solve some FLP problems, but they assume some initial conditions of the problem. Like wise, some fuzzy production planning problems have been addressed by[19], [20], [18], [15], [2], [1], and [9].

Figueroa & López [6], [7], and Figueroa [5], [8] defined the CMF of a fuzzy set to deal with non linear fuzzy left hand side parameters and linear right hand side parameters of an LP problem. Here we report a method to solve production planning problems under fuzzy uncertainty.

This paper is divided into five principal sections. In Section 1 the Introduction and Motivation is presented. In Section 2 the Fuzzy Mixed Production Planning problem (FMPP) is described. In Section 3, the CMF concept and an iterative algorithm for solving this problem are presented. In Section 4, an application example is solved, and finally in Section 5 some concluding remarks are suggested.

1.1 A discussion

Uncertainty in production planning appears in different ways, and the analyst must deal with uncertainty in the production rates, standard unitary manufacturing times, demands, availability of some resources, etc. The main question is: How to make a decision when both left and right parameters of a constraint have uncertainty?

This question appears in different scenarios: when the analyst have to define standard processing times, measure the capabilities of the system or fore cast the potential demands. Hence, the analyst should take a decision about how to handle uncertainty in different parameters of the problem, so the use of fuzzy sets for representing uncertainty is an available choice. Here, we use an algorithm which deals with fuzzy parameters on both sides of a constraint at once. This algorithm uses a single global defuzzification degree for all fuzzy parameters, which seems to be a simple and effective way to find a global solution of the problem.

The analyst could use different defuzzification degrees for solving the problem, but a simpler approach consists in finding a single degree which can be easily interpreted and applied, instead of multiple degrees which can lead to inappropriate solutions, or a complex interpretation.

2. An FLP model for mixed production planning

Mixed production planning (MPP) tries to find optimal production quantities of products "j" at different times "k". LP models are efficient at solving the problem in terms of the capacities of the system and the demand of each product. Time units are widely used to express the capacities of the system. However, different units can be used to measure capacities such as energy, materials, space, money units, etc. All of them are included in a model where these capacities and demands are expressed as constraints of an LP model whose solution is the optimal mix of products regarding an objective function.

A multiple decision criteria model is defined depending on the go al of the system, with different objective functions $f(x_{jk}(\cdot))$, where $x_{jk}(\cdot)$ is the set of all decision variables which compose the goal. So the crisp MPP model is defined asfollows:

$$Opt \; f(x_{jk}(\cdot)) \tag{1}$$

$$s.t.$$

$$\sum_{j=1}^{n} ts_{ijk}\, x^r_{jk} \leqslant Ac^r_{ik} \quad \forall \; i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{2}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\, Op_i\, x^r_{jk} \leqslant Wc^r_k \quad \forall \; k \in \mathbb{N}_K \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} e_{jkl}\, [x^r_{jk} + x^o_{jk}] \leqslant Ae_{kl} \quad \forall \; k \in \mathbb{N}_K; l \in \mathbb{N}_L \tag{4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} rm_{jkr}\, [x^r_{jk} + x^o_{jk}] \leqslant Am_{kr} \;\; \forall \; k \in \mathbb{N}_K; r \in \mathbb{N}_R \tag{5}$$

$$\sum_{j=1}^{n} ts_{ijk}\, x^o_{jk} \leqslant Ac^o_{ik} \quad \forall \; i \in \mathbb{N}_m; k \in \mathbb{N}_K \tag{6}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ts_{ijk}\, Op_i\, x^o_{jk} \leqslant Wc^o_k \quad \forall \; k \in \mathbb{N}_K \tag{7}$$

$$\sum_{j=1}^{n} a_j\, q_{jk} \leqslant A_k \quad \forall \; k \in \mathbb{N}_K \tag{8}$$

$$x^s_{jk} \leqslant As_{jk} \quad \forall \; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{9}$$

$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} \leqslant d^{(+)}_{jk} \;\; \forall \; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{10}$$

$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} \geqslant d^{(-)}_{jk} \;\; \forall \; j \in \mathbb{N}_n; k \in \mathbb{N}_K \tag{11}$$

$$x^r_{jk}; x^o_{jk}; x^s_{jk}; q_{jk}, s_{jk} \geqslant 0 \tag{12}$$

Index sets:

$\mathbb{N}_m$ is the set of all "i" resources, $i \in \mathbb{N}_m$, $\mathbb{N}_m = 1, 2, \cdots, m$.
$\mathbb{N}_n$ is the set of all "j" products, $j \in \mathbb{N}_n$, $\mathbb{N}_n = 1, 2, \cdots, n$.
$\mathbb{N}_K$ is the set of all "k" periods, $k \in \mathbb{N}_K$, $\mathbb{N}_K = 1, 2, \cdots, K$.
$\mathbb{N}_L$ is the set of all "l" energy units, $l \in \mathbb{N}_L$, $\mathbb{N}_L = 1, 2, \cdots, L$.
$\mathbb{N}_R$ is the set of all "r" raw materials, $r \in \mathbb{N}_R$, $\mathbb{N}_R = 1, 2, \cdots, R$.

Decision variables:

x^r_{jk} : Quantity of product "j" to be manufactured in regular time per period "k".
x^o_{jk} : Quantity of product "j" to be manufactured in overtime per period "k".
x^s_{jk} : Quantity of product "j" to be manufactured by outsourcing per period "k".
q_{jk} : Quantity of inventory of product "j" to be held per period "k".
s_{jk} : Quantity of shortage of product "j" to be supported per period "k".

Parameters:

ts_{ijk} : Unitary standard production time that product "j" uses from the "i" resource per period "k".
Op_i : Amount of workers needed to operate the resource "i".
e_{jkl} : Amount of the "l" energy units used to manufacture product "j", per period "k".
rm_{jkl} : Amount of the "r" raw material units used to manufacture product "j", per period "k".
a_j : Amount of space used to hold a unit of product "j".
$d^{(-)}_{jk}$: Minimum demand of product "j" per period "k".
$d^{(+)}_{jk}$: Maximum demand of product "j" per period "k".
Ac^r_{ik} : Available capacity on regular time of resource "i" per period "k", expressed in hours.
Wc^r_k : Available capacity on regular time of workforce per period "k", expressed in hours.
Ac^o_{ik} : Available capacity on overtime of resource "i" per period "k", expressed in hours.
Wc^o_k : Available capacity on overtime of workforce per period "k", expressed in hours.
Ae_{kl} : Availability of the energy type "l" per period "k", expressed in energy units.
Am_{kr} : Availability of the raw material type "r" per period "k", expressed as units as Kg, Lt, etc.
As_{jk} : Availability of outsourced product "j" per period "k".
A_k : Available space units per period "k".

In the MPP, all parameters are crisp numbers, so its solution has no uncertainty. On the other hand, all production times, its constraints, and the demands of the MPP can be defined by fuzzy numbers, so the MPP becomes a more complex problem since both sides of its constraints has fuzzy numbers. From now, we state the equations (4), (5), (9), (8) as a set of crisp constraintsdenotedby a($x_{jk}(\cdot)$) ≤ $b(\cdot)$ where $(\cdot)$ involves all variables and index sets from (2) to (12).

2.1 MPP as a multiple criteria problem

The MPP model can use multiple decision criteria affecting the fuzzy model since it changes the shape of the fuzzy set of optimal decisions. These decision criteria are functions of $x_{jk}(\cdot)$, thatis f ($x_{jk}(\cdot)$). In other words, (1) can be replaced by any of the following equations:

Cost minimization:

$$\min f(x^r_{jk}, x^o_{jk}, x^s_{jk}; Cp^r_{jk}, Cp^o_{jk}, Cp^s_{jk}) = \sum_{j=1}^{n}\sum_{k=1}^{K} Cp^r_{jk}\, x^r_{jk} + Cp^o_{jk}\, x^o_{jk} + Cp^s_{jk}\, x^s_{jk} + h_{jk}\, q_{jk} + o_{jk}\, s_{jk} \tag{13}$$

where Cp^r_{jk} ,Cp^o_{jk} and Cp^s_{jk} are the unitary production cost for Normal time, the Overtime and the Outsourced product "j" per period "k" respectively, h_{jk} is the unitary holding cost of product "j" per period "k", and o_{jk} is the unitary opportunity cost of product "j" per period "k".

Profit maximization:

$$\max f(x^r_{jk}, x^o_{jk}, x^s_{jk}; Sp_{jk}) = \sum_{j=1}^{n}\sum_{k=1}^{K} Sp_{jk}\, [x^r_{jk} + x^o_{jk} + x^s_{jk}] \tag{14}$$

where Sp_{jk} is the unitary sell price of product "j" per period "k".

Utility maximization:

$$\max f(x^r_{jk}, x^o_{jk}, x^s_{jk}; Cp^r_{jk}, Cp^o_{jk}, Cp^s_{jk}, Sp_{jk}) = \sum_{j=1}^{n}\sum_{k=1}^{K} Sp_{jk}\,[x^r_{jk} + x^o_{jk} + x^s_{jk}] - [Cp^r_{jk}\,x^r_{jk} + Cp^o_{jk}\,x^o_{jk} + Cp^s_{jk}\,x^s_{jk} + h_{jk}\,q_{jk} + o_{jk}\,s_{jk}] \quad (15)$$

where Cp^r_{jk}, Cp^o_{jk} and Cp^s_{jk} represent the unitary production cost for the Regular time, the Overtime and the Outsourced product and Sp_{jk} is the unitary sell price of product "j" per period "k".

Resources utilization maximization:

$$\max f(x^r_{jk}, x^o_{jk}, x^s_{jk}; ts_{ijk}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} ts_{ijk}\,[x^r_{jk} + x^o_{jk}] \quad (16)$$

Workforce utilization maximization:

$$\max f(x^r_{jk}, x^o_{jk}, x^s_{jk}; ts_{ijk}, Op_i) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} ts_{ijk}\,Op_i\,[x^r_{jk} + x^o_{jk}] \quad (17)$$

Now, if we consider ts_{ijk}, Ac^r_{ik}, Wc^r_k, Ac^o_{ik}, Wc^r_k, $d^{(+)}_{jk}$ and $d^{(-)}_{jk}$ as fuzzy numbers, then the model be comes non linear, so its solution can not be found directly by convex algorithms.

2.2 The fuzzy MPP (FMPP) model

In this fuzzified version of the MPP model, ts_{ijk}, $d^{(+)}_{jk}$, $d^{(-)}_{jk}$, Ac^r_{ik}, Wc^r_k, Ac^o_{ik}, and Wc^o_k are considered as fuzzy sets, namely $\tilde{ts}_{ijk}$, $\tilde{d}^{(+)}_{jk}$, $\tilde{d}^{(-)}_{jk}$, $\tilde{A}c^r_{ik}$, $\tilde{W}c^r_k$, $\tilde{A}c^o_{ik}$, and $\tilde{W}c^o_k$ respectively. Now, the general model for this version called FMPP, is as follows.

$$Opt\ \ f(x_{jk}(\cdot)) \quad (18)$$
$$s.t.$$
$$a(x_{jk}(\cdot)) \leqslant b(\cdot) \quad (19)$$
$$\sum_{j=1}^{n} \tilde{t}s_{ijk}\,x^r_{jk} \leqslant \tilde{A}c^r_{ik} \quad \forall\ i \in \mathbb{N}_m; k \in \mathbb{N}_K \quad (20)$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{t}s_{ijk}\,Op_i\,x^r_{jk} \leqslant \tilde{W}c^r_k \quad \forall\ k \in \mathbb{N}_K \quad (21)$$
$$\sum_{j=1}^{n} \tilde{t}s_{ijk}\,x^o_{jk} \leqslant \tilde{A}c^o_{ik} \quad \forall\ i \in \mathbb{N}_m; k \in \mathbb{N}_K \quad (22)$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{t}s_{ijk}\,Op_i\,x^o_{jk} \leqslant \tilde{W}c^o_k \quad \forall\ k \in \mathbb{N}_K \quad (23)$$
$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} \leqslant \tilde{d}^{(+)}_{jk} \quad \forall\ j \in \mathbb{N}_n; k \in \mathbb{N}_K \quad (24)$$
$$x^r_{jk} + x^o_{jk} + x^s_{jk} + q_{j,k-1} + s_{jk} - q_{jk} - s_{j,k-1} \geqslant \tilde{d}^{(-)}_{jk} \quad \forall\ j \in \mathbb{N}_n; k \in \mathbb{N}_K \quad (25)$$
$$x^r_{jk}; x^o_{jk}; x^s_{jk}; q_{jk}, s_{jk} \geqslant 0 \quad (26)$$

where $\tilde{ts}_{ijk}$ are LR fuzzy sets, each $\tilde{d}^{(-)}_{jk}$ is an L fuzzy set and $\tilde{d}^{(+)}_{jk}$, $\tilde{Ac}^{r}_{ik}$, $\tilde{Wc}^{r}_{k}$, $\tilde{Ac}^{o}_{ik}$, and $\tilde{Wc}^{o}_{k}$ are R fuzzy sets, and all remaining parameters still the same.

Uncertainty can appear in different ways, so the analyst can reduce its impact in some cases, or simply he should deal with. In this approach $\tilde{d}^{(+)}_{jk}$, $\tilde{d}^{(-)}_{jk}$, $\tilde{Ac}^{r}_{ik}$, $\tilde{Wc}^{r}_{k}$, $\tilde{Ac}^{o}_{ik}$, and $\tilde{Wc}^{o}_{k}$ are defined as linear fuzzy sets with parameters $\check{d}^{(+)}_{jk}$, $\check{d}^{(-)}_{jk}$, $\check{Ac}^{r}_{ik}$, $\check{Wc}^{r}_{k}$, $\check{Ac}^{o}_{ik}$, $\check{Wc}^{o}_{k}$, $\hat{d}^{(+)}_{jk}$, $\hat{d}^{(-)}_{jk}$, $\hat{Ac}^{r}_{ik}$, $\hat{Wc}^{r}_{k}$, $\hat{Ac}^{o}_{ik}$, $\hat{Wc}^{o}_{k}$, and $\tilde{ts}_{ijk}$ can take any shape.

3. Solving the FMPP

Figueroa & López [6], [7], and Figueroa [5], [8] proposed the use of the CMF to solve fuzzy mathematical programming models with fuzzy joint parameters, so we applied their results to the FMPP. Abrief explanation of the CMF and the algorithm used here is shown as follows.

3.1 The Cumulative Membership function (Figueroa [6])

Definition 3.1 (CMF function) *Consider a fuzzy number defined on a set $\tilde{A}$ with known bounds a and b and a centralinterval $c \in [c_1, c_2]$ contained into the power set $\mathbb{S}$ which satisfies $\mu_{\tilde{A}}(c)=1 \forall c$. Using the l(x),c(x),r(x) decomposition of a membership function, the cumulative membership function (CMF)[1] is defined as:*

$$\psi_{\tilde{A}}(x) = Ps_{\tilde{A}}(X \leqslant x) \tag{27}$$

$$= \oint_{-\infty}^{x} \mu_{\tilde{A}}(t)\, dt \tag{28}$$

Orin a l (x), c (x), r (x) decomposition:

$$\psi_{\tilde{A}}(x) = \oint_{-\infty}^{x} l(t)\, dt\,,\ -\infty \leqslant t \leqslant x\ ;\ x \in [-\infty, c_1] \tag{29}$$

$$\psi_{\tilde{A}}(x) = \oint_{-\infty}^{c_1} l(t)\, dt + \oint_{c_1}^{c_2} c(t)\, dt + \oint_{c_2}^{x} r(t)\, dt, \quad c_2 \leqslant t \leqslant x\ ;\ x \in [c_2, \infty] \tag{30}$$

This definition represents the possibility that the elements X of the support of a subset $A \in \mathbb{S}$ fall into $X \leq x$, $Ps_{\tilde{A}}(X \leq x)$. Note that in the probabilistic case F $(\infty)=1$ while in the possibilistic case $1 < \psi_{\tilde{A}}(\infty) < \wedge$, where $\wedge$ is a finite value defined itself as the *Total Area* of $\mu_{\tilde{A}}(x)$.

$$\Lambda = \oint_{-\infty}^{\infty} \mu_{\tilde{A}}(t)\, dt \tag{31}$$

As always when $\mu_{\tilde{A}}(x)$ has a closed form to compute its integral, it is easy to obtain $\psi_{\tilde{A}}(x)$ exactly, while if it has no closed form, $\psi_{\tilde{A}}(x)$ is only obtainable by numeric methods.

[1] Here *t* is a dummy variable used for integration purposes.

Note that $\mu_{\tilde{A}}(\infty) > 1$ is an important issue to be solved. This is an interpretation problem since the definition of a *normalized* fuzzy set determines that sup $\mu_{\tilde{A}}(x) = 1$ and the above definition does not have this property.

An easy way to normalize $\psi_{A}(x)$ is dividing it by $\wedge$, as shown as follows:

$$\psi_{\tilde{A}}(x) = \frac{1}{\Lambda} \oint_{-\infty}^{x} \mu_{\tilde{A}}(t)\, dt \tag{32}$$

where $\psi_{\tilde{A}}(-\infty)=0$ and $\psi_{\tilde{A}}(\infty)=1$. The set $\psi_{\tilde{A}}$ coming from a triangular fuzzy set is depicted in Figure 1.

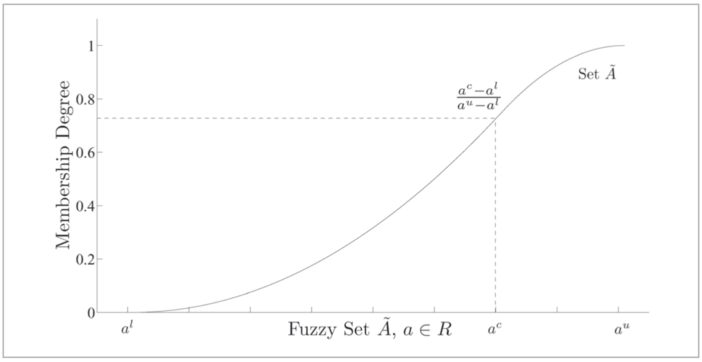

Figure 1. CMF For a Triangular Fuzzy Set, $\psi_{\tilde{A}}$

The CMF projects a fuzzy set into a monotonically function which returns simpler α-*cuts*[2]. This means that any LR fuzzy set which has at least two bounds per each α-*cut* can be transformed into a CMF with α-*cuts* that always has only one either left or right boundary as shown in Figure 1. Then, the CMF allows us to avoid the use of expected value, chance constrained or binary decision based methods for handling this problem.

3.2 The soft constraints method

A classical approch for solving FLP with linear fuzzy constraints($Ax \geq \tilde{b}$)[3] in the form $\min_{x \in \mathbb{R}} \{z = c'x \mid Ax \geq \tilde{b}\}$ was proposed by Zimmermann [24], [25]. This algorithm computes a fuzzy setof solutions $\tilde{z}(x^*)^{x \in \mathbb{R}}$, finding a maximal joint α-*cut* for $\tilde{z}(x^*)$ and $\tilde{b}$, as shown as follows:

1. Calculate an inferior bound called *z minimum* ($\check{z}$)by solving an LP model with $\check{b}$
2. Calculate a superior bound called *z maximum* ($\hat{z}$)by solving an LP model with $\hat{b}$
3. Define a fuzzy set $\tilde{z}(x^*)$ with bounds $\check{z}$ and $\hat{z}$ and linear membership function. This is the degree that any feasible solution has regarding the optimization objective.
4. If the objective is to maximize, then its membership function is:

2 An α-*cut* allows us to find operation point sinterms of *X* which are required by LP methods, as function of its membership degree which is enclosed into 0≤1. In other words, we can easily decompose any fuzzy set in to its α-*cuts*.

3 A denotes the crisp matrix of technological coefficients known as the *Left Hand Side* parameters of the constraints of an LP model.

$$\mu_{\tilde{z}}(x;\check{z},\hat{z}) = \begin{cases} 1, & c'x \geqslant \hat{z} \\ \dfrac{c'x-\check{z}}{\hat{z}-\check{z}}, & \check{z} \leqslant c'x \leqslant \hat{z} \\ 0, & c'x \leqslant \check{z} \end{cases} \tag{33}$$

5. Thus, solve the following LP model

$$\begin{aligned} &\max\ \{\alpha\} \\ &s.t. \\ c'x + c_0 - \alpha(\hat{z}-\check{z}) &= \check{z} \\ Ax + \alpha(\hat{b}-\check{b}) &\leqslant \hat{b} \\ x &\geqslant 0 \end{aligned} \tag{34}$$

where α is the overall satisfaction degree of all fuzzy sets, $\check{b}$ and $\hat{b}$, $\check{b} \leq \hat{b}$ are parameters of $\tilde{b}$, $(\check{b}, \hat{b} \in \mathbb{R})$.

3.3 Iterative algorithm

Figueroa [5], [8] proposed an iterative method which obtains an optimal solution by iterative computations of an initial α-*cut* for A which is updated by the optimal α-*cut* obtained by the classical FLP model, until α and λ^* levels converge to an stable value, as summarized as follows.

Algorithm 1 Iterative FLP algorithm.

```
Set α, n = 1
Compute ψ_{Ã_ij} For each Ã_ij fuzzy set.
Initialize with λ*_0 = α.
for λ*_n ≠ λ*_{n-1}, n > 0 do
   return ^α ψ_{Ã_ij} For each ψ_{Ã_ij} set.
   Solve the FLP model presented in Section 3.2.
   return λ*_n
   if λ*_n ≠ λ*_{n-1} then
      Set n = n + 1 and λ*_n = α
   else
      Stop the algorithm.
   end if
end for
return n, λ*_n, ^α ψ_{Ã_ij}, x*_j, and z*
```

The main idea here is to apply the Zimmermann's method together with ${}^{\alpha}\psi_{\tilde{A}ij}$ to the left side parameters $\tilde{A}_{ij}$, obtaining its projections over $x \in \mathbb{R}$ Then, the algorithm tries to find an α-*cut* which is both a satisfaction degree of each constraint and a defuzzification degree of each parameter of its left side.

Therefore, α-*cut* is an equilibrium degree between the operation point the systems hould use ${}^{\alpha}\psi_{\tilde{A}ij}$ and theamount of resources the analyst should plan ${}^{\alpha}\mu_{\tilde{b}i}$ to get a global optimal solution, using a single defuzzification degree.

Algorithm 1 deals with any nonlinear fuzzy set for A, finding a solution. Although it converges to stable solutions, Figueroa [5] identified a special case where this algorithm can become cyclical but stable. This means that λ^*_n can cycle, but in a stable way.

Application example

We introduce an example to show how the model and algorithm work together. The example is about 4 products manufactured in 3 stations in 4 periods, based on crisp model shown in Section 3.2 and solved by the Algorithm 1. A production, inventory and backorder strategy is applied to a utility maximization objective. For simplicity, we only consider resources capacity and constrained demands, using the parameters shown in Tables I, II and III.

Table I. Fuzzy capacities AC_{ik}

i,k	$\check{A}c^r_{ik}$	$\hat{A}c^r_{ik}$	$^{\alpha}\tilde{A}c^r_{ik}$
1,1	7700	8300	8008.9
1,2	5500	6400	5963.4
1,3	5200	5800	5508.9
1,4	6900	7800	7363.4
2,1	3200	4100	3663.4
2,2	3100	3800	3460.4
2,3	3600	4000	3806.0
2,4	2100	3000	2563.4
3,1	1800	2400	2108.9
3,2	1900	2100	2003.0
3,3	2200	3200	2714.9
3,4	2600	3100	2857.5

Table II. Unitary Standard Production Time, ts_{ijk}

$ts_{ijk}\ \forall\ k=1,2,3,4$				
i,j	$\tilde{ts}_{i1k}$	$\tilde{ts}_{i2k}$	$\tilde{ts}_{i3k}$	$\tilde{ts}_{i4k}$
$\tilde{ts}_{1jk}$	G(0.15,0.05)	T(0.2,0.25,0.5	T(0.1,0.25,0.45)	T(0.05,0.15,04)
$\tilde{ts}_{2jk}$	T(0.25,0.1,0.4)	G(0.1,0.02)	G(0.12,0.04)	G(0.09,0.01)
$\tilde{ts}_{3jk}$	G(0.12,0.03)	T(0.15,0.28,0.55)	T(0.05,0.21,0.45)	G(0.34,0.1)

Table III. Production, inventory, backorder costs and fuzzy Demands

Unitary Costs				Fuzzy demands			
j,k	Cp^r_{jk}	Sp_{jk}	h_{jk}	$\check{d}^{(-)}_{jk}$	$\hat{d}^{(-)}_{jk}$	$\check{d}^{(+)}_{jk}$	$\hat{d}^{(+)}_{jk}$
1,1	225	350	12	3700	3900	9249	9723
1,2	165	300	14	3200	3400	8128	8606
1,3	160	280	13	2400	2700	6834	7593
1,4	105	220	15	3400	3500	8107	8339
2,1	205	320	12	2650	2800	6657	7014
2,2	175	300	14	3100	3400	8390	9130
2,3	160	270	13	1750	1900	4641	5007
2,4	225	330	15	2600	2800	6785	7270
3,1	53	190	12	1950	2100	5088	5451
3,2	74	220	14	3700	3900	9249	9723
3,3	103	250	13	2450	2600	6208	6566
3,4	138	320	15	4000	4100	9456	9687
4,1	160	350	12	2650	2800	6657	7014
4,2	90	270	14	3300	3400	7882	8114
4,3	65	210	13	1900	2100	5222	5719
4,4	50	240	15	2400	2500	5859	6093

In Table II we denote *T*(*a,b,c*) as a triangular fuzzy number with parameters *a,b* and *c*, $a<b<c$, and *G* (*a,b*) is agaussian fuzzy number with mean a and spread *b*. $\tilde{Ac}^{r}_{ik}$, $\tilde{d}^{(-)}_{jk}$ and $\tilde{d}^{(+)}_{jk}$ are R fuzzy sets with linear membership function as the Algorithm 3.2 states (See Zimmermann [24]).

Remark 4.1 *There is an infinite amount of predefuzzified values of $\tilde{ts}_{ijk}$ that the analyst can choose. Note that as ts_{ijk} is small, then the system can produce more units using the same resources. This leads us to think that the maximum utility of the system is found when using $\min_{ts\in\mathbb{R}} ts_{ijk}$, but it could be an unfeasible solution in practice.*

In this way, the analyst could select a value of $\tilde{ts}_{ijk}$ which reaches an equilibrium between the use of resources and the desired objective, not only an unreachable value which at a first glance leads to a higher optimum.

4.1 Results of the proposed algorithm

Solving the FMPP using the Algorithm 1, we found that $\check{z} = 6'105.244$ is obtained using $\check{d}^{(+)}_{jk}$ and $\hat{z} = 8'710.403$ is obtained through $\hat{d}^{(+)}_{jk}$. The optimal uncertainty degree is $\alpha^*_n = 0.5149$ and the final maximal utility of the system afteris $z^* = 7'446.640, 4$, which is depicted in Figure 2. A detailed report of the solution is shown in Table V.

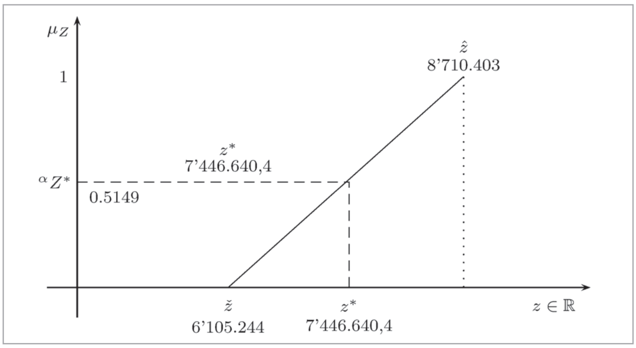

Figure 2. Fuzzy optimal set $\tilde{z}(x^*)$, and the defuzzified solution z^*

The behavior of λ^*_n is shown in Table IV. Note that λ^*_n goes to a single value which is a global defuzzification degree itself. Thus, λ^* is the point where all fuzzy parameters get a global optimal solution projected into $\tilde{Z}$.

The algorithm was started with $\alpha^*_n = 0.5$ and it found a stable value of $\lambda = 0.5149$. So $\tilde{ts}_{ijk}$, $\tilde{d}^{(-)}_{jk}$, $\tilde{d}^{(+)}_{jk}$, and $\tilde{Ac}^{r}_{ik}$, and should be defuzzified at this value. A detailed report of the final solution is shown in Tables V and VI. The method was implemented in MatLab using an Intel i3 processor, 6 Gb RAM and 500 HDD Windows-based machine; all computations were obtained in about 6 seconds.

Table IV. Summary report of the behavior of the algorithm

Iteration	$\check{z}$	$\hat{z}$	λ^*_n	z^*
1	6282371	8863725	0.5126	7605573
2	6133152	8733863	0.5145	7471217
3	6110093	8714477	0.5148231	7450890
4	6106175	8711185	0.5148852	7447456
5	6105424	8710554	0.514897	7446797
6	6105280	8710433	0.5148994	7446671
7	6105249	8710408	0.5148998	7446644
8	6105246	8710405	0.5148999	7446642
9	6105244	8710403	0.5149	7446640

Table V. Summary report of the model

j,k	x^{r*}_{jk}	I^*_{jk}	S^*_{jk}	$^{\alpha}\tilde{d}^{(-)}_{jk}$	$^{\alpha}\tilde{d}^{(+)}_{jk}$
1,1	9478.94	0	0	3797	9493.06
1,2	7967.08	4664.1	0	3297	8374.12
1,3	2554.47	0	0	2545.5	7224.81
1,4	3451.49	0	0	3448.5	8226.46
2,1	5828.03	0	0	2722.8	6840.82
2,2	2393.35	3802.98	0	3245.5	8771.03
2,3	4423.73	2596.49	0	1822.8	4829.45
2,4	2702.98	0	0	2697	7034.73
3,1	2027.24	0	0	2022.8	5274.91
3,2	0	0	0	3797	9493.06
3,3	1933.72	2002.98	0	2522.8	6392.33
3,4	4051.49	0	0	4048.5	9574.94
4,1	3193.68	0	0	2722.8	6840.82
4,2	3351.49	0	0	3348.5	8001.46
4,3	0	0	0	1997	5477.91
4,4	2451.49	0	0	2448.5	5979.49

Table VI. Defuzzified values of ts_{ijk}

i,j	ts_{i1k}	ts_{i2k}	ts_{i3k}	ts_{i4k}
ts_{1jk}	0.15187	0.30926	0.26573	0.19398
ts_{2jk}	0.25225	0.10075	0.12149	0.09037
ts_{3jk}	0.12112	0.32111	0.23420	0.34374

5. Concluding Remarks

The FMPP deals with fuzzy constraints, fuzzy demands and fuzzy production times, using convex fuzzy optimization techniques leading to global optimum solutions.

λ reaches a stable value, which is useful to defuzzify all the fuzzy production times, finding a crisp value where a global solution of the problem exists. This is a reference point where the analyst can operate the system using the defuzzified parameters, achieving better results.

The proposed method reaches a balance between $\psi_{\tilde{\Lambda}}$ and $\mu_{\tilde{b}}$ by projecting $x \in \mathbb{R}$ to an optimal solution of the MPP problem with both fuzzy constraints and production times. Although there is an infinite amount of possible choices of ts_{ijk}, the proposed algorithm tries to find a single defuzzification value for all fuzzy parameters, as a reference point for the analyst. However, the analyst can use other values of ts_{ijk} with better z^*, so we recommend to use our proposal in cases where the analyst has no prior information about $\tilde{A}$.

Finally, fuzzy optimization is an active area of research were new models are continuously arising. As proposed by Figueroa [3], [4], in future work we plan to use Type-2 fuzzy sets for handling higher uncertainty levels, as a new step on fuzzy optimization.

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