From Philosophical to Mathematical Inquiry in the Classroom

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Abstract:
This paper discusses some major similarities and differences between community of philosophical inquiry (CPI) and community of mathematical inquiry (CMI), and offers a few examples of the implementation of CMI in the context of a school mathematics classroom. Three modes of CMI are suggested. The first mode facilitates inquiry into mathematical problems - that is, it provides a medium for "doing and talking mathematics." In this case, CMI is primarily an avenue for problem solving—defining problems, interpreting them, working with different methods to solve them, reflecting on suggested alternative methods, verifying solutions, and drawing conclusions. The second mode leads us to "talk *about* mathematics" through collaborative inquiry into mathematical concepts such as axioms, theorems, algorithms, infinity, and the posing of philosophical questions that concern mathematics as a system--particular structures and rules and their relation to human experience. The third mode makes use of CMI for meta-inquiry into our collective experience in "doing and talking mathematics" and "talking about mathematics," and may be characterized as "talking *about doing* mathematics."

Keywords: mathematics education; community of mathematical inquiry; mathematics teaching and learning

Da investigação filosófica à investigação matemática na sala de aula.

Resumo:
Esse artigo discute algumas das maiores similaridades e diferenças entre a comunidade de investigação filosófica (CPI) e a comunidade de investigação matemática (CMI), e oferece alguns exemplos da implementação da CMI no contexto do ensino da matemática em sala de aula. Três modelos da CMI são sugeridos. O primeiro modelo facilita a investigação em problemas matemáticos- ou seja, oferece um meio para "fazer e falar matemática". Neste caso, a CMI é primariamente uma avenida para resolução do problema – definindo os problemas, interpretando-os, trabalhando com diferentes métodos para solucioná-los, refletindo nos métodos alternativos sugeridos, verificando soluções e desenhando conclusões. O segundo modo nos conduz a "falar *sobre* matemática" através da investigação colaborativa dentro dos conceitos matemáticos, tais como axiomas, teoremas, algoritmos, infinitude, e a apresentação das questões filosóficas que diz respeito à matemática como um sistema – estruturas particulares e regras e a relação destas com a experiência humana. O terceiro modo faz uso da CMI para uma meta-investigação na nossa experiência coletiva em "fazer e falar matemática" e "falar sobre matemática", e pode ser caracterizado como "falar *sobre fazer* matemática"

Palavras-chave: educação matemática; comunidade de investigação matemática; ensino e aprendizagem da matemática.

De la investigación filosófica a la investigación matemática en el aula.

Resumen:
Este artículo discute algunas de las mayores similitudes y diferencias entre la comunidad de investigación filosófica (CPI) y la comunidad de investigación matemática (CMI), y ofrece algunos ejemplos de la implementación de la CMI en el contexto de la enseñanza de la matemática en el aula. Se sugieren tres modelos de la CMI. El primer modelo facilita la investigación en problemas matemáticos- o sea, ofrece un medio para "hacer y hablar matemática". En este caso, la CMI es primariamente una avenida para la resolución del problema – definiendo los problemas, interpretándolos, trabajando con diferentes métodos para solucionarlos, reflexionando sobre los métodos alternativos sugeridos, verificando soluciones y sacando conclusiones. El segundo modo nos lleva a "hablar *sobre* matemática" a través de la investigación colaborativa dentro de los conceptos matemáticos, tales como axiomas, teoremas, algoritmos, infinitud, y la presentación de las cuestiones filosóficas que aluden a la matemática como un sistema – estructuras particulares y reglas y la relación de estas con la experiencia humana. El tercer modo hace uso de la CMI para una meta-investigación en nuestra experiencia colectiva en "hacer y hablar matemática" y "hablar sobre matemática", y puede ser caracterizado como "hablar *sobre hacer* matemática"

Palabras-clave: educación matemática; comunidad de investigación matemática; enseñanza y aprendizaje de la matemática.

FROM PHILOSOPHICAL TO MATHEMATICAL INQUIRY IN THE CLASSROOM

Nadia Stoyanova Kennedy

Mathematics teaching and learning is still plagued by inherited, traditional models and is still largely understood as the "transmission" of facts, rules, or disjointed concepts from the teacher to the learner, who is expected to find the connections between the concepts on her own, whether now or at some future moment of insight. This and other didactic and pedagogical problems of mathematics learning for which the discipline is well known,[1] can be traced to a set of much larger epistemological and ontological beliefs, which have come increasingly to be challenged over the course of the last half century.

One of the greatest challenges to the traditional model is presented by constructivists theories, which understand mathematics learning as an active form of sense-making which is intrinsically oriented toward conceptual development and the making of connections among concepts rather than as the accumulation of facts, disjointed notions, or as the adoption of fixed truths. Doing mathematics--whether as learner, teacher, applied practitioner or researcher--is nowadays largely understood to be a meaningful, active enterprise which involves reflecting on others' opinions, exploring working hypotheses and assumptions, making inferences and offering justifications (e.g. NCTM, 2000; Schoenfeld, 2005). Along with this, there has been change of focus from individualistic learning to learning in the social context of the classroom. The latter is coming to be recognized as offering possibilities for rich pedagogical activities and creative approaches in mathematics teaching and learning (e.g. Hiebert et al., 1997; Watson & Mason, 2005).

The constructivist vision, which is shared by leading mathematics educators around the world (for example Lampert, 2001; Ball, 1995, Boaler & Humphreys, 2005; Schoenfeld, 1992), encourages the creation of learning environments that recognize and utilizes students' informal knowledge and spontaneous concepts, and expects students work primarily in collaboration with their peers on mathematical tasks that are designed to encourage individual and group inquiry, the communication of mathematical ideas, and the communal exploration of proposed conjectures The purpose and the promise of such a vision is to aid students in building, strengthening, and connecting various representations of mathematical ideas, and in developing further depth and sophistication in their understanding of mathematical concepts and their relations to the world. However, there is a need for working models that incorporate this educational vision, and which can serve as a guide for mathematics teachers.

Philosophy for Children (P4C)--given its distinctive curricular and pedagogical approach, designed to lead students toward a collaborative encounter with the elements of philosophy through narrative texts and communal dialogue--is one program that might contribute to this search for working models. The program's published curriculum does not offer any mathematical texts, nor does it state any particular goals related specifically to mathematical thinking or knowledge apart from its clear emphasis on critical thinking. However, P4C's theory and methodology do in fact offer strong possibilities for creative adaptation in the context of mathematics teaching and learning. The particular methodological framework of P4C—that is, community of philosophical inquiry (CPI)—involves the conscious construction of a classroom event structure that is emergent, participative, dialogical, and egalitarian. A more broadly conceived and defined pedagogical model of community of inquiry has in fact been used in mathematics education in different versions (e.g. Ball, 1995; Lampert, 1990, Schoenfeld, 1989; Cobb et al, 2001; Goos, 2004; Boaler & Humphreys, 2005), and in math education the term usually refers to a setting for mathematical practice in the classroom that engages the learning community in doing mathematics collaboratively. Beyond that, it varies widely in specific goals and characteristics.

The form of community of mathematical inquiry (CMI) that I and several colleagues are working to develop embodies most of the essential characteristics of a community of philosophical inquiry as conceived by P4C, but introduces some further field-specific differences. In what follows I will delineate some major similarities and differences between CPI and CMI, and then offer a few examples of the implementation of CMI in the context of a school mathematics classroom.

One main objective of both is the construction of meaning and the formation of concepts, not through transmission, individual reflection or debate, but through what is referred to as "building on each other's ideas." The ideal inquiry proceeds through a form of argumentation which is inherently dialogical and thus by implication, a dialectical process which moves forward through working with conflicting ideas and attempting to resolve the tensions or contradictions encountered. Epistemologically and in terms of learning theory it operates on the basis of distributed cognition—or knowledge which is a group rather than an individual construction.

In both CMI and CPI, any given argument is built on or emerges as a counter argument to a previous one. As such, argumentation in community of inquiry is inherently both chaotic and teleological (Lipman, 1991). It can be influenced by any single element of the cognitive system—for example by any single participant—as well as by any element in the cognitive medium, for example the initial problem under consideration, specific examples and counterexamples offered by the participants, or by the presence of conscious or unconscious assumptions in the group dialogue.

One chief pedagogical feature of the constructive process of community of inquiry is that it operates in the collective zone of proximal development, which acts to "scaffold" concepts, skills and dispositions for each individual. The scaffolding process in CMI, just as in CPI, functions through subprocesses or participant moves such as clarification, reformulation, summarization and explanation, as well as through challenging and disagreement or doubting someone's position, which forces the group participants to further articulate their ideas. Thus scaffolding is understood to consist of and lead to the "externalization" of concepts through their emergence in the community process, which then makes possible their "internalization" by the individual, thus exemplifying Vygotsky's (1978) two chief claims about learning, viz. 1) "An operation that initially represents an external activity is reconstructed and begins to occur internally"; and 2) "An interpersonal process is transformed into an intrapersonal one" (pp. 56-57). It could be argued that community of inquiry represents the ideal situation for the intrapersonal appropriation of the interpersonal—or "internalization"--not only on the conceptual but on the behavioral level, i.e. in the development of habits of both cognitive and behavioral self-control and self-regulation.

Community of inquiry as an open emergent system is continually mediating further cognitive advancement through the re-externalization of the internal in the ongoing discourse of the community, followed by further internalization, and so on in an ascending spiral of development (Vygotsky, 1978). The skills and dispositions of argumentation that are elicited through group interaction—for example, making a proposition or hypothesis, offering a counter-example, or reasoning analogically or syllogistically--are, on this account, internalized by each member in a dialogical setting and transformed into personal skills and dispositions which, over time, feed back into the group process. More specifically, we would like to point out two specific cognitive and behavioural outcomes that result from this interplay of internalization-externalization processes: 1) ongoing collective and individual conceptual transformations that can be registered in CMI, 2) continual modelling in the group of thinking and argumentation moves and patterns, which results in thinking and argumentation discourse routines, and dispositions, and 3) further development over time of the thinking and argumentation moves into more stable thinking habits and argumentation schemas (see Kennedy, 2005).

Thus, drawing from P4C theory, my own mathematics teaching experience and experiments with CMI, three modes of incorporation of CI methodology in mathematics teaching and learning suggest themselves. The first indicates that we might use community of inquiry methodology to inquire into mathematical problems pure and simple, that is, to "do and talk mathematics." In this case, CMI is primarily an avenue for problem solving—defining problems, interpreting them, working with different methods to solve them, reflecting on suggested alternative methods, verifying solutions, and drawing conclusions. The second mode leads us to "talk *about* mathematics" through collaborative inquiry into mathematical concepts such as axioms, theorems, algorithms, infinity, and the posing of philosophical questions that concern mathematics as a system--particular structures and rules and their relation to human experience. The third mode makes use of CMI for meta-inquiry into our collective experience in "doing and talking mathematics" and "talking about mathematics," and may be characterized as "talking *about doing* mathematics." Such experience in meta-analysis is a powerful tool for bringing into conscious awareness issues which could otherwise have stayed even unnoticed. By way of example, some students are afraid of participating in group discussions for fear that they might be wrong, until it comes to the group's attention how a mistake or a misconception could be pivotal for the development of a more sophisticated conceptual understanding. What follows is a more detailed elaboration of these three modes of CMI, followed by examples taken from my work with upper elementary school students.

CMI is in fact an ideal working environment for "doing and talking mathematics." Different versions of community of inquiry have been used by mathematics education researchers, but all have capitalized on the potential of collaborative thinking and communication in working with students on solving mathematical problems (e.g. see Lampert, 1990; Goos, 2004; Boaler & Hemphreys, 2005). Most documented research in this mode of CMI portrays collective work on well-defined mathematical problems. While we agree that utilizing a collaborative inquiry model to facilitate mathematical problem solving is important, I want to suggest that there is also much to gain from inquiry into math problems that are more ambiguous, open to interpretation, and which might call for definitions of certain concepts or terms. It is the task of the facilitator to select mathematical problems that are not computational and do not offer immediately clear paths for their solution, or which may have more than one answer. A major goal of this form of inquiry is not so much the "right answer" to the mathematics problem as a significant "residue" that is a result of the collective experience of group deliberation (Davis & Simmt, 2003). This "residue" could be anything from a more sophisticated understanding of a concept to an understanding that a mathematical problem could be "read" and interpreted in different ways, to the experience of the group as a resourceful pool of ideas and a powerful interlocutor that helps the individual in reshaping her ideas. This aligns with another primary goal for the facilitator in the collective problem-solving process, no matter the type of math problems the group works with: to encourage students to problematize mathematics, to develop an attitude of healthy scepticism that prompts the questioning of implicit assumptions and the critical evaluation of suggested methods and solutions. CMI is an ideal context for pursuing such a goal, for it offers the facilitator opportunities to model more general paths of approach to problems as well as specific problem solving strategies for students. Such a discursive context also encourages the emergence of new attitudes towards mathematics and of enhanced skills and dispositions of collaborative deliberation, such as questioning techniques or a sense for measured participation in the group dialogue. This CMI format positions the participants as "insiders" of mathematics as a system of rules—they are expected to know major mathematics principles and rules, and tackle the mathematics problem according to rules that are foundational for the system.

As for the second mode, "talking *about* mathematics," we have a rich exemplar in P4C, which has taught us to generate and facilitate critical discussions focused on certain philosophical questions and themes. In fact the infusion of a philosophical dimension into mathematical thinking and communication in the classroom might provide a new context for philosophical-mathematical investigations into the logic of mathematics as an axiomatic system—its structure, power, and limitations--and into the connections between mathematics and human experience. Philosophical inquiry focused on the former category might encourage a search for meanings in mathematical concepts that have not traditionally been offered for discussion in school mathematics—e.g. infinity, axioms, algorithm, or the nature and use of mathematical models. Additionally, a group inquiry into the meaning of terms that are not mathematical but reveal new meaning when put in a mathematical context might enrich and deepen students' understanding—for example concepts such as "elegant," " obvious," and "thinking," as embodied in inquiry questions such as "What is mathematical thinking?" "What is an 'elegant' solution?", or "What is 'obvious' in math?"

Other philosophical-mathematical investigations raise questions that are focused on the connection between mathematics and human experience. Some examples of such questions are: "When can we say that we "understand" a mathematical concept? "What makes a mathematical problem hard?" "Is mathematics a mirror of the real world, a map, a model, or something else?" "Is there anything that math is not good for?" "Can something be finite and infinite at the same time?" "Can we make mathematical models of human relations?" "Where does math exist?" "How can we trust in math that is not experienced?" Such philosophical discussions complement more concrete mathematical investigations in that they allow one's own construction of meanings and understandings of concepts and connections to be applied that are not readily available or encouraged if they have not been a focus of conscious, reflective philosophical-mathematical inquiry. Thus, the introduction of a philosophical dimension into mathematical inquiry offers the possibility of a deeper and more spacious epistemological approach to mathematics, and richer experiences in teaching and learning school mathematics.

Finally, in the mode I have labelled "talking *about doing* mathematics," the CI format is directly applicable to evaluative or meta-discussions that follow events in either the first or the second modes. Here participants reflect on their collective experience—whether in solving mathematical problems or deconstructing and reconstructing philosophical-mathematical concepts. Such meta-discussions may oblige participants—including the teacher or group leader--to evaluate aspects of their own participation in the group inquiry, and to examine and reflect on the normative characteristics of the emergent event of collective inquiry. Such reflection is desirable not only for increasing participants' awareness of the complexity of group dynamics, but in the development of metacognitive skills, and for awareness of and control over the kind and quality of one's own participation. Meta-discussions offer a context for the development of deeper insights into the processes of knowledge-construction, in particular the importance of discussing mathematical mistakes and misconceptions in furthering each individual's conceptual framework. Furthermore, meta-discussions model the skills and dispositions associated with self-evaluation, and thereby further facilitate autonomous thinking and the capacity for self-motivated learning and critical reflection.

Having described three modes of inquiry appropriate for CMI, I want to offer two actual examples of the first two, in the form of transcripts taken from two discussions among upper elementary students in a local public school in northern NJ, USA. They comprise sections excerpted from 20 weekly sessions, 40 minutes each, conducted by myself with one group of fifth graders over the course of one school year. The excerpts are accompanied by my own analysis, in an attempt to demonstrate a correspondence between the ideas discussed above and the actual events that unfolded during the group inquiries.

Vignette1: Talking and Doing Mathematics: Inquiry into mathematical problems

The following discussion revolved around a problem proposed by the facilitator, as follows: A frog finds herself at the bottom of a 30-foot well. Each hour she climbs 3 feet, and slips back 2 feet. How many hours would it take her to get out? One interpretation of the problem with the respective solution was offered immediately, along with its respective set of unstated assumptions. This interpretation instantly came into conflict with an alternative one which was based on a different set of assumptions. Another response came as a disagreement with this interpretation, leading again to the first interpretive frame.

Samantha: She climbs 3 and goes down 2 feet. So every hour she climbs only one foot. So she'll need 30 hours to come out of the well.
Victor: I disagree with Samantha. She thinks that it requires one step every hour, but what about after 27 hours? She won't have more to climb.
Nellie: I agree with the first suggestion—a step an hour.
Facilitator: Do you mean a foot an hour?
Nellie: Yes, a foot an hour. So she'll need 30 hours, because I think that she'll go back, otherwise it will be outside the problem.

At this early point the conversation had already arrived at what Dewey (1910) called a "forked-road" situation— a situation that is ambiguous, that presents a dilemma, that proposes alternatives, and which keeps us "in the suspense of uncertainty" and drives inquiry forward. The "forked-road" is a metaphor that highlights the collective inquiry process initiated by a carefully chosen mathematical problem, and highlights the moment of inception of the action of the collective drama that induces further commitment to the inquiry process. It is also an index of what Leon Festinger (1975) describes as "cognitive dissonance"—a perceived inconsistency that is arrived as a result of a fallacious entailment at two contradictory or contrary propositions which cannot both be true. Cognitive dissonance, Festinger assures us, cannot be eliminated or reduced by avoidance but only by changing one of the dissonant elements (for example, the modification of a fallacious inference, or of incorrect premises, incorrect assumptions, or inadequate beliefs)—or, if we consider it from a dialectical perspective, by finding a third alternative through reaching a kind of a synthesis of the two. In short, the "forked-road" situation calls for a reverse examination of the students' reasoning so far, and careful reflection not only on the inferences but also on the premises and their warrants that are involved.

In this case the students were already familiar enough with the facilitator's tactics to know that the "right answer" would not be provided, and that it was their job to arrive at an answer through group deliberation. Their way of proceeding with this deliberation was transparent enough—a detailed search for more clues combined with a search for flaws in each others' reasoning. The intuitive search for more clues in fact resulted in tracing out the territory of the problem and establishing its boundaries, and a dialectical interplay is apparent between the process of framing the problem and the search for flaws—that is, they inform each other. The following excerpt illustrates this process of framing and reframing:

Victor: It doesn't say whether she'll decide to go back again once she's out.
Rush: I think she's going to go back.
Samantha: But why does she have to go back? Look, she tries to go out.
Chas: But the question was, when first is she going to be out of the well? Right?
Facilitator: The problem says "How many hours would it take her to get out"? How do we interpret that?
Chas: So they mean *first*.
Asia: It could be that once she's out of the water, she never goes back. But it could be also that she gets out and then she goes back. So I'm saying that it could be both ways.
Facilitator: Samantha?
Samantha: I think it can take 28 hours, because if this is the well [she makes a drawing]. This is 27 feet above the bottom, and she'll need 3 feet more, which is one hour more.
Facilitator: Now we have two suggestions. How are we going to evaluate them?
Laura: Actually all of them might work.
Rush: Isn't she going to need some sleep?
Bud: It's out of the problem. Most likely she wants to go out as soon as possible.
Nellie: But it doesn't matter whether she's out or not, she still has to climb back…'cause we can't go outside the problem.
Facilitator: Nellie is saying that the frog will return back to the well even when she's already out. There is another group of students who are saying that once the frog is out, she wouldn't need to go back. What are we going to decide on that?
Bill: I think that Victor is probably right, because once she's out, why would she jump back into?

Victor: The question says "When it's first going to be out." She might go back, but the question is saying first. And it's after 27 hours.

Facilitator: O.K. We can clarify the question as "When is the frog *first* going to be out of the well"? Now let's go from here and check the calculations. Is it 27 hours, 30 hours, or 28 hours that she would need? We have several suggestions so far.

Rush: I agree with Victor. Because if you're in a well you want to climb and get out. Do you think you will jump back? You get out. Period. This is her main goal. Nothing else.

Facilitator: O.K. We're going to focus on when she will first be out of the well. And let's do the calculations.

The phenomenon of framing the problem—to which we will return later—is interesting from a logical point of view. In the excerpt above we can trace the discussion as it alternates between framing the problem and making an inference. For instance, in order to be able to infer how much time the frog will need to get out of the well, it was necessary to redefine the problem question as "When is the frog *first* going to be out of the well"? It should also be noted that this redefinition increased the students' sense of ownership of the interpretation of the problem, and thereby acted to facilitate their further deliberation.

It was Samantha who presented the very first proposition, then abandoned it together with the first interpretive frame, and proposed one which corresponded to the second. The change in the set of assumptions that she underwent was quite clearly articulated, which suggests that she hadn't considered any other alternative at the beginning, when she assumed that the frog would follow the same pattern of climbing. But after reflecting on the alternative presented by Victor, she took it as the more plausible one, which suggests that her deliberative style already included an implicit criterion of reasonableness—i.e. that assumptions could be evaluated, and more adequate ones replace less adequate. Her comment "But why does she have to go back? Look, she tries to go out" appears to supports that hypothesis.

The temporal moment marked by the end of the excerpt still held the discussion in suspense between the two frames for interpreting the problem. At this point, the second frame encompassed two separate candidates for a solution: one, proposed by Victor, stating that the frog will need 27 hours to go out, and the other, proposed by Samantha, who claimed that the frog would climb out of the well in 28 hours. The cognitive conflict was still in place, but the first frame was no longer favored. Those students who had adhered to it in the beginning—e.g. Samantha, Rush, and Bill--had already declared their change in position. Since both the first frame and its underlying assumption seemed to be rejected, the focus was expected to fall on making a decision between the two propositions associated with the second frame, but it turned out that the climax had not yet been reached. Another development was waiting to unfold. Sally proposed a third frame grounded in a third kind of assumption about the complex behavioral pattern of the frog.

Sally: I'm just thinking about …from the 27th feet, you're saying she'll climb up 3 feet and she's out, but then she'll be exactly levelled with the . . . ground. . . and she'll need a little more strength to climb out of the well, but she might not have this strength and she'll slip back.
Samantha: So it will depend I guess on her strength, but how can we know?
Jimmy: Wait, you're saying 28, but it's in case she first climbs 3 feet to reach the ground. And what if she climbs a little bit and then slides back, then again climbs a little bit, then slides down, then she wouldn't really climb 3 feet, would she?
Victor: The problem is saying she climbs 3 feet and slides back *then* 2 feet.
Bill: Well that's the way we understood it, but it's not quite clear.

Sally's unexpected complication of the problem was followed by yet another unexpected proposition by Jimmy, whose fourth interpretive frame offered an even more sophisticated suggestion about the behavioral patterns of frogs. But now, restricted by time, the inquiring community decided that, except for the first frame, the other three seemed to be viable frames for interpretation, considering the fact that the problem didn't offer more information about the putative climbing patterns of the frog. The discussion concluded with a conditional statement which reflected all three of the possibilities under consideration.

Facilitator: Who would summarize the conclusions that we've reached so far? Jimmy?

Jimmy: If she climbs the way we thought she did in the beginning, most likely 28 hours will be enough, but if she doesn't first climb these 3 feet and then slides back 2 feet and does it differently..
Bud: …which in fact is most likely…
Jimmy: …then she might need 30 hours.

In summary, this excerpt portrays the way in which the group dealt with solving the mathematical problem that was posed. We can see the "forked-road" situation and the group's struggle to interpret the problem and find appropriate ways of thinking about it and searching for the answer, and we can trace the four interpretive frames and the competition of the contrary propositions associated with these frames. The problem also offers a high level of uncertainty and ambiguity, which forced the inquiring community to zig-zag between the negotiation of the problem boundaries and reflection on each others' reasoning before an agreement about the solution could be reached.

The "residue" that one would hope the students were left with after experiencing collective mathematical problem solving in a CI context like the one documented above is multifaceted. It is about helping students gain understanding that doing mathematics is a sense-making process; that mathematical problems are matters of interpretation and require careful examination of the data given; and that any inferences made are based upon implicit assumptions which also call for examination. Some other facets have to do with understanding the relationship between mathematics and uncertainly, and the role that the one who poses or solves a mathematical problem plays in defining and interpreting the problem. And yet another facet—although not the last—has to do with understanding the role of the community as an interlocutor, as a generator of ideas, and as a reflector and corrector of one's reasoning and perspective.

Vignette 2: Talking about Mathematics: Philosophical-mathematical inquiry

A philosophical-mathematical discussion can occur in at least two ways: a) it can be "staged"—meaning planned in detail by the facilitator, who has prepared a list of philosophical questions and starts the discussion with one of those questions, chosen by the group.[2] Similarly, it may follow the P4C tradition and use a narrative text that offers or suggests one or more implicit philosophical-mathematical questions, and ask of the group to generate its own questions and choose one from them for a start; b) the philosophical-mathematical discussion emerges from a more strict mathematical discussion. In this case the facilitator has a choice whether to embrace the emergent philosophical impulse and allow the discussion to unfold, or to forestall it by adhering strictly to the mathematical inquiry as framed--although it is quite difficult to draw a line between philosophy and mathematics proper. Based on my own classroom discussions with children, I believe that the group impulse to detour from a strict mathematical inquiry into a philosophic-mathematical one tends to occur at moments when the inquiry stagnates. Such moments are marked by a sense of lacking the conceptual tools or skills or understanding that are necessary to continue. Such shifts allow for a qualitatively different type of exploration--one that takes a philosophical rather than structuralist approach. In fact such internal loops of philosophical inquiry complement the mathematical inquiry in a ways that suggest that they may be sources for the development of more refined cognitive tools and deeper understandings, which equip the group for further advancement in the mathematical inquiry per se.

The episode that follows is an example of a philosophical-mathematical inquiry into the concept of infinity that emerged from a mathematical exploration based on the following question: Given the infinite sets [1, 2, 3, 4, 5, . . .] and [2, 4, 6, 8, 10, . . .]. Do both sets have an equal or a different number of elements? Prior to this the term infinity had not yet been discussed, although the concepts of finite and infinite sets had been explored in a previous discussion[2]. After a long discussion three conjectures concerning the comparison of the two infinite sets were on the table: 1) the first set has double the elements of the second set—the positive integers are double the positive even integers—and therefore the sets have different numbers of elements; 2) both sets have the same number of elements i.e. there are as many positive integers as positive even integers; and 3) a comparison between the elements of both sets is impossible, because "we don't exactly know what happens with the numbers down the number line." Neither of the camps stating these positions was able to persuade the other, and the discussion seemed to be stuck in repetition.

Then a student spontaneously offered the question "Is infinity a number?" and that changed the subject of the discussion; thus, it may be interpreted at least in part as a verbalization of the group's search for way out of the stagnant situation. The facilitator—whether she was aware of the potential of such exploration or intuitively allowed it--encouraged this philosophical inquiry.

Nelly: Is infinity a number?
Facilit.: O.K. And some people are not only using the word "infinite," but also "infinity" and what does that mean? And is it a number?
Chas: I think when people say *infinity* that means that they were too tired making more numbers, and they were too lazy to name any more numbers, and….and also there would have been a lot of work.
Voices: How do you know?
Chas: Well, nobody bothers to go to infinity.
Jimmy: Infinity is not a number.
Chas: Infinity is not a number, a… it's just in there…
Sally: If numbers are going forever I guess it's a number.
Chas: It's not exactly a number, that's the name of the rest of the numbers, ….[He means that numbers that are denoted by ellipses] and they might skip a number when they go through, they might go odd, odd, even, even, odd, even. It's a problem that they can change.

It seems that at this point there were different understandings present as to how to describe or imagine infinity. Some thought that it was a number, others that it wasn't. Chas's conception was that it was the name of the numbers which were denoted by the series of dots in [1, 2, 3, 4,…] "It's not exactly a number, that's the name of the rest of the numbers,…and they might skip a number when they go through. They might go odd, odd, even, even, odd, and even. This is the problem— that they can change." Then yet another idea was offered by Rush, viz. that infinity is a number, but a unique one.

Rush: ….because the number infinity, it's not like 1000 billion, or 1000 trillion, it's not like that and there is a whole bunch of numbers inside infinity, so we don't know whether inside they're skipping odd, even, even, even, odd.

This is an interesting and "wild" idea—of infinity being a special number that contains "a bunch of numbers." But then Claire challenged the idea that infinity is a number at all:

Claire: And what number comes before infinity? I don't really think that infinity is a number. Or a bunch of numbers, what Rush is saying.
Sally: He said that all the numbers are packed inside.

What Claire appeared to mean was that the idea of infinity being a number of any kind was absurd because it immediately implied finitude and this would lead to a self-contradiction. She was inferring that if infinity is a number it must have an antecedent, and since we don't know what that number is, then infinity cannot be a number. This intervention caused a turn in the discussion that led to an agreement among the students in terms of their current understanding of infinity:

Jimmy: I think infinity is not a number, we just say this way—endless.
Chas: That's what we said.
Mark: Everybody seems to be saying this.
Victor: We are making our own definitions, 'cause the dictionary is written by people and I think they may not be always right. Infinity is just numbers that are too big.
Jimmy: Infinity just means expanding and expanding and expanding. It means *infiniteness*.
Sally: O.K. That means that there are no numbers packed in infinity, but we just have infinite numbers packed in these sets [in the infinite sets].

Thus, after a series of sequences involving the shaping and reshaping of ideas, the current understanding of infinity was changed and came to mean: 1) "infiniteness," "endless," an "expanding and expanding" process—a conception they had already come up with a bit earlier in the discussion; and 2) "infinity is not an actual number," and "there are no numbers packed in infinity," but if we deal with infinite sets "we have infinite numbers packed in these sets." The second in fact represented a new understanding which, integrated into the first, expanded the existing conception of infinity. This richer conception of infinity helped students move along with the primary discussion on the comparison of the infinite positive integers and the infinite positive even integers. After the discussion moved back to the mathematical inquiry the group agreed that "neither of the sets has more numbers." Even the conjecture that we can't decide which set has more elements because we don't know how the elements "behave down the number line," was re-evaluated. It was finally concluded that ". . . it's still the same number [of elements in the two sets]. Even if they [the numbers] did skip, uh…we will still have endless numbers in both rows so…so eventually…even if they do skip it's not very important anyway."

Conclusion

The broadest claim of this paper is that there are insights to be gained from the Philosophy for Children program and its methodology community of inquiry, and that these insights emerge as much from its philosophy as from its practice. The advantages to be gained by introducing argumentation discourse as a central mode of mathematics pedagogy seem obvious, above all the extent to which collective inquiry and deliberation act to foster transformation in students' mathematical conceptual development and understanding, and in their ability to conceive mathematics as a discursive structure--i.e. as a language open to individual and communal mediation.

But the grand idea in Philosophy for Children, and the one that makes it so compelling, powerful, and transformative—is its insistence on inquiry as both praxis and epistemological stance. If inquiry is adopted as an approach in mathematics teaching, it will entail a profound change in the way both teachers and students engage with and make sense, not just of math activities, but of the world, the other, and oneself. The three modes of community of mathematical inquiry described might find their place as regular forms of classroom practice that morph naturally into one another, since inquiry cannot be restricted by formal boundaries or even artificially contained within specific disciplines. Mathematical teaching and learning would then come to mean inquiry into a wide variety of problems and situations—most of which cross disciplinary boundaries--with the tools of mathematics, as well as inquiry into mathematical ideas and concepts on a broad philosophical level—whether aesthetic, ontological, epistemological, logical or ethical.

Community of inquiry is a form of practice deeply informed by positive humanistic belief, and its adoption by any discipline—whether mathematics, social studies, science or physical education--could profoundly transform teaching and learning into a synergetic form that unites theory and practice, philosophy and application, argumentation and calculation in the concrete, problem-based context of the classroom. Given both the nature of the discipline and the pedagogical traditions which still dominate mathematics education, any addition to or alteration of its canonical structures poses a profound challenge. But I would like to think and hope that a theoretical and practical exploration such as this one suggests that community of mathematical inquiry--carried out in a context of communal deliberation with an emphasis on argumentation—has the potential of developing into a form of sustainable classroom practice that is capable of transforming the field of mathematics education in a profoundly positive manner.

ENDNOTES

1. For example, the valuing of memorization over independent thinking; the emphasis on knowing procedures rather than generating algorithms; the assumption that the mathematics teacher is the one knowledge-authority in the classroom; learning viewed as the passive reception of knowledge rather than active constructive engagement; teachers still understanding themselves as solely responsible for student learning rather than responsible for creating environments for active student participation in which responsibility for learning is shared.

2. For a detailed analysis of this discussion, see Kennedy (Summer, 2005).

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