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An approach for solving Goal Programming problems using Interval Type-2 Fuzzy goals

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Abstract

This paper presents a proposal for solving goal problems involving multiple experts opinions and perceptions. In goal programming problems where no statistical data about their goals exist, the use of information coming from experts becomes the last reliable source. This way, we propose an approach to model this kind of goals using Interval Type-2 fuzzy sets, and a simple method for finding an optimal solution based on previous methods that have been proposed for classical fuzzy sets.

Key words: Fuzzy linear programming, Interval Type-2 Fuzzy sets, Goal programming.

Resumen

Este trabajo presenta un acercamiento a la solución de problemas de programación por metas que incluyen la opinión y percepción de múltiples expertos. En problemas de metas que no tienen información estadística adecuada para definir los valores meta, el uso de información proveniente de expertos se convierte en la última fuente confiable de información. Así pues, proponemos una aproximación al modelado de este tipo de problemas utilizando conjuntos difusos de Intervalo Tipo-2, y un método sencillo para encontrar soluciones usando métodos propuestos por otros autores para conjuntos difusos clásicos.

Palabras claves: Programación lineal difusa, Conjuntos difusos Tipo-2 de intervalo, Programación por metas.

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1 Introduction

Decision making in practical applications faces multiple issues, including human being interaction and social behavior. Some problems are built over the base of having multiple goals involving multiple experts that try to solve the same problem with different objectives. To solve this kind of situations, goal programming provides a first tool to find crisp solutions.

To handle the problem of having both multiple experts and uncertainty around the exact value of a desired goal, fuzzy sets appear as a useful tool for handling numerical uncertainty coming from experts. Fuzzy goal programming has been proposed by Narasimhan [15], and later developed by Yang [20], Turgay & Taşkın [18], Li & Gang [12], Hu, Zhang & Wang [9], Khalili-Damghani & Sadi-Nezhad [10], in both theoretical and practical situations.

In decision making, Qin & Liu [17], Zhang & Zhang [21], and Chen & Ting [4] have already used Type-2 fuzzy systems to handle uncertainty coming from multiple experts, so its use in goal programming seems to be feasible.

Based on the model of Narasimhan [15], Yang [20] has proposed a model with fewer variables which obtains the same solution, so what we propose in this paper is to extend their results to a case where multiple experts deal with multiple goals by using Interval Type-2 fuzzy sets to handle linguistic/numerical uncertainty coming from experts and Linear Programming (LP) methods for handling goal programming.

The paper is organized as follows: Section 1 introduces the main problem. Section 2 presents some basics on fuzzy sets. In Section 3, goal programming LP model is referred. Section 4 presents the Yang [20] proposal for fuzzy goal programming. Section 5 contains the proposal; Section 6 shows an application example; and finally Section 7 presents the concluding remarks of the study.

2 Basic on Fuzzy sets

According to Klir & Yuan [11], the membership function of a fuzzy set A is denoted by $\mu_A : X \rightarrow [0, 1]$. \mathcal{P} is the class of all crisp sets, \mathcal{F}_1 is the class of all fuzzy sets, and \mathcal{F}_2 is the class of all Type-2 fuzzy sets.

2.1 Interval Type-2 Fuzzy Sets (IT2FS)

In general, a Type-2 fuzzy set is simply a function that transforms a set A into the set of fuzzy sets defined over $[0, 1]$, this is $\tilde{A} : X \rightarrow \mathbf{F}[0, 1]$, where $\mathbf{F}[0, 1]$ is also known as the secondary membership function of \tilde{A} . An Interval Type-2 Fuzzy set (see Mendel [13]) is an ordered pair $\{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where A is a linguistic label \tilde{A} that represents uncertainty about the

word A . Its mathematical definition is:

$$\begin{aligned} \tilde{A} &= (x, \mu_{\tilde{A}}(x)) : x \in X \\ \tilde{A} &= \int_{x \in X} \int_{u \in J_x} 1/(x, u), J_x \subseteq [0, 1] \end{aligned} \tag{1}$$

where $u \in J_x \subseteq [0, 1]$ is the domain of uncertainty around A .

Alternatively, an IT2FS can be fully characterized using two primary membership functions: *Lower Membership function (LMF)* and *Upper Membership Function (UMF)* in which are contained all embedded fuzzy sets A_e which composes the *Footprint of Uncertainty (FOU)*. Although there are other notations to refer to IT2FSs (see Mendel [14], Türksen [19], and Pagola et al. [16]) who recognize equivalences between Mendel and mathematical standard set notations, we use Mendel notations (see Mendel [13]) due to its interpretability and completeness.

2.2 Why Fuzzy Sets?

The main reason for using fuzzy sets is its ability to handle uncertainty coming from human perceptions, which is a common issue in decision making. On the other hand (numerical uncertainty), fuzzy sets can handle imprecision about X which commonly appears when no historical/statistical data is available, so the estimation of the parameters of the problem is based on approximate information coming from the experts of the problem.

3 Goal programming

The basic goal programming model proposed by Charnes, Cooper & Wagner [1], [2] tries to minimize deviations from different goals (desired objectives) through minimizing the absolute deviations d_k of the constraints of the problem $A_k x$ regarding its desired value (a.k.a goal) B_k in the format $\min\{D = \sum_{k=1}^n |A_k x - B_k|\}$. This model is equivalent to the following LP model (see Charnes, Cooper & Wagner [1], [2]):

$$\begin{aligned} \min D &= \sum_{k=1}^n d_{k1} + d_{k2} \\ &s.t. \\ A_k x + d_{k1} - d_{k2} &= B_k, \\ A'_k x &\leq B'_k \\ x, d_{k1}, d_{k2} &\geq 0; \forall k \in \mathbb{N}, \end{aligned} \tag{2}$$

where $B_k \in \mathbb{R}$ is the aspiration level, $d_{k1}, d_{k2} \in \mathbb{R}$ are negative and positive deviations from the goal B_k , A_k is the set of n constraints related to goals, A'_k is a set of crisp constraints of the problem, B'_k is its set of boundaries, and $x \in \mathbb{R}^m$ is the set of decision variables of the problem. A negative deviation quantifies a lack of satisfaction of the desired aspiration level, and a positive deviation quantifies an excess over the desired aspiration level.

4 Fuzzy Goal Programming

Although the first fuzzy goal programming has been proposed by Narasimhan [15], Narasimhan & Hanna [7], Yang [20] has proposed a model with fewer variables which obtains the same solution of [7], [15]. Yang's proposal starts by defining the membership function of the fuzzy goal B_k namely μ_{B_k} , as follows:

$$\mu_{B_k} = \begin{cases} 1 - \frac{G_k(x) - b_k}{b_{k2}}, & \text{if } b_k \leq G_k(x) \leq b_k + b_{k2}, \\ 1 & \text{if } G_k(x) = b_k, \\ 1 - \frac{b_k - G_k(x)}{b_{k1}}, & \text{if } b_k - b_{k1} \leq G_k(x) \leq b_k, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where $k \in \mathbb{N}$ denotes the k_{th} goal, $G_k(x)$ is the k_{th} constraint to be fulfilled, $b_k \in \mathbb{R}$ is the aspiration level of the k_{th} goal, and d_{k1} and d_{k2} are the maximum negative and positive deviations from b_k , respectively. Its LP model is

$$\begin{aligned} \min \quad & \sum_{k=1}^n d_{k1} + d_{k2} \\ \text{s.t.} \quad & \\ & A_k x + d_{k1} - d_{k2} \cong \tilde{B}_k, \\ & A'_k x \leq B'_k \\ & x, d_{k1}, d_{k2} \geq 0; \forall k \in \mathbb{N}, \end{aligned} \quad (4)$$

where $\tilde{B}_k \in \mathcal{F}_1$ the fuzzy aspiration level, $d_{k1}, d_{k2} \in \mathbb{R}$ are negative and positive deviations from the goal b_k , A_k is the set of n constraints related to fuzzy goals, A'_k is a set of crisp constraints of the problem, B'_k is its set of boundaries, and $x \in \mathbb{R}^m$ is the set of decision variables of the problem.

Finally, the proposal of Yang [20] is based on a simpler LP model in which $G(x) \equiv A_k x$, as follows:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \\ & \lambda \leq 1 - \frac{A_k x - b_k}{b_{k2}}, \quad \forall k \in \mathbb{N} \\ & \lambda \leq 1 - \frac{b_k - A_k x}{b_{k1}}, \quad \forall k \in \mathbb{N} \\ & A'_k x \leq B'_k \\ & \lambda \in [0, 1], x \geq 0. \end{aligned} \quad (5)$$

where $\lambda \in [0, 1]$ is the global satisfaction degree of all goals.

This approach uses λ as a global variable that represents the satisfaction of every fuzzy goal μ_{B_k} . The first constraint represents the satisfaction degree for $b_k \leq G_k(x) \leq b_k + b_{k2}$ (see Eq.

(3)), and the second constraint represents the satisfaction degree for $b_k - b_{k1} \leq G_k(x) \leq b_k$ (see Eq. (3)). As x is a free variable in this model, it operates over A_kx and finally moves λ to its maximum value.

5 Goal programming with Interval Type-2 fuzzy aspiration levels

Disagreement among people who are involved into decision making is a common issue in real scenarios. Some people is *pessimistic* while others are *optimistic* about different goals. This leads to have different perceptions coming from different experts, so we handle those perceptions using two functions *LMF* and *UMF* which are defined as follows:

$$UMF \equiv \bar{\mu}_{\bar{b}_k} = \begin{cases} 1 - \frac{G_k(x) - b_k}{\bar{b}_{k2}}, & \text{if } b_k \leq G_k(x) \leq b_k + \bar{b}_{k2}, \\ 1 & \text{if } G_k(x) = b_k, \\ 1 - \frac{b_k - G_k(x)}{\bar{b}_{k1}}, & \text{if } b_k - \bar{b}_{k1} \leq G_k(x) \leq b_k, \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

$$LMF \equiv \underline{\mu}_{\underline{b}_k} = \begin{cases} 1 - \frac{G_k(x) - b_k}{\underline{b}_{k2}}, & \text{if } b_k \leq G_k(x) \leq b_k + \underline{b}_{k2}, \\ 1 & \text{if } G_k(x) = b_k, \\ 1 - \frac{b_k - G_k(x)}{\underline{b}_{k1}}, & \text{if } b_k - \underline{b}_{k1} \leq G_k(x) \leq b_k, \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

where $\bar{\mu}_{\bar{b}_k}$ defines the UMF of the k_{th} goal, and $\underline{\mu}_{\underline{b}_k}$ defines the LMF of the k_{th} goal. A graphical display of a Interval Type-2 fuzzy goal is shown in Figure 1.

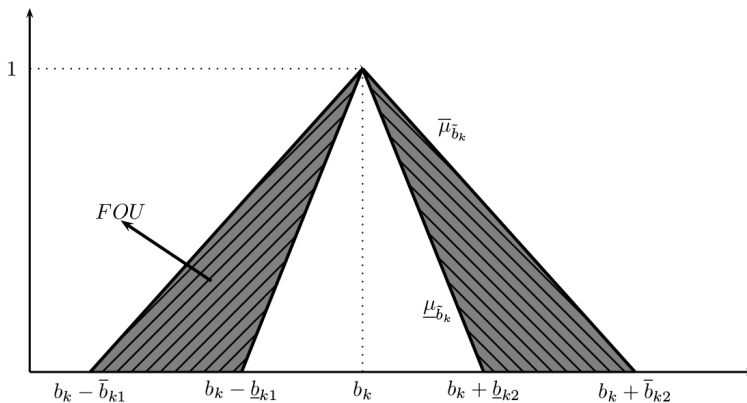


Figure 1. Interval Type-2 Fuzzy goal

Based on (3) and (5), we extend its results to a Interval Type-2 fuzzy environment. Thus, we define the Interval Type-2 fuzzy aspiration level as \tilde{b}_k which leads to the following LP model:

$$\begin{aligned} \min \quad & \sum_{k=1}^n d_{k1} + d_{k2} \\ \text{s.t.} \quad & A_k x + d_{k1} - d_{k2} \approx \tilde{b}_k, \\ & A'_k x \leq b'_k \\ & x, d_{k1}, d_{k2} \geq 0; \forall k \in \mathbb{N}, \end{aligned} \tag{8}$$

where $\tilde{b}_k \in \mathcal{F}_2$ is the Interval Type-2 fuzzy aspiration level, $d_{k1}, d_{k2} \in \mathbb{R}$ are negative and positive deviations from the goal B_k , A_k is the set of n constraints, and $x \in \mathbb{R}^m$ is the set of decision variables of the problem.

Therefore, we extend the proposal of Yang [20] to a Interval Type-2 fuzzy model using a two-step method that finds two different λ values, one for $\bar{\mu}_{\tilde{b}}$ and one for $\underline{\mu}_{\tilde{b}}$. To do so, we have to solve the following two LPs:

$$\begin{aligned} \max \quad & \bar{\lambda} \\ \text{s.t.} \quad & \bar{\lambda} \leq 1 - \frac{A_k x - b_k}{\bar{b}_{k2}}, \\ & \bar{\lambda} \leq 1 - \frac{b_k - A_k x}{\bar{b}_{k1}}, \\ & \bar{\lambda} \in [0, 1], x \geq 0; \forall k \in \mathbb{N}, \end{aligned} \tag{9}$$

$$\begin{aligned} \max \quad & \underline{\lambda} \\ \text{s.t.} \quad & \underline{\lambda} \leq 1 - \frac{A_k x - b_k}{\underline{b}_{k2}}, \\ & \underline{\lambda} \leq 1 - \frac{b_k - A_k x}{\underline{b}_{k1}}, \\ & \underline{\lambda} \in [0, 1], x \geq 0; \forall k \in \mathbb{N} \end{aligned} \tag{10}$$

where $\bar{\lambda}$ is the overall upper satisfaction degree of the goals, and $\underline{\lambda}$ is the overall lower satisfaction degree of the goals. $A_k x$ is the k_{th} technological constraint, and $\bar{b}_{k1}, \underline{b}_{k1}, \bar{b}_{k2}, \underline{b}_{k2}$ are the admissible deviations from b_k .

Our approach finds two values: $\min \{\underline{\lambda}\} = \underline{\lambda}^*$ and $\max \{\bar{\lambda}\} = \bar{\lambda}^*$ that represent pessimistic and optimistic perceptions about b_k , and also compose the interval $[\underline{\lambda}^*, \bar{\lambda}^*] = \{\lambda^* \in [0, 1] \mid \underline{\lambda}^* \leq \lambda^* \leq \bar{\lambda}^*\}$ of satisfaction of all experts.

6 Experimentation and Results

6.1 Interval Type-2 Fuzzy Goals

As application example we use the proposed by Narasimhan [15] and extended by Chen & Tsai [3] which is composed by three fuzzy goals, as shown as follows:

$$\begin{aligned} G_1 : 80x_1 + 40x_2 &\cong 630, \\ G_2 : x_1 &\cong 7, \\ G_3 : x_2 &\cong 4, \end{aligned} \tag{11}$$

where x_1 and x_2 are the manufacturing quantities of two products which regard to three goals: G_1 is a profitability goal, and G_2, G_3 are the expected selling quantities per product. The maximum deviations from $G_k = \{630, 7, 4\}$ are symmetrically handled where $b_{k1} = b_{k2} = \{10, 2, 2\}$.

We use those values as the LMF of an extended problem e.g $\underline{b}_{k1} = \underline{b}_{k2} = \{10, 2, 2\}$, and the UMF is defined using $\bar{b}_{k1} = \bar{b}_{k2} = \{15, 3, 3\}$. Now, their LPs are based on Eqs. (9) and (10):

$$\begin{aligned} &\max \bar{\lambda} \\ &\quad s.t. \\ \bar{\lambda} &\leq -\frac{16}{3}x_1 - \frac{8}{3}x_2 + 43, \\ \bar{\lambda} &\leq \frac{16}{3}x_1 + \frac{8}{3}x_2 - 41, \\ \bar{\lambda} &\leq -\frac{x_1}{3} + \frac{10}{3}, \\ \bar{\lambda} &\leq \frac{x_1}{3} - \frac{4}{3}, \\ \bar{\lambda} &\leq -\frac{x_2}{3} + \frac{7}{3}, \\ \bar{\lambda} &\leq \frac{x_2}{3} - \frac{1}{3}, \\ \bar{\lambda}, x_1, x_2 &\geq 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\max \underline{\lambda} \\ &\quad s.t. \\ \underline{\lambda} &\leq -8x_1 - 4x_2 + 64, \\ \underline{\lambda} &\leq 8x_1 + 4x_2 - 62, \\ \underline{\lambda} &\leq -0.5x_1 + 4.5, \\ \underline{\lambda} &\leq 0.5x_1 - 2.5, \\ \underline{\lambda} &\leq -0.5x_2 + 3, \\ \underline{\lambda} &\leq 0.5x_2 - 1, \\ \underline{\lambda}, x_1, x_2 &\geq 0, \end{aligned} \tag{13}$$

The solution of the model (12) is $\bar{\lambda}^* = 0.76$ reached by $(x_1, x_2) = (6.28, 3.28)$ and the solution of (13) is $\underline{\lambda}^* = 0.64$ with $(x_1, x_2) = (6.28, 3.28)$. A graphical description of the results can be seen in Figures 3, 4 and 5 respectively (see Appendix 1). The optimal value of the goals 1, 2 and 3 are 633.6, 6.28 and 3.28 respectively for both $\bar{\lambda}^*$ and $\underline{\lambda}^*$.

Both models reach the same values of the decision variables x_1 and x_2 which is a logical solution since all deviations are defined as L_1 distances using $\bar{\lambda}$ and $\underline{\lambda}$ as linear functions of d_{k1} and d_{k2} . The optimal values $\bar{\lambda}^* = 0.76$ and $\underline{\lambda}^* = 0.64$ are global satisfaction degrees, which means that all three goals are satisfied at the same level.

6.2 Type-1 embedded goals

Now, we solve another example to illustrate how an embedded Type-1 fuzzy set (B_e) into $FOU(\tilde{b}_k)$ works. To do so, we have selected the deviations for goals 1, 2 and 3 as 12.5, 2.5, 2.5 which corresponds to the middle point of the support of $FOU(\tilde{b}_k)$ as shown in Figure 2:

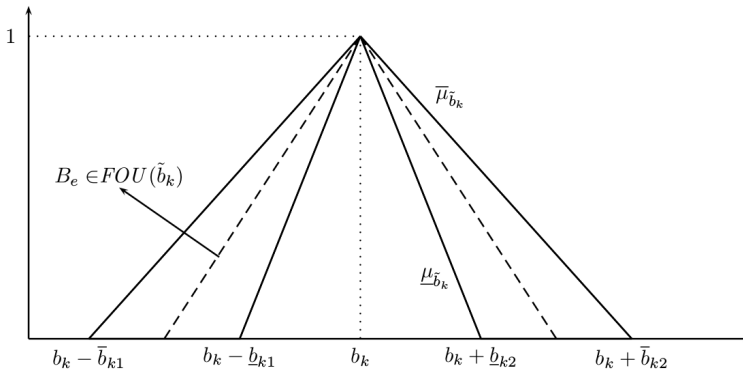


Figure 2. Type-1 embedded set B_e

The LP formulation based on Eq. (5) is:

$$\begin{aligned}
 & \max \lambda \\
 & \text{s.t.} \\
 & \lambda \leq -6.4x_1 - 3.2x_2 + 51.4, \\
 & \lambda \leq 6.4x_1 + 3.2x_2 - 49.4, \\
 & \lambda \leq -0.4x_1 + 3.8, \\
 & \lambda \leq 0.4x_1 - 1.8, \\
 & \lambda \leq -0.4x_2 + 2.6, \\
 & \lambda \leq 0.4x_2 - 0.6, \\
 & \lambda, x_1, x_2 \geq 0,
 \end{aligned}
 \tag{14}$$

The crisp solution to the problem (14) given Type-1 fuzzy goals is $\lambda^*=0.712$ which fits into the obtained range $[\underline{\lambda}^* = 0.64, \bar{\lambda}^* = 0.76]$ obtained through (9) and (10). Also note that any

$B_e \in \mathcal{F}_1$ embedded into $\text{supp}(\tilde{b}_k)$ leads to an optimal satisfaction degree λ^* that fits into the range $[\underline{\lambda}, \bar{\lambda}]$ as described in the Appendix 2. By continuity of LP models (see Dantzig [5], Hladík [8], and Fiedler et al [6]), if $\tilde{b}_k \in \mathcal{F}_2$ is continuous then its UMF leads to an optimal solution namely $\bar{\lambda}^*$ and every $B_e \in \text{FOU}(\tilde{b}_k)$ leads to an optimal satisfaction degree, namely $\lambda_e^* \in [\underline{\lambda}, \bar{\lambda}]$.

7 Concluding Remarks

We have presented and solved an extension of the fuzzy goal programming basic model proposed by Narasimhan [15], Yang [20], and Chen & Tsai [3] to a Interval Type-2 fuzzy environment, which includes linguistic uncertainty and numerical imprecision coming from multiple experts opinions and perceptions.

Our approach gives the model flexibility to find other kind of solutions in cases where the system has no the ability to fulfill all goals. As higher λ^* as closer to the goal the model is. $\bar{\lambda}$ and $\underline{\lambda}$ describe overall optimistic and pessimistic satisfaction degrees regarding different experts of the system.

There is a relationship among λ^* , d_{k1} and d_{k2} since as wider \tilde{b}_k as higher λ^* is, which means higher satisfaction values. In the first example if d_{k1} and d_{k2} are increased in a 50% then λ^* is increased only in 12%, and if d_{k1} and d_{k2} are decreased in a 50% then λ^* is decreased in 36% (LP formulations for 50% decreased d_{k1} , d_{k2} , $\bar{\lambda}$ and $\underline{\lambda}$ are shown in Appendix 2). Finally, all experts are satisfied into the range $[\underline{\lambda} = 0.64, \bar{\lambda} = 0.76]$.

The second example shows an embedded set A_e into $\text{FOU}(\tilde{b}_k)$ whose optimal satisfaction degree λ^* fits into the range $[\underline{\lambda}, \bar{\lambda}]$ as described in the Appendix 2. This helps decision making when having multiple experts and helps to see how different selections of A_e affect the problem.

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A Appendix 1

This appendix contains the results of the optimization process for the first Interval Type-2 fuzzy goals example.

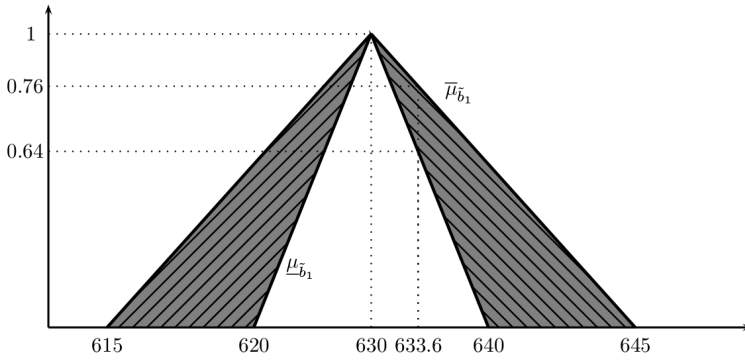


Figure 3. Interval Type-2 Fuzzy Goal 1

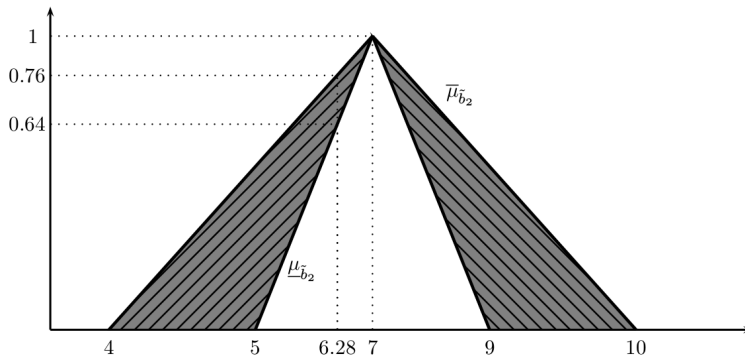


Figure 4. Interval Type-2 Fuzzy Goal 2

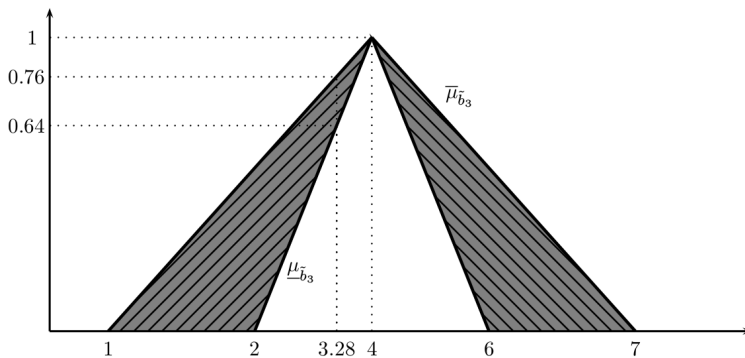


Figure 5. Interval Type-2 Fuzzy Goal 3

B Appendix 2

This appendix shows the model of the Type-1 example shown in Section 6 where $\underline{b}_{k1} = \underline{b}_{k2} = \{5, 1, 1\}$ and $\bar{b}_{k1} = \bar{b}_{k2} = \{10, 2, 2\}$:

$$\begin{aligned} & \max \underline{\lambda} \\ & \text{s.t.} \\ & \underline{\lambda} \leq -16x_1 - 8x_2 + 127, \\ & \underline{\lambda} \leq 16x_1 + 8x_2 - 125, \\ & \underline{\lambda} \leq -x_1 + 8, \\ & \underline{\lambda} \leq x_1 - 6, \\ & \underline{\lambda} \leq -x_2 + 5, \\ & \underline{\lambda} \leq x_2 - 3, \\ & \underline{\lambda}, x_1, x_2 \geq 0, \end{aligned}$$

$$\begin{aligned} & \max \bar{\lambda} \\ & \text{s.t.} \\ & \bar{\lambda} \leq -8x_1 - 4x_2 + 64, \\ & \bar{\lambda} \leq 8x_1 + 4x_2 - 62, \\ & \bar{\lambda} \leq -0.5x_1 + 4.5, \\ & \bar{\lambda} \leq 0.5x_1 - 2.5, \\ & \bar{\lambda} \leq -0.5x_2 + 3, \\ & \bar{\lambda} \leq 0.5x_2 - 1, \\ & \bar{\lambda}, x_1, x_2 \geq 0, \end{aligned}$$

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