Logicism Revisited

OTÁVIO BUENO
University of South Carolina

To Vanja, Vadá, Roque and Carol,
for all their help and support

Abstract

In this paper, I develop a new defense of logicism: one that combines logicism and nominalism. First, I defend the logicist approach from recent criticisms; in particular from the charge that a crucial principle in the logicist reconstruction of arithmetic, Hume's Principle, is not analytic. In order to do that, I argue, it is crucial to understand the overall logicist approach as a nominalist view. I then indicate a way of extending the nominalist logicist approach beyond arithmetic. Finally, I argue that a nominalist can use the resulting approach to provide a nominalization strategy for mathematics. In this way, mathematical structures can be introduced without ontological costs. And so, if this proposal is correct, we can say that ultimately all the nominalist needs is logic (and, rather loosely, all the logicist needs is nominalism).

1. Introduction

This paper provides a return to an old idea, logicism, from a new perspective, that of nominalism. Roughly speaking, according to the logicist (e.g. Frege [1884] or Russell [1903]), mathematical structures can be reduced to logic alone, and in this way, logic provides the basis for the reconstruction of the whole of mathematics. In Frege's view, the basic principle from which such a reconstruction is achieved (in the case of arithmetic) is provided by the claim that every concept has an extension (constituted by the objects that fall under that concept). Let's call this the *comprehension principle*. As

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is well known, the main problem faced by this proposal (at least as originally articulated by Frege) is that the proposed reconstruction of arithmetic turned out to be inconsistent, since it is open to Russell's well-known paradox. Thus, as Frege conceived it, logicism cannot get off the ground.

But is this conclusion really correct? I don't think so, and some recent works have actually challenged it. As Wright [1983] and Boolos [1998] have argued, Frege's approach *can be made consistent*. Frege's only use of the comprehension principle was to derive the so-called *Hume's Principle*. According to this principle, two concepts are equinumerous if and only if there is a one-one correspondence between them. And using Hume's Principle as the basic principle of logicism (together with the notions of concept, object and extension), Frege's approach to the foundations of arithmetic can be entirely reconstructed. In other words, the idea is to reject the comprehension principle, adopt Hume's Principle as the basic principle of the logicist view of arithmetic, and argue that Hume's Principle is analytic.

In this paper, I assess this overall approach to logicism. First, (a) I defend the approach from recent criticisms; in particular from the charge that Hume's Principle is *not* analytic. In order to do that, as will become clear, (b) it is crucial to understand the overall approach as a *nominalist* view. (c) I then indicate a way of supplementing the approach so that it can be extended beyond arithmetic. Finally, given (b), I argue that (d) a nominalist can use the resulting approach to provide a nominalization strategy for mathematics. In this way, mathematical structures can be introduced without ontological costs. And so if this proposal is correct, we can say that ultimately all the nominalist needs is logic.

2. Setting out the scene

2.1. The task at hand

The aim of this project, then, is to develop a new logicist approach that is compatible with nominalism. Initially, the idea is to extend the approach to accommodate at least arithmetic, but ultimately any reasonable nominalist view needs to accommodate substantial parts of mathematics, including set theory. This is a tall order, and one that wouldn't be reasonable to expect to be completed here. I am only concerned with indicating the main features of the approach, and motivating why, at least in principle, it can be done. In particular, what we need to achieve are the following tasks: (i) to emphasize that all that is required from logicism is that mathematical notions be *definable* from logical notions and other analytic concepts; (ii) to show that Hume's Principle is analytic (despite some criticisms to the effect that it isn't); and (iii) to establish that arithmetic notions can be defined in purely logicistic terms (that is, only in terms of logic and analytic definitions).

The characterization of logicism is no trivial matter.[1] Some accounts emphasize that logicism is tied to the notion of *provability* from logical and analytic principles (instead of definability). This is, of course, not an unreasonable account of logicism, in particular given the role played by proofs in mathematics. But this was the notion of logicism that many people (including Carnap) took to be refuted by Gödel's theorem. Even if this assessment turns out to be not completely justified, other things being equal, it would be better to have an account of logicism that is not immediately open to this charge.[2]

The main idea underlying the present proposal is that if mathematical notions can be characterized just in terms of logic and analytic definitions, such notions require no ontological commitment, and so they are nominalistically acceptable after all. As I will defend here, logicism, properly developed, provides an argument *for* nominalism about mathematics.

Of course, in the way developed by Frege and Russell (and, more recently, Hale and Wright [2001]), logicism is thoroughly platonistic, given that it presupposes concepts (Frege), classes (Russell), and abstract objects (Hale and Wright). What the present view aims to achieve is to preserve the 'content' (as it were) of the original logicist project without incurring into any ontological commitments to abstract entities. I take it that this nominalist reading of logicism is how logicism was read by one of its most interesting defenders: Carnap. Here is what I have in mind.

Even though Carnap has never been simply a logicist, his use of logicism—together with other nominalization strategies—has always been motivated by nominalistic considerations (see, in particular, Carnap [1934]). In particular, logicism is amenable to a nominalistic reading according to which if mathematical notions can be characterized only in terms of logic and analytic definitions, then mathematics as such doesn't require a commitment to abstract entities. It is this nominalistic reading of logicism that I emphasize here even though I will be making some moves that Carnap himself might not be happy with.

The main challenge facing the present proposal is to provide suitable replacements for Fregean concepts and Russellian classes—the two main reasons why traditional logicism is committed to platonism. Fregean concepts and Russellian classes are, of course, abstract entities, and while concepts and classes are certainly helpful for the logicist, they are not of much use for the nominalist, given that they are abstract. As I will argue below, a different framework is required, and it can be provided. Before doing that, let me first consider a preliminary issue.

2.2. What is logicism?

Logicism is intuitively presented as the claim that mathematics can be reduced to logic plus definitions. The notion of reduction here can be elaborated in three different ways (see Sainsbury [1979], pp. 272–4):

(a) Every mathematical truth can be expressed in a language whose expressions are logical (that is, whose expressions are formulated in terms of logic alone). In other words, all mathematical truths can be formulated as true logical propositions.

Note that a true logical proposition may not be a logical truth. For example, 'There are at least three objects' is clearly a true logical proposition: it is true, and it can be formulated in terms of logic alone. However, it is not a logical truth. A logical truth, as Russell stressed, is "true in virtue of its form" (Russell [1903], p. xvi), and one typically requires that it be necessarily true.

(b) Every true logical proposition, which is a translation of a mathematical truth, is a *logical truth.* (Given that a true logical proposition may not be a logical truth in general, it becomes clear that (b) is stronger than (a).)

(c) Every mathematical truth, after being formulated as a logical proposition, can be *deduced* from a small number of logical axioms and rules.

As Sainsbury indicates, logicism is often presented as the conjunction of (a) and (c). For example, in *The Principles of Mathematics*, Russell claims:

> All pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and [. . .] all its propositions are deducible from a very small number of fundamental logical principles. (Russell [1903], p. xx)

It is important to distinguish (b) and (c), at least with hindsight, given Gödel's incompleteness theorem (Sainsbury [1979], p. 273). Roughly speaking, no consistent logical system, which is strong enough to formulate arithmetic, allows the derivation of all mathematical truths. So the logicist who only asserted (a) and (c) will have to revise (c), and will allow that only a *substantial amount* of mathematical truths are derivable. (Let us call (c′) this revised version of (c).)[3] However, with (c′), the logicist is in no position to distinguish mathematical truths from true logical propositions that are *not* mathematical truths. In order to indicate that the mark of mathematical truth is *logical* truth, the logicist has to assert (b). And that is why he or she has to keep the three dimensions of logicism distinct. Ultimately, the logicist asserts (a), (b) and (c′).

There is a further aspect of logicism that will be crucial in the discussion that follows. With the conjunction of (a), (b) and (c′), the logicist can claim that mathematics can be reformulated as a deductive-definitional development of logic (see Bohnert [1975], p. 184). But this leaves open the issue of whether logic, and hence mathematics, is *analytic*.[4] If we add to (a)–(c′) the claim (d) that logic is indeed analytic, we have what Bohnert calls *strong logicism* (*ibid.*). Alternatively, if we only keep (a) plus the claim (d), we have what I will call *weak logicism*.

Now, Russell was certainly *not* a logicist in the strong sense. In his view, the axioms of infinity and choice were *not* logical truths. He took these principles as hypotheses (see Russell [1919]; and Sainsbury [1979], pp. 305–7). As a result, (b) can't be asserted in general, since there are translations of mathematical truths into a logical language (such as the axiom of infinity) that are *not* logical truths.

How about Frege? Was Frege a logicist? Well, if Frege weren't a logicist, who else would be? But it's *likely* that Frege was also not a *strong* logicist, and the reason for that has to do, once again, with the axiom of choice. As Heck points out in an illuminating paper (Heck [1998]), Frege discovered an axiom of *countable* choice, and realized that he was unable to establish it in his system.[5]

As Frege points out, in his review of Cantor's *Zur Lehre vom Transfiniten*:

> Mr. Dedekind gives us as the characteristic mark of the infinite that it is similar to a proper part of itself [...], after which the finite is defined as the non-infinite, whereas Mr. Cantor tries to do what I have done: first to define the finite, after which the infinite appears as the non-finite. Either plan can be carried through correctly, and it can be proved that the infinite systems of Mr. Dedekind are not finite in my sense. This proposition is convertible; but the proof of it is rather difficult. (Frege [1892], p. 180)

The reason for this difficulty is not hard to find. After all, as we now know, the proof of the result mentioned by Frege could only be carried out assuming the axiom of countable choice. As Heck indicates, Frege identified the axiom that would be required to formulate the proof. However, the axiom didn't seem to be something that one could simply accept as a principle of logic. Being unable to establish this axiom in his system, Frege considered three options: (i) *He could reject the axiom*. But the problem with this option is that if Frege were to reject the axiom he couldn't prove that Dedekind's notion of infinite was equivalent to his own. Alternatively, (ii) Frege considered the possibility of *accepting the axiom as a truth of some particular science*, taking it to be a *part of mathematics not reducible to logic*. But the problem with this option is that it would entail the rejection of logicism as a general approach to mathematics, given that the option explicitly recognizes the limitation of logicism. Finally, (iii) Frege considered the possibility of *accepting the axiom as a truth of logic*, and *deriving it from some logical principle*. The problem is that this was exactly what Frege was unable to do! Frege's inability to derive the axiom from some logical principle was exactly what led him to raise these three options.

The reason why I am mentioning these options is to illustrate the difficulty of properly characterizing Frege's logicism. One thing seems to be clear, though. Similarly to Russell, Frege wasn't a *strong* logicist. A strong logicist would clearly accept option (iii). If this were not a possibility, though, he or she would never even consider option (ii), which would amount to the rejection of (strong) logicism. In the worst case, a strong logicist might even accept option (i). The fact that Frege didn't seem to be able to decide among which option to accept (if any) clearly illustrates the difficulty of properly characterizing the nature of Frege's logicism. In any case, as I tried to indicate, whatever commitment to logicism Frege had, it wasn't a commitment to strong logicism, for the reasons just indicated. I take it to be more plausible to consider Frege as a weak logicist.

In what follows I will consider, as a working definition of logicism, the intuitive one mentioned above: logicism is the claim that mathematics can be reduced to logic plus analytic definitions. And for the purposes of the discussion that follows I will focus in particular on logicism in the weak sense. This will be enough for our present needs.

3. The logicist framework and its defence

3.1. The logicist framework

A immediate challenge emerges then: How could anyone even begin to think about defending something like Frege's logicism if Frege's attempted logicist reconstruction of arithmetic was inconsistent? This is, of course, the usual response to the project of returning to logicism. And the refutation of Frege's logicism is easy to show, or so the argument goes, given that Russell's paradox established that Frege's attempt to formulate arithmetic in logicist terms was inconsistent.

And if the account is inconsistent, it cannot be accepted. Although I disagree with the latter claim (even though I admit that this is controversial point), it's enough to indicate that a defense of Frege's proposal can be provided with far less controversial assumptions.

As is well known, the trouble maker in the case of Frege's logicism was Basic Law (V). This principle can be formulated, in a second-order language, in the following way:

$$\forall F \forall G({}'F = {}'G \leftrightarrow \forall x(Fx \leftrightarrow Gx)),$$

where F and G are concepts and '′' is the notation for the *extension of a concept*. So Basic Law (V) states that the extension of the concept F is the same as the extension of the concept G if and only if the same objects fall under the concepts F and G. As is easy to see, this immediately raises Russell's paradox if we consider the concept 'is not a member of itself'.

However, we can save Frege from contradiction by noting that Frege made a *limited* use of Basic Law (V). This is a point made by Wright in a seminal work (Wright [1983]; see also Boolos [1998]). The main idea is that Frege's only use of Basic Law (V) was to obtain Hume's Principle, namely, the claim that two concepts, F and G, are equinumerous (in symbols, '$\#F = \#G$') if and only if there is a one-one correspondence between them (i.e. '$F \approx G$'). Or formally:

$$\forall F \forall G(\#F = \#G \leftrightarrow F \approx G).$$ [6]

And by using Hume's Principle (together with second-order logic), it is possible to reconstruct arithmetic in a purely logicist way. I will provide some details about how this reconstruction goes shortly.

A crucial feature of Hume's Principle is that it can be expressed in pure second-order logic in such a way that the only additional term is the numerical operator '#':[7]

$$\begin{aligned}\forall F \forall G[\#F = \#G \quad \leftrightarrow \quad & \exists R[\forall x \forall y \forall z(Rxy \;\&\; Rxz \rightarrow y = z) \;\&\\ & \forall x \forall y \forall z(Rxz \;\&\; Ryz \rightarrow x = y) \;\&\\ & \forall x(Fx \rightarrow \exists y(Gy \;\&\; Rxy)) \;\&\\ & \forall y(Gy \rightarrow \exists x(Fx \;\&\; Rxy))]].\end{aligned}$$

In this way, it becomes clear that Hume's Principle may not be as contentious as it may initially seem to be—in particular for those who consider second-order logic to be *logic* (see Bueno [1999] and Shapiro [1991]).

To give a "taste" of how arithmetic can be developed using Hume's Principle as the sole extra-logical axiom, consider the following definitions and theorems (where '#' is the numerical operator and '[ ]' stand for concepts).[8]

Definition 1. $0 = \#[x : x \neq x]$. (That is, 0 is the number of the concept *being non-self-identical.*)

From this natural definition, we can easily derive that the number of a concept F is 0 if and only if there are no objects that follow under that concept:

Theorem 1. $\#F = 0 \leftrightarrow \forall x \neg Fx$.

Proof. Since $0 = \#[x : x \neq x]$ (definition 1), from Hume's Principle it follows that $\#F = 0$ if and only if $F \approx [x : x \neq x]$. Given that $\forall x \neg x \neq x$, $F \approx [x : x \neq x]$ iff $\forall x \neg Fx$.

Once 0 has been defined, we can then define 1 as follows:

Definition 2. $1 = \#[x : x = 0]$. (That is, 1 is the number of the concept *being identical with the number* 0.)

The following simple theorem then follows immediately:

Theorem 2. $0 \neq 1$.

Proof. Since $1 = \#[x : x = 0]$ (definition 2), 0 is the only object that falls under the concept 1. Given that there is exactly one object falling under the concept 1, and none falling under the concept 0, these two concepts are *not* in one-one correspondence. By Hume's Principle, their numbers, 1 and 0, are not identical.

Having proved that 0 and 1 are different numbers, we can then define 2:

Definition 3. $2 = \#[x : x = 0 \vee x = 1]$. (That is, 2 is the number of the concept *being identical with* 0 *or* 1.)

The definitions and theorems can then continue to be developed in the obvious way. Moreover, it is also possible to define addition in the language under consideration:

Definition 4. *Let* $\#F = m$ *and* $\#G = n$, *and suppose that* $\neg\exists x(Fx \wedge Gx)$. *In this case,* $m + n = \#[x : Fx \vee Gx]$. (In other words, if the number of Fs is m, the number of Gs is n, and no object falls under both of the concepts F and G, then $m + n$ is the number of Fs or Gs.)

The basic facts about addition can be established using the above definition. Multiplication can also be defined, and all natural numbers can be characterized. In fact, all the rest of arithmetic can be carried out just on the basis of Hume's Principle and logic. In this way, we obtain what has been called 'Frege's Theorem' (see Boolos [1998], and Hale and Wright [2001]):

Theorem 3 (Frege's Theorem) *Second-order logic plus Hume's Principle as the only additional axiom is enough for the derivation of second-order arithmetic.*

As a result of Frege's Theorem, at least a weak version of logicism can be articulated along strictly Fregean lines. In other words, there is no need to assume the inconsistent Basic Law (V). The crucial work is done by Hume's Principle (and second-order logic, of course).

However, for the resulting view to qualify as a logicist proposal, the status of Hume's Principle becomes crucial. If the principle is a substantial (say, mathematical) assertion, there is no hope to defend the resulting view as a version of logicism. The 'logicist' would then be simply reducing one part of mathematics to another. In other words, unless Hume's Principle is analytic, there is no hope for logicism to get off the ground. The crucial difficulty then is to determine whether Hume's Principle is *analytic* or not.

3.2. Is Hume's Principle analytic?

Boolos [1998] provided a forceful answer to this question. In his view, Hume's Principle is *not* analytic. This means that, despite Frege's Theorem, logicism ultimately cannot be made to work. After all, if Hume's Principle is *not* analytic, Frege's Theorem won't establish that second-order arithmetic could be reduced to logic and analytic definitions alone—arithmetic has presuppositions that go *beyond* logic. To support his challenge, Boolos provides two important arguments:

A. *The argument from content and ontological commitment.* Boolos invites us to consider Frege's Theorem for a moment. What the theorem establishes is that Hume's Principle entails the existence of infinitely many objects (namely, natural numbers). The challenge then is: How can any such principle be considered *analytic*? As Boolos points out:

> Frege Arithmetic is equi-interpretable with full-second-order arithmetic, 'analysis', and hence equi-consistent with it. [...] My worry about content is that HP [Hume's Principle], when embedded into axiomatic second-order logic, yields an incredibly powerful mathematical theory. (Boolos [1998], p. 304)

The trouble here is that if we take analytic principles to be content-free—in the style of simple claims, such as 'All bachelors are unmarried'—then it seems implausible that Hume's Principle qualifies as an analytic statement. It simply has too much content.

Moreover, since the theory that results from Hume's Principle is so powerful, a serious worry about the theory's consistency emerges. This is the point of Boolos's second argument:

B. *The consistency argument.* How can we know that the arithmetic yielded by Hume's Principle (also called 'Frege Arithmetic') is consistent? According to Boolos:

> [...] (it is *not* neurotic to think) we don't *know* that second-order arithmetic, which is equi-consistent with Frege Arithmetic, is consistent. Do we really know that some hotshot Russell of the 23rd

> Century won't do for us what Russell did for Frege? The usual argument by which we think we can convince ourselves that analysis is consistent—"Consider the power set of the set of natural numbers..."—is flagrantly circular. Moreover, although we may think Gentzen's consistency proof for PA provides sufficient reason to think PA consistent, we have nothing like a similar proof for the whole of analysis, with full comprehension. We certainly don't have a constructive consistency proof for ZF. And it would seem to be a genuine possibility that the discovery of an inconsistency in ZF might be refined into that of one in analysis. [...] Uncertain as we are whether Frege Arithmetic is consistent, how can we (dare to) call HP [Hume's Principle] analytic? (Boolos [1998], p. 313)

Boolos's second argument seems to presuppose the following condition for the analyticity of a given principle: If we know that P is analytic, then we know that P is consistent. Given that we don't know whether this is the case with Hume's Principle, we don't know that the principle is analytic—or, at least, unless we know that Frege Arithmetic is consistent, we cannot claim that we know that Hume's Principle is analytic.

Both of Boolos's arguments are important and intriguing, and if successful, they would raise doubts about the analyticity of Hume's Principle. I don't think, however, that the worries the arguments raise are decisive. It is still possible to defend logicism against Boolos's challenge.

A′. *Resisting the argument from content and ontological commitment.* The claim that Hume's Principle cannot be analytic—given that it entails the existence of infinitely many objects—can be responded. After all, as a worry about content and ontological commitment, this is only a problem for the *platonist*. It is the platonist who insists that, in doing arithmetic, we are ontologically committed to a domain of abstract objects (natural numbers), and that, given Hume's Principle, the content of arithmetic requires the existence of infinitely many such objects.

However, as I will argue below, with a *nominalist* reading of Hume's Principle, the issue about content and ontological commitment simply vanishes. If Hume's Principle doesn't entail the existence of infinitely many abstract entities, it is far more plausible to argue that it is indeed analytic. (Of course, to recapture arithmetic as we know it, an infinity of objects will be required. But, as we will see, such objects are not abstract, according to the nominalist.)

B′. *Resisting the consistency argument.* Boolos's challenge here, as we saw, is that we don't know that Frege Arithmetic is consistent (recall that Frege Arithmetic is equi-consistent with second-order arithmetic). Moreover, the knowledge of the consistency of Frege Arithmetic is required to assert the analyticity of Hume's Principle. Boolos's argument then crucially depended on the claim that "(it is *not* neurotic to think) we don't *know* that second-order arithmetic [...] is consistent" (Boolos [1998], p. 313).

In response, I would insist that it *is* neurotic to think that second-order arithmetic may turn out to be inconsistent—particularly using Boolos's argument that connects the (possible) inconsistency of Frege Arithmetic with the (possible) inconsistency of ZF. After all, there is simply *no* evidence to the effect that ZF and second-order arithmetic are inconsistent—quite the contrary. The possibility that "some hot-shot Russell of the 23rd Century [will] do for us what Russell did for Frege" is far more remote than the experience, shared throughout the mathematical community by almost one century now, of using ZF and second-order arithmetic without the discovery of any inconsistency. The lack of a Gentzen's type consistency proof for second-order arithmetic or of a constructive consistency proof for ZF seems of little concern when we are confronted by one hundred years of successful use of these two theories—again with no inconsistency. For these reasons, the claim that "we don't *know* that second-order arithmetic is consistent" is simply unsubstantiated. And without this claim, the consistency argument cannot be accepted.[9]

I'm not claiming that it is a simple matter to establish the consistency of ZF or of analysis—of course it's not. My only point here is that Boolos's didn't provide a case to cast genuine doubt on the consistency of these theories. And this is enough to resist his criticism.

3.3. The meaning of analyticity

But in what sense can we say that Hume's Principle is analytic? For the purposes of the present discussion, an intuitive understanding of analyticity will do. For example, a principle is analytic if it is true in virtue of the meaning of its constitutive terms. This is *not* a definition of analyticity; it's only *one way* of indicating how the term is being used. In any case, this seems to be the working notion of analyticity that Boolos himself uses in his discussion (see Boolos [1998]). So let us grant, for the sake of the argument, that the notion is clear enough (at least for our present purposes).

Having granted this point, can we identify a *criterion of analyticity*? What we are looking for is a criterion that would help us decide whether a given sentence is analytic or not. Here is a possible candidate:

(C) A sentence *S* is analytic if and only if the negation of *S* is inconsistent.

Although (C) is far from perfect, it does have one interesting feature. A number of mathematical axioms don't seem to be analytic—and the above criterion doesn't classify them as such. After all, the negation of such axioms is typically *not* inconsistent. Consider, for example, the fifth postulate of Euclidean geometry, the axiom of choice, and the axiom of foundation in set theory. None of them seems to be analytic—and none of them is, according to (C). For there are non-Euclidean geometries, set theories in which the axiom of choice fails, and non-well-founded set theories. I take it that these examples provide support for (C).

Two caveats, though: First, if some mathematical axioms are not analytic, given that their negation is not inconsistent, what is the status of mathematical *theorems*? Of course, the negation of a mathematical *theorem* is always inconsistent—in the system in which the theorem was established. So, to make sense of criterion (C), it's important to separate the role played by axioms (in a mathematical theory) from that played by theorems—even though the distinction between axioms and theorems is not absolute. (After all, we can always consider the theorems of a mathematical theory as axioms of another theory.)

Second caveat: Criterion (C) can only be adequately applied after the underlying logic has been determined. The notion of inconsistency presupposed in (C) depends, of course, on the logic used.

Some logics may validate inferences that are not valid in other logics, and so inconsistencies obtained in one logical system may not be obtained in others. Rather than indicating a weakness of criterion (C), the dependence on logic is actually the strength of the criterion. Analyticity, being a linguistic notion, depends on the logic one adopts—and that's exactly what (C) highlights.

The crucial question now is to determine whether Hume's Principle satisfies criterion (C). Well, this seems to be the case—at least in the context of second-order logic. Suppose that Hume's Principle fails in second-order arithmetic. That is, suppose that there is a one-one correspondence between concepts *F* and G, but *F* and G are *not* equinumerous. If this is the case, then *there is* a nonstandard model of arithmetic in which, despite the one-one correspondence between *F* and G, there are, say, more *F*s than Gs. But, as is well known, there are *no* non-standard models in second-order arithmetic (see, e.g., Shapiro [1991] for details). So the negation of Hume's Principle clearly yields an inconsistency, and criterion (C) is satisfied.

Of course, I don't offer the above argument as a conclusive proof of the analyticity of Hume's Principle. It's not. The argument only indicates one road that could be taken to *motivate* the analyticity of the principle. And if this motivation works, perhaps it's not so implausible to defend the idea that Hume's Principle may be analytic after all.

4. Toward a nominalist logicism

Two questions should be considered at this point: (a) Is it possible to provide a nominalist reading of Hume's Principle? Is it possible to put forward a nominalist reading of logicism? (b) Why should one provide such readings? In response to (b), the motivation comes from the fact that, as mentioned above, the analyticity of Hume's Principle seems to *depend* on a nominalist reading of the principle. This is the point I made to resist Boolos's argument from content and ontological commitment. But to be able to make this defense one needs to carry out a positive answer to (a). In what follows I will outline a strategy to do that.

The major difficulty to develop this project—namely, to provide a nominalist reading of Hume's Principle—comes from the following challenge posed by Hale and Wright [2001]. Consider the formulation of Hume's Principle in second-order logic given above:

$$\begin{aligned}\forall F \forall G[\#F = \#G \quad \leftrightarrow \quad & \exists R[\forall x \forall y \forall z(Rxy \,\&\, Rxz \rightarrow y = z) \,\&\\ & \forall x \forall y \forall z(Rxz \,\&\, Ryz \rightarrow x = y) \,\&\\ & \forall x(Fx \rightarrow \exists y(Gy \,\&\, Rxy)) \,\&\\ & \forall y(Gy \rightarrow \exists x(Fx \,\&\, Rxy))]].\end{aligned}$$

The trouble here, as Hale and Wright point out, is that

> The nominalist must find a [. . .] reconstrual of the surface syntax of the left-hand [side of Hume's Principle] which avoids discerning any reference to or quantification over *abstracta* there, but does construe [its meaning] as compositional, and does sustain the [equivalence]. It is very doubtful if any such account is possible. (Hale and Wright [2001], pp. 354–5)[10]

To *resist this challenge*, it is crucial to take the *right*-hand side of Hume's Principle seriously, and explore the resources of the underlying second-order logic. As discussed already, the right-hand side of Hume's Principle is a pure second-order sentence, stating the existence of a one-one correspondence R between F and G. The question here is: How should we understand the quantification over the function R? Isn't R an abstract object after all?

To answer this challenge, it becomes crucial to provide a nominalist reading of the second-order quantification over R. (It's ironic that part of the nominalization to be suggested here comes from Boolos's own work, who *challenged* the analyticity of Hume's Principle to begin with!) As Boolos clearly indicated (in his [1984] and [1985]), monadic second-order quantification can be understood in terms of *plurals*.[11] Boolos's point is that the ontological commitment of this logic does not go beyond that of *first-order* logic, given the introduction of plural quantifiers. In other words, quantification over monadic predicates (such as 'is a critic') can be seen as a counterpart of a plural quantification in natural language. So, instead of understanding the monadic second-order existential quantifier, $\exists X$, as 'there is a class', in Boolos's view, it can be read as the natural language plural quantifier 'there are (objects)'. For instance, the well-known Geach-Kaplan sentence, 'Some critics admire only one another', despite not being equivalent to any first-order sentence,[12] can be straightforwardly symbolized in second-order logic.[13] However, Boolos stresses ([1984], p. 449), in doing so we do not commit ourselves to the existence of additional items beyond those to which we are already committed. In order to understand this sentence (and, more generally, in giving a semantics for monadic second-order logic), we do not have to postulate, in addition to critics, a class of critics (or of whatever other objects we might be concerned with), and so our ontological commitments do not go beyond those of first-order logic.[14]

The plural interpretation can be extended to the *dyadic* case (the case that is needed to interpret Hume's Principle) by adopting an elegant proposal developed by Burgess, Hazen and Lewis (in the appendix of Lewis [1991]). The idea is to use the framework articulated in terms of plural quantification and mereology (the study of the *part-whole* relation), and by mimicking a paring function in terms of mereological atoms, to obtain the resources of full second-order quantification on nominalist grounds.[15] In this way, the strength of full second-order quantification can be obtained in a way that is still compatible with nominalism. And so it is possible to interpret Hume's Principle without unacceptable ontological commitments.

The crucial idea of the present proposal is that, instead of quantifying over *concepts* (which are abstract entities), the nominalist logicist will be quantifying over *mereological atoms*—items that are, of course, nominalistically acceptable. But the following worry may be raised: can we really say that we are reducing arithmetic to logic if we are quantifying over mereological atoms? In what sense is arithmetic really analytic if ultimately we quantify over such atoms?

The problem with this consideration is that if sound, it would entail too much. After all, this sort of consideration about logicism would establish that not even Frege's original project (if made consistent) could be characterized as logicist. For if we are quantifying over *concepts*, and if concepts are not what arithmetic is about, how can we really be logicist about arithmetic? On Frege's proposal, we are presupposing a domain of concepts as part of the logical construction of arithmetic. But it may be argued that this is not what arithmetic is about. In other words, if the original worry about using a mereological setting to develop a nominalist logicism is warranted, one could easily transform this complain into a worry about whether Frege's original approach was logicist at all. And this conclusion is just way too strong.

In any case, it's important to note that the use of plural quantification and mereology as a framework to nominalize Hume's Principle is only the first step of the nominalization process. After all, the only thing accomplished by such a move is to alleviate the ontological commitments made at the right-hand side of Hume's Principle. What about the *left-hand* side? (Recall that this was exactly Hale and Wright's challenge.) The idea here, again, is to quantify over mereological atoms—as a replacement for concepts. Talk of number in the right-hand side is then replaced by talk of mereological notions. In this way, even first-order quantifications will not be ontologically committing to abstract entities (only to mereological entities, which are nominalistically acceptable).

Given this proposal, what should we say about the analyticity of Hume's Principle? As noted above, it's important to recognize that the analyticity of this principle doesn't depend on certainty. A principle is analytic, as indicated above, if it is true only in virtue of the meaning of its constitutive terms. Once this point is recognized it's easy to understand the assumption underlying Boolos's criticism (of the analyticity of Hume's Principle). Boolos's worry is that a substantial amount of mathematical knowledge seems to be presupposed to determine the truth of Hume's Principle, and so it is difficult to maintain that the latter principle is analytic.

One of the motivations to develop the present nominalist version of logicism is exactly to overcome this worry. Coupled with the above nominalization strategy, it becomes clear that Boolos's worry is not justified. After all, given the above strategy, it is simply *not* the case that one presupposes a substantial amount of mathematical knowledge to defend the analyticity of Hume's Principle. This is the whole point of introducing second-order mereology (and plural quantification) into logicism. After all, the present proposal insists that the resources introduced are nominalistically acceptable, and they allow one to understand how arithmetic can be reconstructed without presupposing abstract entities. In this way the content and the ontological commitments of arithmetical theories are well within the reach of what the nominalist accepts.

It might be argued that neo-Fregeans, such as Hale and Wright [2001], could easily overcome Boolos's worry. Neo-Fregeans insist that the use of an abstraction principle (such as Hume's Principle) to reconstruct arithmetic doesn't raise any worry on the ontological and epistemological fronts. As they point out, the objects thus introduced, although arguably abstract, have no mysterious or problematic features (see Wright [2001]). Mathematical objects are introduced as the objects that satisfy abstraction principles, and in this way their introduction is quite straightforward. These objects are, of course, abstract. But the simplification of the resulting epistemology of mathematics is considerable, since our knowledge of these objects is now simply obtained via the introduction of appropriate abstraction principles. And this is a perfectly simple and well-understood procedure, according to the neo-Fregean. So what's the problem?

This neo-Fregean move provides an ingenious combination of a platonist ontology with a "weak" epistemology, in the sense that no substantial assumptions about mathematical knowledge seem to be required for the proposal to work—or so claims the neo-Fregean. Undoubtedly, this is an interesting feature of the neo-Fregean approach. It is also a feature that the proposal has in common with other recent platonist views, in particular with Balaguer's full-blooded platonism (see Balaguer [1998]). In the case of Balaguer's proposal, every consistent mathematical theory is true of some region of the mathematical realm. In this way, although the metaphysics of mathematics is still composed of abstract entities, the epistemology of mathematics is characterized in terms of knowledge of the consistency of mathematical theories. If these two proposals (the neo-Fregean and the full-blooded platonist ones) could be made to work, they would certainly represent a substantial simplification of mathematical epistemology for the platonist.

But I don't think these proposals can be made to work. The overall problem they face is that knowledge of the consistency of mathematical theories represents a *substantial* kind of mathematical knowledge, in particular if we are considering non-trivial mathematical theories, such as ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) and second-order arithmetic ("analysis"). To establish the consistency of these theories is no simple matter, and it can only be done by supposing the consistency of theories that are way stronger and far more controversial (in the case of ZFC, for example, hypotheses about inaccessible cardinals need to be introduced).

This *doesn't* mean, of course, that we can't have any knowledge of the consistency of mathematical theories. My only point here is that, as opposed to what Balaguer, Hale and Wright seem to suggest, knowledge of the consistency of the above mathematical theories *is* a *substantial* piece of *mathematical* knowledge. After all, to establish the consistency of such mathematical theories we need to construct convenient models of set theory (or analysis), and the existence of such models is no trivial fact. In this way, it's not clear that Balaguer and the neo-Fregeans actually provide a proper defense of a platonist epistemology. The former presupposes that one establishes the consistency of ZFC; the latter presuppose the analyticity of claims resulting from abstraction principles—claims that, as Hale and Wright acknowledge, involve reference to abstract entities, and thus have a substantial mathematical content. No trite assumptions!

But no such problems are faced by the present proposal. As indicated above, abstract entities are not presupposed by the present proposal, and so the reconstructed theories are not committed to such entities nor have they *mathematical* content. So, from the viewpoint of the nominalist logicist, the use of abstraction principles, such as Hume's Principle, is neither epistemologically nor ontologically contentious. Moreover, as opposed to full-blooded platonism, to establish the consistency of mathematical theories plays no *epistemological* role in the present proposal. Mathematical epistemology emerges as the outcome of the framework sketched above, where the crucial role is played by logic and mereology in the reconstruction of arithmetic.

5. How far can we go?

Even if it is granted that the nominalization strategy outlined above can be made to work, a pressing question still remains: how far can this nominalization strategy go? Using the above strategy, can we nominalize other mathematical theories? For example, can we nominalize real analysis, functional analysis, topology, and set theory? To answer these questions, let me indicate two possibilities:

(a) If the neo-Fregeans *can* provide logicist reformulations of mathematical theories (see Hale and Wright [2001] for the case of real analysis), the nominalist logicist *also* can. That's the point of the parasitic strategy suggested here! The nominalist logicist simply reinterprets the outcome of the neo-Fregean reconstruction in terms of plural quantification and mereology.

(b) If the neo-Fregeans *cannot* provide logicist reformulations of such theories, the nominalist logicist still has an option: *reverse mathematics*. This is a program initially conceived by Harvey Friedman, and which he and his collaborators have been developing for a number of years (for an excellent overview, see Simpson [1998]). Just as second-order arithmetic can be derived from Hume's Principle in second-order logic (Frege's Theorem), Hume's Principle can be derived back from second-order arithmetic. This derivation of axioms from theorems is the crucial feature of *reverse mathematics*.

The point of using reverse mathematics in this context is this: reverse mathematics helps one to determine what are the minimum assumptions required to establish a given result. Once these assumptions become clear, the nominalist logicist can use the information obtained to generate abstraction principles (similar to Hume's Principle in the case of arithmetic). These abstraction principles are then used to provide the relevant reconstruction of the target mathematical theory. Once this is done, the nominalization strategy discussed above can then be applied to the abstraction principles employed, which shows that the ontological commitments in question are nominalistically acceptable.

Of course, I'm only indicating here a schema to implement the suggested nominalization strategy. This is only the first step—but a necessary one—to develop a stable version of logicism. Once the step is taken, the details need to be fleshed out, which is something I plan to do in future works.

6. Conclusion

Where do things stand now? A defense of a framework for a nominalist-logicist view of arithmetic—and possibly other mathematical theories—was provided. If the framework suggested here can be implemented, arithmetical structures can be introduced without ontological costs, since no quantification over abstract entities is required. And so we can say that ultimately all the nominalist needs is logic—logic, logic, and logic![16]

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Otávio Bueno
Department of Philosophy
University of South Carolina
Columbia, SC 29208, USA
obueno@sc.edu

Notes

[1] For an interesting discussion, see Sainsbury [1979]. I will briefly address the issue in the next section.
[2] Of course, there is the related charge that not every notion may be definable in a given system. But if this is the case of *every* mathematical notion *vis-à-vis* logical and analytic principles, then logicism loses much of its motivation.
[3] Of course, (c′) is weaker than (c), but despite being somewhat vague, it is strong enough to be taken as a logicist proposal. After all, we are still able to claim that the mathematical propositions derived from the logicist's stock are *logical* propositions. With hindsight, Gödel's incompleteness result shouldn't be surprising for the logicist. After all, due to Gödel's result, the logicist's crucial tool for the reformulation of mathematics—higher-order logic—is ultimately incomplete. As is well known, Frege used *second-order* logic in his logicist program whereas Russell employed *type theory*, both of which are incomplete in the standard semantics. (Of course, they become complete if we introduce Henkin models; see Shapiro [1991].)
[4] It should be noticed that, at least until 1903, Russell thought that logic was *synthetic*. In his view, given that Kant established that arithmetic is synthetic, and given Frege's logicist program of reducing arithmetic to logic, we should conclude that logic is also synthetic, rather than that arithmetic is analytic. As he points out in *The Principles of Mathematics*: "Kant never doubted for a moment that the propositions of logic are analytic, whereas he rightly perceived that those of mathematics are synthetic. It has since appeared that logic is just as synthetic as all other kinds of truth" (Russell [1903], p. 457; see also Russell [1912], Chapter VIII; for a discussion, see Dreben [1990], pp. 86–7, and 93, note 67).
[5] In the following discussion, I draw on Heck's thoughtful 1998 paper.

[6] It was Boolos who first introduced the term 'Hume's Principle' to refer to the above second-order sentence (see Boolos [1998]). Despite some vehement protests (Dummett [1998], pp. 386–7), the name is due to an interesting passage from *The Foundations of Arithmetic*, in which Frege attributes to Hume a way of determining identity among numbers that closely resembles Hume's Principle:

Hume long ago mentioned such a means [to determine identity among numbers]: "When two numbers are so combined as that the one has always a unit answering to every unit of the other, we pronounce them equal" (*Treatise*, Bk. I, Part iii, Sect. I). This opinion, that numerical equality or identity must be defined in terms of one-one correlation, seems in recent years to have gained widespread acceptance among mathematicians. (Frege 1884, pp. 73–4)

According to Dummett, Frege was making little more than a joke when he wrote the above passage (Dummett [1998], p. 386). Whether this is the case or not, the terminology is now widespread, and I'll be using it in what follows.

[7] This point is made over and over again in the literature (see, e.g., Boolos [1998] and Heck [1998]).

[8] I'm following here Boolos's presentation (see Boolos [1998]). For further details, see Boolos [1998], Heck [1993], Hale and Wright [2001], and the papers collected at Demopoulos (ed.) [1995] (especially Parts II and III), and at Schirn (ed.) [1998] (especially Part IV).

[9] It's important *not* to confuse analyticity with certainty, although some points raised by Boolos *may suggest* such a confusion—in particular when he tied his worries about the analyticity of Hume's principle to our lack of *certainty* as to whether second-order arithmetic is consistent. This is, for example, the way in which Wright reads Boolos's worry (see Wright [2001]). Having said that, I don't think that Boolos himself was committed to the claim that analyticity entails certainty (even though some of his claims may suggest that much). The issue Boolos raised was clear enough: how can a notion whose consistency we haven't been able to show be deemed analytic? The point hasn't to do with certainty, but with the perhaps not unreasonable assumption—given the parties involved in the debate—that analytic statements should be shown to be consistent. But as I tried to indicate above, I don't think that Boolos's worry is justified.

[10] It should be noted that Hale and Wright's point is not restricted to Hume's Principle, but it applies to *any* abstraction principle used as a device to reconstruct mathematics on logicist grounds. Hume's Principle is, of course, the abstraction principle adopted in the case of arithmetic, and

given that it is the focus of our current discussion, I changed Hale and Wright's quotation accordingly.

[11] Monadic second-order logic is a second-order logic that allows quantification only over *monadic* predicate variables.

[12] Kaplan's proof of this fact is presented in Boolos [1984], pp. 432–3.

[13] Supposing that the domain of discourse consists of critics, and *Axy* means '*x* admires *y*', the Geach-Kaplan sentence becomes: $\exists X(\exists x Xx \wedge \forall x \forall y((Xx \wedge Axy) \rightarrow x \neq y \wedge Xy))$.

[14] Boolos's idea is that the informal metalanguage, in which we give the semantics for (monadic) second-order logic, contains the plural quantifier 'there are objects', which is then used to interpret the second-order monadic quantifiers. (For further discussion, see Boolos [1998], and Shapiro [1991], pp. 222–6.)

[15] For details about the framework, see Lewis [1991]. A thorough discussion of mereology is presented in Simons [1987].

[16] An earlier version of this paper was presented at the Second International Conference Principia (held in Florianópolis, August 6–10, 2001). I wish to thank the audience for really stimulating discussions. In particular, comments made by Tom Baldwin, Oswaldo Chateaubriand, Mark Colyvan, Newton da Costa and Décio Krause were especially helpful.