Russell's Early Type Theory and The Paradox of Propositions

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Abstract

The paradox of propositions, presented in Appendix B of Russell's The Principles of Mathematics (1903), is usually taken as Russell's principal motive, at the time, for moving from a simple to a ramified theory of types. I argue that this view is mistaken. A closer study of Russell's correspondence with Frege reveals that Russell came to adopt a very different resolution of the paradox, calling into question not the simplicity of his early type theory but the simplicity of his early theory of propositions.

1. Introduction

When Russell published his logicist treatise *The Principles of Mathematics* (1903) he had to leave the issue of the paradoxes largely unresolved. Only in Appendix B of that work did he make an attempt at resolving the paradox of the Russell class. These rather sketchy remarks appended to the *Principles* are the first statement in print of a simple theory of types.

It seems that Russell was convinced that in Appendix B he had found the right kind of approach to the paradoxes. But he was also painfully aware that the theory outlined there would have to be modified considerably. Although his first, simple version of type theory does indeed suffice to block the reasoning that leads to the paradox of the Russell class, new problems arise. The problems culminate in the paradox of propositions. This is a problem that appears to run in exact parallel to the paradox of the Russell class. It seemed therefore desirable to Russell that a single solution to both paradoxes be

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found. Since the simple theory of types (ST) does not offer such a solution it is commonly believed that the paradox of propositions was Russell's principal motive—at least at the time when he had just finished writing the *Principles*—for searching for and eventually formulating a ramified theory of types (RT). Russell commentators unanimously agree—to the best of my knowledge—that if a single *movens* from a simple to a ramified theory of types is to be identified in Russell's work, then it would have to be the paradox of propositions, also known as the Appendix B paradox. I shall argue below that this is a myth. There is no textual evidence in its favour, there is significant circumstantial evidence against it, and, perhaps most decisively, Russell's last words on the matter (in a letter to Frege) explicitly indicate a solution that has nothing to do with ramification.

In the next section I shall present the very first version of ST as it occurs in the *Principles*. I shall state the paradox of propositions and explain why it cannot be resolved within ST. Then I shall recount the various ways in which Russell tried to find a solution to the paradox of propositions which would run in parallel to his solution to the class paradox.

Next I turn to the Russell-Frege correspondence of 1902 and 1903. Apart from Appendix B this is the only place in Russell's writings in which the paradox is mentioned. It turns out that the paradox was one of the principal topics in the correspondence between Frege and Russell during these years. I shall reconstruct Frege's suggestion as to how the paradox may be resolved and show that it is unsuccessful. Finally I shall describe Russell's own, rather unspectacular solution.

In conclusion I shall argue that although RT *can* be used to block the paradox of propositions, this is is not the solution Russell came to adopt. The paradox of propositions does not mark a point of transition from a simple to a ramified theory of types. Its significance in Russell's *œuvre* lies elsewhere. Russell came to realize that the paradox was rooted in his largely unarticulated theory of propositions. Though wishing to agree with Frege that propositions are not linguistic entities, Russell had adopted in practice a rather syntactic criterion for individuating propositions. As a consequence of the paradox the problem of propositions moved up on his philosophical agenda and remained an important concern for many decades to come.

2. The very first theory of types

In Appendix B of *The Principles* Russell sketches for the first time in print his doctrine of types as follows

> Every propositional function ϕx—so it is contended—has, in addition to its range of truth, a range of significance, i.e. a range within which x must lie if ϕx is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second is that ranges of significance form types, i.e. if x belongs to the range of significance of ϕx, then there is a class of objects, the type of x, all of which must also belong to the range of significance of ϕx, however ϕ may be varied; and the range of significance is always either a single type or a sum of several whole types. (523)

Russell immediately adds that the second point "is less precise than the first". The remainder of the appendix is concerned with explaining in more detail the shape of the required theory, how it blocks the class paradox and why it cannot be considered a successful attempt at resolving the paradoxes in general. In fact what Russell presents in Appendix B is not specific enough to be called a type theory. Instead Russell lists a number of conditions on any theory intended to resolve the paradoxes by stratifying expressions according to logical types in an as yet rather loose sense. Russell emphasizes more than once that this is all very tentative and in the end he will point out that the conditions stated do not delineate an adequate framework for resolving the problems at hand.

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Let D be a given domain of discourse which provides the possible values for propositional functions. Russell's basic idea is a simple one. Although all items in the domain may be grammatically of the right kind to fill an argument place in a propositional function, not all such fillings make good logical sense. Example: the class of teaspoons in the drawer is an individual and individuals may be said to have properties. Thus, where ϕ is any property, we may ask whether the class of teaspoons in the drawer possesses ϕ or not. But there is room for disagreement here. Consider for example the property of being made of metal. It may well be questioned whether it makes sense to ask whether a class is made of metal or not. The question may be asked with respect to each individual member of the class of teaspoons but not as regards the class "as one", as Russell says. Teaspoons but not classes of teaspoons fall within the range of items of which the property of being made of metal can significantly be asserted. Thus with each propositional function ϕ we associate

> a range of *truth*, True(ϕ),
> a range of *falsehood*, False(ϕ), and
> a range of *significance*, Sign(ϕ).

These concepts are subject to a number of constraints, some of them so obvious, that Russell does not mention them explicitly. Among them are, first, that truth and falsity ranges are mutually exclusive; second, that the two exhaust the range of significance; third, that the latter must be part of the given domain; and, fourth and fifth, that the falsity range of ϕ is the truth range of $\neg\phi$ and that the truth range of ϕ is the falsity range of $\neg\phi$.

A. $\mathrm{True}(\phi) \cap \mathrm{False}(\phi) = \varnothing$
B. $\mathrm{True}(\phi) \cup \mathrm{False}(\phi) = \mathrm{Sign}(\phi)$
C. $\mathrm{Sign}(\phi) \subseteq D.$
D. $\mathrm{False}(\phi) = \mathrm{True}(\neg\phi)$
E. $\mathrm{True}(\phi) = \mathrm{False}(\neg\phi)$

As an immediate consequence of conditions (B) and (D–F) we note that any value significant for a given propositional function is also significant for the negation of that function, i.e.

$$\mathrm{Sign}(\phi) = \mathrm{Sign}(\neg\phi).$$

The task of a theory of types is to systematically construct the respective ranges of significance of propositional functions. Let us here restrict attention to the monadic case, i.e. the case of propositional functions of one free variable only.

At the ground floor level there is the type of atomic or simple items, sometimes called individuals. Let us denote this type by *o*.

The type o provides the raw material for the next type, the type (o) of classes consisting of or properties applying to individuals. Next comes the type $((o))$ of classes of classes or properties of classes, and so on. By talking the powerset operation at each level the linear hierarchy of simple (or "minimal", as Russell says) types is generates. Simple types are classes all of whose members are of the same type.

Apart from classes of a simple type there are mixed classes too. Consider e.g. "Heine and the French" understood as the class consisting of the individual Heine and the class of Frenchmen. What should be the type of such a mixed class? Russell answers that "it must, if the Contradiction is to be avoided, be of a different type both from classes of individuals and from classes of classes of individuals" (*Principles*, 524).

It will serve our present purposes best if we put this condition by using a lambda-abstract. Let the abstract $[\lambda y.y \in x]$ denote the property of being a member of the possibly mixed class x. Then Russell requires that for an item to be a possible member of the class x—to be within the abstract's range of significance—it must not be of the same type as that class, in short:

F. if $z \in \mathrm{Sign}([\lambda y.y \in x])$, then $\mathrm{Typ}(z) \neq \mathrm{Typ}(x)$.

One might perhaps expect a stronger condition here, namely that the type of z must be *lower* than that of x. But such a condition is nowhere present in Appendix B; for the purpose of solving the paradox of the Russell class it is also unnecessary.

Given that, as remarked above, a propositional function has the same range of significance as its negation, (F) has a natural companion, i.e.

F′. if $z \in \mathrm{Sign}([\lambda y.y \notin x])$, then $\mathrm{Typ}(z) \neq \mathrm{Typ}(x)$.

Finally, the construction of ranges of significance from types is completed with a condition designed to ensure that ranges of significance just are types. Let $x \sim y$ express that the items x and y are (or that the variables x and y range over items) of the same type. Then the final condition Russell mentions is this:

G. If $t \in \mathrm{Sign}(\phi)$ and $s \sim t$, then $s \in \mathrm{Sign}(\phi)$.

Thus, every range of significance is a type. Russell also speculates about the converse, i.e. whether every type is the range of significance of some propositional function. But his brief discussion of the matter (*Principles*, 525) remains somewhat inconclusive.

As already remarked, the crucial condition for blocking the road to the Russell paradox is the condition (F), respectively (F′). Consider the Russell class r defined by the equivalence

$$x \in r \leftrightarrow x \notin x. \qquad \text{(r)}$$

If we substitute r for x in (r), then we obtain the contradictory equivalence

$$r \in r \leftrightarrow r \notin r.$$

Russell's diagnosis in Appendix B of what has gone wrong may be presented as follows. The defining condition,

$$x \notin x, \qquad \text{(r1)}$$

is equivalent to

$$[\lambda y.y \notin x]x. \qquad \text{(r2)}$$

Given that $\text{Typ}(x) = \text{Typ}(x)$, (F′) allows to infer that

$$x \notin \text{Sign}([\lambda y.y \notin x]). \qquad \text{(r3)}$$

Thus the defining condition for the Russell class does not pass the test for significance and so the doctrine of types offers an explanation for why there can be no such thing as the Russell class.

Condition (F) is an early version—perhaps the earliest published version—of what has come to be known as the *Vicious Circle Principle* (VCP). This early version of the principle requires only that a totality of a given type must not include members of the same type, thus *prima facie* not ruling out the possibility that a class may include members of a higher type than the type of the class in question. But this "possibility" is certainly a spurious one, not to be considered seriously. The picture which emerges from the theory sketched in Appendix B of the Principles is a strictly hierarchical one in which entities at a higher level are built up from the material available at a lower lever.

The source of the paradoxes is located in an illicit attempt at reaching out for entities not yet available. This is the core of the variant solutions which Russell proposed during his lifetime. Looking back in the late 50s he wrote

> I lay no stress upon the particular form of the doctrine [of types] which is embodied in *Principia Mathematica*, but I remain wholly convinced that without *some* form of the doctrine the paradoxes cannot be resolved. (1959, 79)

* * *

Any theory of types within the outlines presented above will solve the paradox of the Russell class. This is the position Russell reached in 1902 when he passed *The Principles* to the Press. But the position is unstable because there remains a problem. It concerns the type of propositions which, as Russell notes, lies outside the hierarchy of types defined so far.

> [F]rom this as starting-point a new hierarchy, one might suppose, could be started; but there are certain difficulties in the way of such a view, which render it doubtful whether propositions can be treated like other objects. (*Principles*, 525)

The principal difficulty for accepting the sort of type theory sketched in Chapter X and in Appendix B of *The Principles* is the paradox of propositions. Russell finished the book on May 23rd of 1902 as he relates in his autobiography. But sometime in August or September of the same year he must have discovered the paradox of propositions.[1]

To state the paradox of propositions Russell begins with the assumption that there is an operation ^ which turns any class m of propositions, finite or infinite, into a new proposition, the product of m, $^\wedge m$. Intuitively, the product of m is the proposition that all members of m are true. Thus, in the finite case, we may think of product as conjunction. In some passages Russell seems to introduce product as a new primitive. But eventually he defines it as follows:

$$^\wedge m := \forall p(p \in m \rightarrow p). \qquad (D^\wedge)$$

Russell assumes that ^ is an injective mapping, i.e. distinct sets have distinct product propositions.

$$^\wedge m = {}^\wedge n \rightarrow m = n. \qquad \text{(inj)}$$

Thus with any product $^\wedge m$ we can uniquely associate the class of its factors, $^{\wedge -1}m$. But this assumption leads to a contradiction by the following Cantor style argument. (Let me "simplify" notation for the moment by building into it the assumption, just made, that ^ is injective so that $^{\wedge -1}$ is a function. Thus instead of '$\in {}^{\wedge -1}m$' I shall simply write '$\in m$'.)

Consider the class w of propositions defined as follows:

$$^\wedge m \in w \leftrightarrow {}^\wedge m \notin m. \qquad \text{(w)}$$

The class w of all product propositions which are not among their own factors is obviously the propositional analogue to the problematic class in Russell's class paradox. With the latter paradox we obtained an impossible equivalence when asking whether the Russell class was a member of itself. With the present paradox we obtain an analogous result when asking whether the product of w is a member of w:

$$^\wedge w \in w \leftrightarrow {}^\wedge w \notin w.$$

But although we can see that the moves are exactly similar in both paradoxes, the doctrine of types, as developed so far, can offer no explanation of what has gone wrong in the case of the paradox of propositions. Just as the theory of types finds fault in the defining condition (r1), $x \notin x$, of the Russell class, so we would expect a similar explanation to show that something has gone wrong in the defining condition of the paradoxical set of propositions,

$$^\wedge m \notin m. \qquad \text{(w1)}$$

Just as (r1) is equivalent to (r2), so (w1) is equivalent to

$$[\lambda x.x \notin m]^\wedge m \qquad \text{(w2)}$$

Now in the case of (r2) the theory of types—more specifically the condition (F′)—tells us that the class x does not fall within the range of significance of $[\lambda y.y \in x]$—this explains what has gone wrong in the Russell paradox. But so far the theory tells us nothing about the type of propositions and consequently we have as yet no way of associating them with ranges of significance. We simply cannot determine the contexts in which $^\wedge m$ may or may not meaningfully occur. This leaves us without an explanation as to what exactly is wrong with the condition (w) that leads to the paradox of propositions.

It is very tempting at this point to conclude from the observation that the two paradoxes run in a striking parallel that they must have parallel solutions.[2] Thus Russell writes:

> The close analogy of this contradiction with the one discussed in Chapter X strongly suggests that the two must have the same solution, or at least very similar solutions. (*Principles*, 528)

Such a parallel solution to the paradox of propositions would have to extend the assignment of types to propositions in such a way that (w1) can be unmasked as violating the significance condition (F) or some closely similar condition. More specifically, we would need a type assignment to $^\wedge m$ such that the following, contraposed instance of (F),

$$\text{if } \mathrm{Typ}(^\wedge m) = \mathrm{Typ}(m), \text{ then } ^\wedge m \notin \mathrm{Sign}([\lambda y.y \in m]), \qquad (*)$$

could be used to infer that $^\wedge m$ cannot meaningfully be a member of the class m of its factors. We could infer this result, if the antecedent of (*) could be made plausible, i.e. if it could be argued that the type of a product proposition must be the same as the type of the class of its factors.

Russell discusses the suggestion only very briefly that the paradox of propositions should be solved by somehow using (*). He clearly indicates that such a resolution recommends itself in view of the strict parallel between the two paradoxes. But he also says that the suggestion seems "harsh and highly artificial" (*Principles*, 528). It is less clear what his reasons for saying so are. Russell seems to believe that such a solution to the paradox would carry a commitment to the view "that logical products must have propositions of only one type as factors" (*Principles*, 528). Whatever his reasons for this belief may have been, Russell saw no serious possibility of somehow using (*) to work around the paradox of propositions.[3]

* * *

In Appendix B Russell considers one other solution to the paradox. He observes that the reasoning would break down immediately, if it were denied that we can uniquely associate with each product proposition a certain class of propositions. To thus deny that product formation, $\wedge$, is injective, is to deny that its inverse, $\wedge^{-1}$, is a proper function. As a consequence the offending set w would be ill-defined by the equivalence

$$\wedge m \in w \leftrightarrow \wedge m \notin \wedge^{-1} m$$

because the term $\wedge^{-1}m$ does not refer. Given Cantor's Theorem and a theory of propositions coarse-grained enough to identify products formed from distinct sets, this solution is inescapable.

As it happened, the only candidate theory of propositions of the required kind which Russell considered was the one urged upon him by Frege. According to this theory propositions denote truth-values and any two sets of true propositions or any two sets each containing a false proposition make for the same product proposition, i.e. the one denoting the True, respectively the False.

Russell had no sympathy for such a coarse-grained theory. Accordingly he wrote that

> [...] such an escape is, in reality, impractible, for it is quite self-evident that equivalent propositional functions are often not identical. Who will maintain, for example, that "*x* is an even prime other than 2" is identical with "*x* is one of Charles II.'s wise deeds or foolish sayings"? (*Principles*, 528)

Behind Russell's rejection of Frege's theory is his own, early theory of propositions. According to this theory propositions are certain complexes of things, those things referred to in a proposition. Thus Russell held that the mountain Montblanc was itself part of the proposition that the Montblanc is in Switzerland, as was Switzerland with every single cow on every single village meadow. In theory propositions should be individuated by their parts and the way these parts are arranged. (In practice, however, Russell individuates propositions by the syntactical form of their "corresponding" sentences.) In particular then, if two propositions have distinct parts, they have to be considered distinct. This conception of a proposition makes it at least plausible to maintain that the elements of a given set of propositions enter its product as parts. And just as distinct elements make for distinct sets, so distinct sets would make for distinct products.

3. The Frege-Russell correspondence

Frege responded to Russell's theory of proposition with an incredulous stare. Given his own theory of sense and reference he simply could not understand the suggestion that, say, a mountain could be part of a proposition in any sense of the notion that Frege was able to contemplate. Frege believed that Russell's theory of propositions was in fact responsible for the paradox. He therefore suggested that once this theory be given up, the paradox of propositions would evaporate.

In a letter of September 1902 Russell had first presented to Frege the paradox of propositions as a paradox about *sentences* ("*Sätze*"). In his answer to Russell, Frege offered three interpretation of the notion of a sentence in the context of the paradox. First, a sentence could just be a *linguistic item*, a chain of visible or audible objects expressing a thought. Second, "sentence" could mean the *thought* expressed by such a linguistic item, i.e. a *proposition*. Third, a sentence could be taken with regard to its *reference* which, according to Frege would be one of the *truth-values*, True or False.

Frege takes it that Russell does not have the purely linguistic interpretation in mind. In his answering letter Russell agrees. But this agreement should not be taken as a matter of course. In getting Russell to agree that propositions are not linguistic items, Frege has subtly but significantly moved Russell away from a syntactic view of propositions clearly present at least in parts of the *Principles*. For example, in a footnote in Appendix B Russell considers the question whether

> ... the logical product of p and q and r differ from that of pq and r. A reference to the definition of the logical product (p. 21) will set

> this doubt to rest; for the two logical products in question, though equivalent, are by no means indentical. (527)

However, the indicated passage on p. 21 does nothing to settle the question. Only when we turn to the definition of logical product on p. 16 do we get some idea of what Russell may have had in mind. Russell's definition of product, $\wedge$, involves propositional quantification. But we can get the gist of it by helping ourselves to a *falsum* constant $\bot$:

$$p \wedge q := (p \rightarrow (q \rightarrow \bot)) \rightarrow \bot.$$

Thus different ways of associating, i.e. bracketing conjuncts correspond to different ways of nesting implicational propositions and so "slightly" different product propositions turn out to be "very" different implicational propositions. In the same way Russell would presumably have judged $p \wedge q$ and $q \wedge p$ distinct propositions since their defining propositions, $(p \rightarrow (q \rightarrow \bot)) \rightarrow \bot$ and $(q \rightarrow (p \rightarrow \bot)) \rightarrow \bot$ are clearly (?) distinct. If there ever was a syntactic view of propositions, this is clearly one.[4] As we shall see, Russell's admittance, in his answer to Frege, that propositions are *not* to be understood as linguistic items, is a first signal that he is prepared to move away from his early syntacticism; it also contains the seed to the solution of the paradox of propositions which he later came to accept.

In his letter to Russell, Frege then goes on to expound in rather general terms his distinctions between the sense expressed and the reference denoted by a sentence and asks: "Now, what does Peano understand by "proposition", a thought or a truth-value? I believe that he does not know and that he uses the term at one time in this at another in that sense."[5] This remark can only be understoood as politely suggesting that Russell is guilty of the same negligence. The negligence is not innocuous because, so Frege, it leads to fallacies of substitution. In so-called oblique contexts, where a sentences denotes the thought expressed by it, we may not freely substitute co-referential sentences. Frege points out that the paradox of propositions does involve such oblique contexts and that steps in the argument that rely on substitution demand particular attention. Frege's letter does not contain a solution the the paradox of propositions but only a hypothesis as to how his distinction between sense and reference may possibly provide such a solution.

Russell, so it seems, thought about Frege's hypothesis but could not see how it should render a solution to the paradox. Russell agrees with Frege that in the paradox propositions should not be interpreted as proxies for truth-values. But he immediately adds that nothing much depends on how the term proposition ought to be understood as long as the proposition $\forall p(p \in m \rightarrow p)$ stands in a one-to-one relation to the set m of propositions.[6]

A few days later Frege eventually attempts a detailed diagnosis of the paradox from the standpoint of his theory of sense and reference. He writes:

> The distinction between sense and reference is important in our case too. It frequently happens that distinct expressions denote the same thing but are not to be freely substituted for each other."[7]

There follows again a general exposition of his semantic theory, particularly aimed at convincing Russell that the reference of a proposition must be a truth-value.

Frege then turns to the paradox and observes that the definition of the problematic class of propositions,

$$p \in w \ :\leftrightarrow\ \exists m(p = (\forall q(q \in m \rightarrow q)) \ \wedge\ p \notin m)$$

involves oblique contexts, even an oblique context within an oblique context. He argues as follows. First, since w is a class of propositions, the variable p must range over propositions—it cannot stand for the reference but only for the sense expressed by a sentence. Second, if p is a thought, then whatever it is claimed to be identical with must also be a thought. Thus, $\forall q(q \in m \rightarrow q)$ must be a thought. But inside this thought occurs another quantification over members of a class of thoughts. Thus the expression $\forall q(q \in m \rightarrow q)$ is doubly oblique. Now Frege contends that there is no good sense in which the class m could be part of a thought such that in any sentence expressing that thought a name for m could be replaced by any co-referential class name. In effect Frege observes that the name m does not occur extensionally throughout the right-hand-side of the above equivalence, thus suggesting that the paradox rests on an illicit step of substitution. In any case, the occurrence of senses inside senses makes matters "very complicated [*sehr verwickelt*]", as Frege writes— "[...] thus I do not quite know how you obtain the equation $^\wedge w \in w = {^\wedge w} \notin w$ and what it is meant to express: coincidence of thoughts or coincidence of truth-values?"[8]

Russell seemed to accept Frege's criticism. In his reply of February 20th, 1903, Russell wrote:

> As regards $\forall p(p \in m \rightarrow p)$, I believe that the class m is itself part of this thought. If this were impossible, your criticism would be justified; but I am unconvinced as to its impossibility.

These remarks may have been courteous to Frege but they are thoroughly misleading. The truth is that the paradox does not rely on illicit substitutions of co-referential expressions in oblique or doubly oblique contexts; and even less so does it only arise in Russell's theory of propositions as certain complexes. To see why this is so, let me first present the paradox once again in a more explicit manner.

We start with the definitional equivalence

$$p \in w \;:\leftrightarrow\; \exists m(p = {^\wedge m} \wedge p \notin m) \tag{1}$$

where, as before,

$$^\wedge m = \forall q(q \in m \rightarrow q). \tag{2}$$

Next comes the assumption that products stand in a one-to-one relation to their classes of factors:

$$^\wedge m = {^\wedge w} \;\rightarrow\; m = w. \tag{3}$$

Now assume that

$$^\wedge w \in w. \tag{4}$$

It follows from (1) and (4) that

$$\exists m({^\wedge w} = {^\wedge m} \wedge {^\wedge w} \notin m). \tag{5}$$

From (3) and (5) we get

$$m = w. \tag{6}$$

And from (3) and (6) we infer by substituting w for m,

$$^\wedge w \notin w, \tag{7}$$

contradicting (4).

Now, to rephrase the argument while paying attention to whether an expression occurs *qua* sense or *qua* reference, let us bracket an expression to signal that it occurs intensionally, i.e. *qua* sense. As observed above, the definition of w should now be

$$[p] \in w \ :\leftrightarrow\ \exists m([p] = [^\wedge m] \ \wedge\ [p] \notin m), \tag{1a}$$

where

$$[^\wedge m] = [\forall q([q] \in m \rightarrow q)]. \tag{2a}$$

Then the argument proceeds as follows:

(3a)	$[^\wedge m] = [^\wedge w] \ \rightarrow\ m = w$	injectiveness
(4a)	$[^\wedge w] \in w$	assumption
(5a)	$\exists m([^\wedge w] = [^\wedge m] \ \wedge\ [^\wedge w] \notin m)$	(1a),(4a) – modus ponens
(6a)	$m = w$	(3a),(5a) – modus ponens
(7a)	$[^\wedge w] \notin w$	(5a),(6a) – substitution

contradicting (4a).

There is nothing wrong with the final step of substitution. The internal structure of products, as made visible in (2a) remains irrelevant. Thus the matter is not made complicated in any essential way by the fact that intensional contexts occur embedded in intensional contexts. There is also nothing wrong with the definition (2a) of $^\wedge m$ and $[^\wedge m]$. In his letter of 21st of May 1903 Frege had pointed out that expressions of the form $[p] \rightarrow q$ are ill-formed because implication is a function of the references, not of the sense of sentences. But $^\wedge m$ is not short for $\forall q([[q] \in m] \rightarrow q)$ which would indeed be ill-formed. Finally, it would be of no use to replace the assumption of injectiveness by

$$[^\wedge m] = [^\wedge w] \ \rightarrow\ [m] = [w].$$

For, since sense determines reference this is an even stronger principle which entails (3a) via the fact that $[m] = [w]$ implies $m = w$.

* * *

Russell had no reason to further inquire into what solution to the paradox Frege had in mind. Perhaps the very day Russell received Frege's letter, on May 24th of 1903, he reports that he had eventually found a solution to the paradox. He discovered this solution in the course of reconstructing the material of the *Principles* on the basis of his no-classes theory of classes.

The basic idea of the no-classes theory is to make mentioning classes entirely superfluous. Instead of writing $x \in m$ Russell would now write ϕx for some characteristic property of the class m. How does this help with the paradox? When reformulating the paradox in the no-class notation Russell discovered that one of its assumption stood in contradiction to a theorem which Frege cites in the appendix to the *Grundgesetze*. Let us retrace the steps that Russell must have taken.

First, the defining equivalence is now

$$\phi p \ :\leftrightarrow \ \exists\psi(p = {}^\wedge\psi \wedge \neg\psi p) \tag{1b}$$

where ϕp does now duty for $p \in w$, and

$${}^\wedge\psi = \forall q(\psi q \rightarrow q). \tag{2b}$$

Now suppose that

$$\phi{}^\wedge\phi. \tag{4b}$$

It follows from (1b) and (4b) that

$$\exists\psi({}^\wedge\phi = {}^\wedge\psi \wedge \neg\psi{}^\wedge\phi). \tag{5b}$$

The conclusion we aim at is

$$\neg\phi{}^\wedge\phi. \tag{7b}$$

The only means of obtaining (7b) from (5b) is by way of

$${}^\wedge\phi = {}^\wedge\psi \rightarrow (\phi{}^\wedge\phi \rightarrow \psi{}^\wedge\phi). \tag{*}$$

(Given the left conjunct of (5b) we could detach the consequent of (*), contrapose and then use the right conjunct of (5b) to infer (7b).) The missing link to justify (*) is the more general principle

$$^\wedge\phi = {}^\wedge\psi \rightarrow \forall x(\phi x \rightarrow \psi x). \tag{3b}$$

This is indeed the exact translation into the no-classes framework of the assumption (3) that product is a one-to-one mapping between propositions and classes of propositions. But now Russell notes the following instance of a theorem from the appendix to the *Grundgesetze* (pp. 258–61):

$$\exists\phi\,\psi(^\wedge\phi = {}^\wedge\psi \wedge \phi{}^\wedge\phi \wedge \neg\psi{}^\wedge\phi)) \tag{†}$$

which contradicts (3b) and (*) in particular.

Thus "the difficulties have been overcome", writes Russell, and these are his last words on the paradox of proposition. Neither in his publications nor in his correspondence does the paradox surface again.

4. Conclusion

Russell's solution of the paradox, though stated within the context of the no-classes theory, does not depend on that theory. This solution —giving up the assumption that product is an injective function— had been available to Russell all along. The no-classes theory only helped him see clearly what was at fault in the paradoxical reasoning. The theory provided a context of discovery not of justification. Russell was to abandon the no-classes theory shortly after the letter to Frege had been written. But there is no reason to suppose that he was thereby also forced to abandon the solution to the paradox. Had he done so, he would certainly have reconsidered the paradox. But there is not the slightest trace of such reconsideration.

In particular, there is no evidence that Russell eventually came to the view that the paradox should be handed over to RT. Though it is true that the definition of the problematic class of propositions could not be stratified in accordance with RT, for Russell this seemed to be a secondary phenomenon. There are two pieces of circumstantial evidence to the conclusion that Russell did not think that the paradox of propositions was of the kind that should be resolved by RT.

First, he simply does not give such a solution himself. He would certainly have done so, had he believed that RT does provide the proper resources for resolving the paradox. But the paradox is not displayed as one of the trophies of RT. Hence, it is plausible to conclude that Russell did not hold the view that RT unmasks the principal defect in the argument to the paradox.

Second, according to Ramsey the proper way of dealing with the paradoxes is to first separate them into those essentially involving semantic notions such as meaning and truth on the one hand, and those pertaining to logic proper, i.e. involving the notions of class, propositional function or relation on the other hand. The semantic paradoxes should then point towards certain constraints on semantic theories, while the logical paradoxes should be treated by purely logical or set-theoretic means, obeying a maxim of minimally mutilating the ensuing body of mathematics. As is well-known, Ramsey's strategy eventually became widely adopted while theories based on a general VCP, such as RT, fell into thorough disfavor.

Russell was vehemently opposed to Ramsey's suggestion, believing it to be unsatisfactory from a philosophical point of view. Under these circumstances nothing would have served Russell's cause better than a paradox that would resist to Ramsey's classification, exhibiting instead a "pure" violation of the VCP. The paradox of propositions could very plausibly be viewed as such a paradox, and many have done so.[9] The paradox is certainly not semantic in character, involving none of the central semantic notions. It may with more plausibility be held to belong to the domain of logic, as far as the notion of a proposition belongs to that domain. But, as we have seen, the simple theory of types (ST)—a tool which Ramsey favored as a remedy to the logical paradoxes—cannot resolve the paradox of propositions. Hence, if Russell believed at all that the paradox ought to be solved by means of RT, he would have had every reason to display it as a show-piece of that theory. But he did not display it as such—he even never mentioned it again after 1903. Hence, it is implausible to hold that Russell thought that the paradox should be resolved within RT.[10]

There remains one problem to be discussed. I have remarked above that Russell's initial acceptance of the assumption that product is injective was based on his theory of propositions as complexes. Did he then reject that theory together with the assumption of injectivity? Not necessarily. I have argued above that Russell's theory of propositions lends some support to the idea that product propositions should be as finely individuated as their respective classes of factors. But Russell's early theory of propositions was never articulated finely enough to justify the assertion that the theory *entailed* the injectivity of product. To take a very simple example, consider the two sets

$$\{p,q\} \text{ and } \{p \wedge q\}.$$

Their products would be

$$^\wedge(\{p,q\}) \text{ and } ^\wedge(\{p \wedge q\}).$$

Now, there is a good sense in which the two products denote the same complexes since they ultimately involve the same constituents, p and q. Yet their sets of factors are distinct; whence injectivity would fail.

To be sure, it is far from certain that Russell would have treated this particular example in the way suggested. The example only serves to illustrate the point that there are version of a Russellian theory of propositions which are incompatible with the assumption that factors can be uniquely recovered from a given product. What Russell did come to reject in the course of corresponding with Frege, was his practice of identifying propositional complexes by way of the syntactic form of "corresponding" sentences. Although this practice is ubiquitous in the *Principles*, Russell never articulated it as a thesis. So, in a sense, he was never committed to a syntacticist concept of propositions.

Though the path to the solution which Russell eventually accepted is somewhat meandering, the solution itself is not very exciting. Nowadays, with a multitude of theories of propositions at hand which would all issue in the same verdict, i.e. that the assumption of injectivity is extremely implausible, it may appear strange that Russell ever accepted that assumption. But at the beginning of the 20th century propositions were a puzzling topic.[11] The paradox of propositions serves well as a condensed illustration of this fact. It does not, however, mark a point of transition from the simple to the ramified theory of types. Instead it occasioned a shift within Russell's theory of propositions as complexes. This theory had remained largely unarticulated in the *Principles*. Russell now saw the need to elaborate the theory in more detail — a task that occupied him for some years to come.[12]

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Notes

[1] This can be inferred from his correspondence with Frege. In his letter of August 8th Russell explains to Frege his solution to the paradox of the Russell class in terms of types. (Russell uses the German words "Grad" and "Typus".) In his next letter to Frege, of September 29th, he reports how his confidence in this attempted solution is now undermined:

> Mein Vorschlag über logische Typen scheint mir jetzt unfähig das zu leisten was ich davon erhoffte. Aus dem Cantor'schen Satze, dass irgendeine Klasse mehr Unterklassen als Gegenstände enthält, kann man immer neue Widersprüche hervorbringen.

Russell then goes on to describe the paradox of propositions. This paradox remains a recurring theme in all letters between Russell and Frege until May 1903.

[2] In fact Russell always maintained this as a methodological maxim and therefore rejected all proposals to divide up and treat in a piecemeal fashion what he considered at bottom manifestations of a single phenomenon. In particular he rejected Ramsey's proposal – later to become a common background assumption of mainstream axiomatic set theory – to separate the semantical from the set-theoretical paradoxes.

[3] For some more thoughts on the matter see Fuhrmann (2003).

[4] Although Church (1984) tries hard to identify a definite theory of propositions in the *Principles*, he observes that Russell apparently had not decided on any particular criterion of identity for propositions. In practice, so Church, Russell lets syntax be his guide: "We see throughout §500 that Russell tends to declare two propositions different if the corresponding sentences have any difference in form not entirely trivial." (519)

[5] Letter to Russell, October 20th, 1902.

[6] Letter to Frege, 12th of December 1902.

[7] Letter to Russell, 28th of December 1902.

[8] *Loc. cit.*

[9] See e.g. Sainsbury (1979), L. Linsky (1983), Goldfarb (1989) and Linsky (1999).

[10] Linsky (1999) gives a careful and detailed analysis of how the paradox of propositions can be resolved in a simple version of RT, essentially Church's (1976, 1984) theory of *r*-types. But the fact that the paradox could be resolved in this way does little to support the claim that Russell thought it should be thus resolved. This fallacy pervades to my knowledge all published comments on the rôle of the paradox in the development of Russell's philosophy.

Thus Linsky (1999, 64) writes: "Russell interpreted this paradox as requiring ramification of the simple theory of types that he had just proposed in the preceding Appendix B to the Principles as a first try at a solution to the paradoxes." But there is not the slightest textual evidence for such an interpretation. Instead in Appendix B Russell explores briefly whether type assignments can be extended to proposition and concludes that the suggestion seems "harsh and artficial". Linsky supports his interpretation by quoting from the letter of September 1902 to Frege where Russell writes "that not only functions but also ranges of values belong to different types". But, first, there is no evidence that Russell is thinking of ramification at this point, and, second and more importantly, the remark is made in the second part of the letter where Russell had changed from the topic of the paradox of propositions to that of the paradox of relations which certainly can be resolved within ST.

[11] Lest it be thought that present days theories of propositions are in much better shape than the ones Russell contemplated in his early days, let me only mention that Russell's paradox of propositions has a natural successor in what might be called Kaplan's Problem; see Davies (1981) and Kaplan (1995). This is a problem which afflicts all theories of propositions which cannot rule out off-hand that for each proposition p there possibly exist a property P which uniquely characterizes p. Suppose we try to assess this claim, i.e.

$$w \models \forall p \Diamond \forall q(\phi p \leftrightarrow p = q) \qquad (*)$$

in the usual possible worlds semantics. Since propositions are construed as sets of possible worlds, a little calculation reveals that (*) amounts to the assertion that for each proposition p there exists a possible world in which p is uniquely characterised by ϕ, which is to say that there can be no more propositions than there are possible worlds:

$$|\text{PROP}| \leq |\text{PW}|$$

But according to standard possible worlds semantics propositions just are sets of worlds, i.e.

$$|\mathrm{PROP}| = 2^{|\mathrm{PW}|}$$

But then $2^{|\mathrm{PW}|} \leq |\mathrm{PW}|$—which in Cantorian set theory cannot be. We are left to either conclude that there can be no such bijective property ϕ, or to work within a set theory with a universal set that does not satisfy Cantor's Theorem, or to resort to built up models of propositions in some layered fashion. In any case the ensuing theory of propositions will not be as simple as usually portrayed.

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