A simple method for construction of higher-order potentials and computation of scattering phase shifts

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Abstract

By judicious exploitation of supersymmetry formalism of quantum mechanics higher partial wave nucleon-nucleon potentials are generated from its ground state interactions. The nuclear Hulthen potential and the corresponding ground state wave function with the parameters of Arnold and Mackellar are used as the starting point of our calculation. We compute the scattering phase shifts for our constructed potentials through Phase Function Method to examine the merit of our approach to the problema

Keywords: Supersymmetry, Nucleon-nucleon potential, Phase function method, Scattering phase shifts.

Resumen

Por la explotación racional del formalismo de la supersimetría de la mecánica cuántica, potenciales parciales superiores de onda nucleón-nucleón, se generan a partir de sus interacciones en estado fundamental. El potencial nuclear Hulthén y la correspondiente función de onda del estado fundamental con los parámetros de Arnold y Mackellar, se utilizan como punto de partida de nuestro cálculo. Calculamos los cambios de dispersión de fase de nuestros potenciales construidas a través del Método de la Función de Fase, para examinar el mérito de nuestro enfoque del problema.

Palabras clave: Supersimetría, Potenciales nucleón-nucleón, Método de la Función de Fase, Dispersión de fases.

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I. INTRODUCTION

Witten [1], first developed the methodology to study the quantum mechanical system governed by an algebra identical to that of supersymmetry in field theory. Since then, it becomes a popular tool to deal with hierarchy problems and calculation of related physical observables. For any Hamiltonian with one degree of freedom, a comparison Hamiltonian can be constructed such that the resulting system as a whole is supersymmetric [1, 2, 3, 4].

The Hamiltonian hierarchy problems in supersymmetric quantum mechanics [SQM] lead to the addition of appropriate centrifugal barriers, and consequently, the higher partial wave potentials are generated fairly accurately in atomic physics.

Naturally, question may arise whether this methodology is equally applicable for nuclear cases or not, as the nucleon-nucleon potentials are highly state dependent. In the recent past, we have studied nucleon-nucleon scattering within the framework of SQM [5, 6, 7] for both energy-dependent and independent interactions and achieved fairly good agreement for higher partial wave phase shifts in the low and intermediate energy range.

However, for high energy range ($E_{Lab} \geq 200\,MeV$) our phase shift values differ from those of standard data [8, 9].

In our earlier publications [5, 6, 7], the (p-p) interactions are generated by simply adding electromagnetic potential to its corresponding nuclear part.

Unlike our previous methods, here we shall generate p-wave potential for the (p-p) system, from its s-wave part with the addition of an appropriate centrifugal barrier term, of which half of the contribution comes from electromagnetic part and other half from nuclear part.

We also define another set of interactions by the addition of proper centrifugal term to its s-wave part, and study their effectiveness in computation of nucleon-nucleon phase shifts through the phase function method (PFM) [10].

In section II, we develop p-wave nuclear potential for (n-p) and (p-p) systems through SQM. We shall demonstrate the usefulness of our constructed potentials in section III by computing the values of p-p and n-p scattering phase shifts by PFM. Finally, in section IV we put some concluding remarks.

II. p-WAVE NUCLEAR HULTHEN POTENTIAL

There exist experimental situations which involve scattering by additive interactions, some of which must for various physical reasons be treated exactly, whereas others may be

relatively small perturbation. A typical example of this kind is the scattering of particles under the combined influence of Coulomb and nuclear forces like proton-proton (p-p) scattering.

In charged particle scattering the long range of the electromagnetic interaction (coulomb) is a source of special difficulties.

It has been argued that pure Coulomb potential never really occurs in nature, and becomes somewhat screened at a certain distance. Thus, for p-p scattering the s-wave effective potential $V_{0P}(r)$ is written as:

$$V_{0P}(r)=V_{0A}(r)+V_{0N}(r) \tag{1}$$

Where:

$$V_{0A}(r)=V_0\,\frac{e^{-r/a}}{(1-e^{-r/a})}, \tag{2}$$

is the s-wave Hulthen or screened Coulomb potential with atomic parameters $V_0 \& a$.

In SQM, the supersymmetric partner H_1 with potential V_1 (p-wave potential) of the Hamiltonian H_0:

$$H_0=-\frac{\partial^2}{\partial x^2}+V_0(x), \tag{3}$$

with its ground state eigen function $\psi_0^{(0)}$, and energy eigen value $E_0^{(0)}$ is given by [1, 3]:

$$H_1=-\frac{\partial^2}{\partial x^2}+V_1(x)\ . \tag{4}$$

With:

$$V_1(x)=V_0(x)-\frac{\partial^2}{\partial x^2}\ln\psi_0^{(0)}. \tag{5}$$

The application of the above relations to the Hulthen potential are now in order.

Since its appearance the Yamaguchi potential [11]:

$$V(r,s)=\lambda\, e^{-\beta(r+s)}, \tag{6}$$

with λ, the strength and β, inverse range parameters becomes a popular tool in dynamical calculations.

The bound state wave function for the Yamaguchi potential is identical to the wave function for the first bound state of the Hulthen potential with range $(\beta-\alpha)^{-1}$ and depth $-(\beta^2-\alpha^2)$ [12].

A nuclear Hulthen potential with these parameters rewritten as:

$$V_{0N}(r)=-(\beta^2-\alpha^2)\frac{e^{-\beta r}}{(e^{-\alpha r}-e^{-\beta r})}, \tag{7}$$

with its ground state solution:

$$\psi_0^{(0)}\sim e^{-\alpha r}-e^{-\beta r}\ . \tag{8}$$

In view of Equations (5) and (7), the wave function in Eq. (8) leads to the supersymmetric partner potential:

$$V_{1N}(r)=V_{0N}(r)+\frac{(\beta-\alpha)^2 e^{-(\alpha+\beta)r}}{(e^{-\alpha r}-e^{-\beta r})^2}\ . \tag{9}$$

In equation (9) the second term simulates the effect of centrifugal barrier apart from a factor of 2. Similarly, for atomic Hulthen potential one has:

$$V_{1A}(r)=V_{0A}(r)+\frac{e^{-r/a}}{a^2(1-e^{-r/a})^2}\ . \tag{10}$$

In view of above the p-wave potential for p-p scattering is defined as:

$$V_{1P}(r)=V_{1A}(r)+V_{1N}(r). \tag{11}$$

The corresponding potential for n-p scattering designated as $V_{1NP}(r)$ is obtained with $V_{0A}(r)=0$ in Eq. (11) so that the centrifugal barrier term from atomic part of the potential comes into effect. The associated phase shifts with supersymmetry generated potentials for the (p-p) and (n-p) systems will be denoted by $\delta_{\ell P}$ and $\delta_{\ell N}$ respectively.

We also propose the following set of potentials by adding the centrifugal term to its s-wave part directly:

$$V_{1NB}(r)=V_{0N}(r)+\frac{2}{r^2}, \tag{12}$$

and

$$V_{1PB}(r)=V_{0A}(r)+V_{1NB}(r)\ , \tag{13}$$

and study its impact on the scattering phase shifts.

In the above the letter B in subscript denotes the potentials where the corresponding centrifugal barrier terms are added directly to the s-wave parts and the associated phase shifts will be designated as $\delta_{\ell PB}$ and $\delta_{\ell NB}$ for (p-p) and (n-p) systems respectively.

III. RESULTS AND DISCUSSION

The phase function method represents an efficient approach to evaluate the scattering phase shifts for quantum mechanical

problems involving local [10] and non local interactions [13, 14].

For a local potential the function $\delta_\ell(k,r)$, called the phase function, satisfy a first order non-linear differential equation given by:

$$\delta_\ell'(k,r) = -k^{-1}V(r)[\hat{j}_\ell(kr)\cos\delta_\ell(k,r) - \hat{\eta}_\ell(kr)\sin\delta_\ell(k,r)]^2 \quad , \tag{14}$$

with $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$, the Riccati Bessel functions. We shall follow the phase convention of Calogaro [10] with Hankel function of first kind, written as $\hat{h}^1_\ell(x) = -\eta_\ell(x) + i\hat{j}_\ell(x)$.

The scattering phase shift $\delta_\ell(k)$ is obtained by solving the equation from origin to asymptotic region with the initial condition $\delta_\ell(k,0) = 0$.

During the solution of the phase equation, $\delta_\ell(k,r)$ is built up by the potential as one moves away from the origin and it reaches its asymptotic value as soon as one gets out of the range of the potential. Obviously, $\delta_\ell(k) = Lim_{r\to\infty}\delta_\ell(k,r)$.

In Figures 1, 2, 3 and 4, we portray the (p-p) and (n-p) potentials as a function of distance for $\ell = 0\,\&1$ partial waves with $\lambda = -5.237\ fm^{-3}$ and $\beta = 1.4054\ fm^{-1}$ for $^1s_0$ scattering and, $\lambda = -7.533\ fm^{-3}$ and $\beta = 1.4054\ fm^{-1}$ for $^3s_1$ scattering [12].

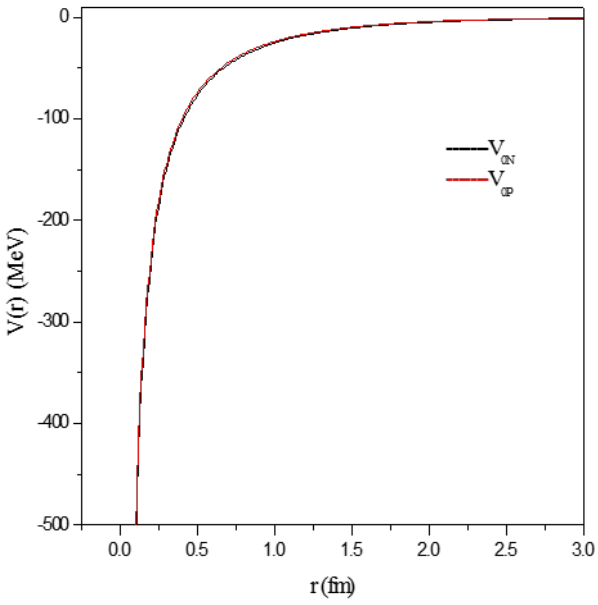

FIGURE 1. $^1s_0$ potential as a function of r.

It is observed that in figures 2 and 4 repulsive cores develop in the generated potentials. These potentials, generated from their $^1s_0$ and $^3s_1$ parts, correspond to $^1p_1$ and $^3p_1$ states respectively.

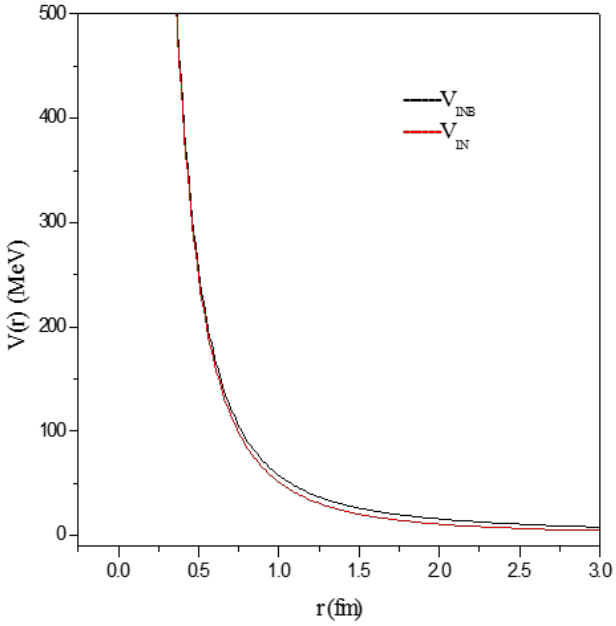

FIGURE 2. Constructed $^1p_1$ potential as a function of r.

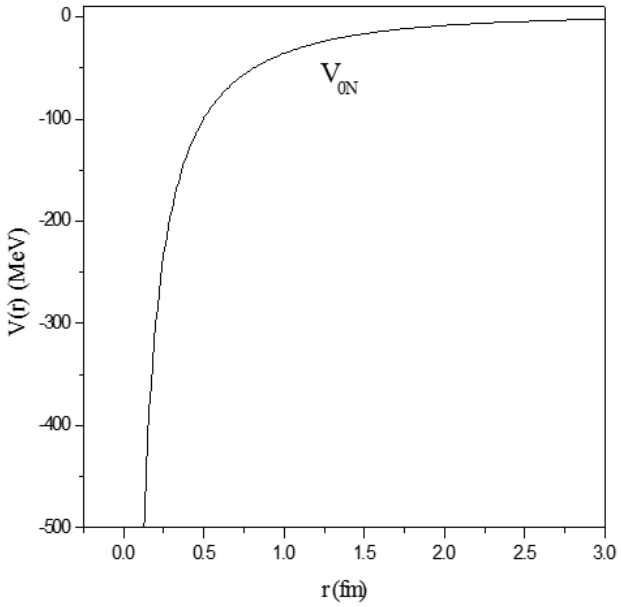

FIGURE 3. $^3s_1$ potential as a function of r.

The corresponding singlet and triplet state phase shifts have been computed using the PFM and plotted in Figures 5, 6, 7 and 8, as a function of laboratory energy up to 300 MeV along with the values of Arndt *et al.* [8] and Gross-Stadler [9] for comparison.

Note that the results for the pure nuclear phase shifts (n-p) have been obtained by turning off the atomic Hulthen

interaction $V_{0A}(r)$ in the associated numerical routine for generating (p-p) phase shifts.

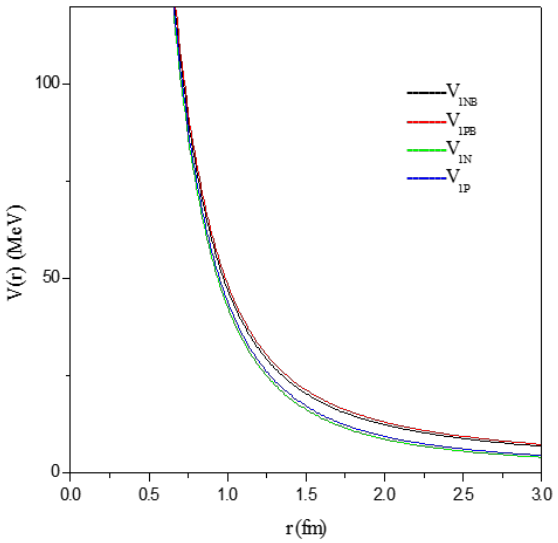

FIGURE 4. Constructed $^3p_1$ potential as a function of r.

In Figure 5, the $^1s_0$ phase shifts δ_{0P} and δ_{0N} for (p-p) and (n-p) systems respectively, agree well with that of references [8, 9], for $E_{Lab} \leq 30\,MeV$. Beyond 30 MeV the phase shifts differ significantly with energy. This is quite expected because $^1s_0$ phase shifts change sign beyond 225 MeV, and a one term potential is not capable of producing such effect.

Therefore, it is expected that our potential for $^1p_1$ state developed from $^1s_0$ part will be able to generate reasonable fit to phase shifts at least up to 30 MeV.

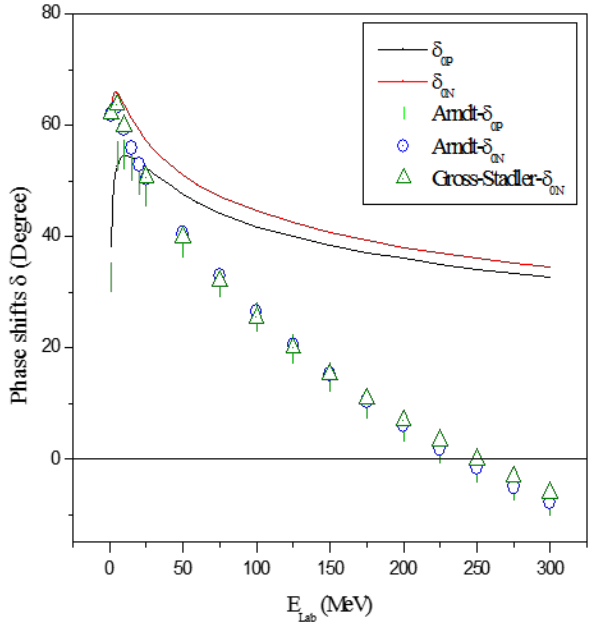

FIGURE 5. $^1s_0$ phase shifts as a function of E_{Lab}.

In Figure 6, the two sets of phase shift values δ_{1NB} and δ_{1N} along with the values of references 8 and 9 are displayed. All our phase shifts show correct trend for $^1p_1$ pure nuclear phase shift but differ in their numerical values.

Among the two sets of phase shift values namely δ_{1NB} and δ_{1N}, δ_{1N} is more consistent than δ_{1NB} up to 250 MeV.

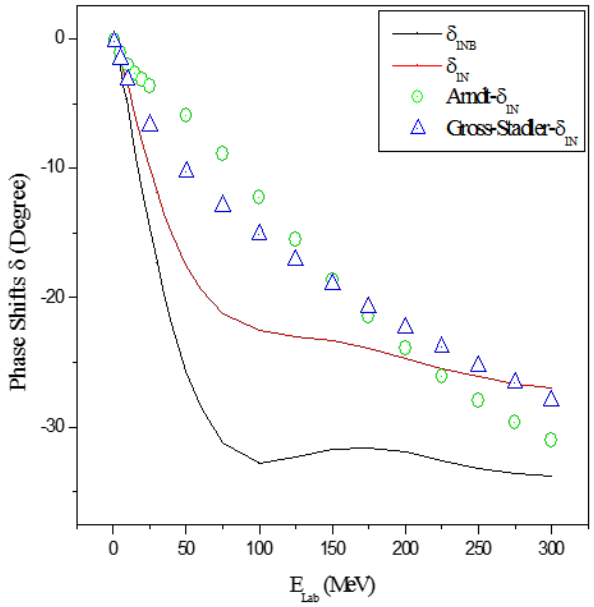

FIGURE 6. $^1p_1$ phase shifts as a function of E_{Lab}.

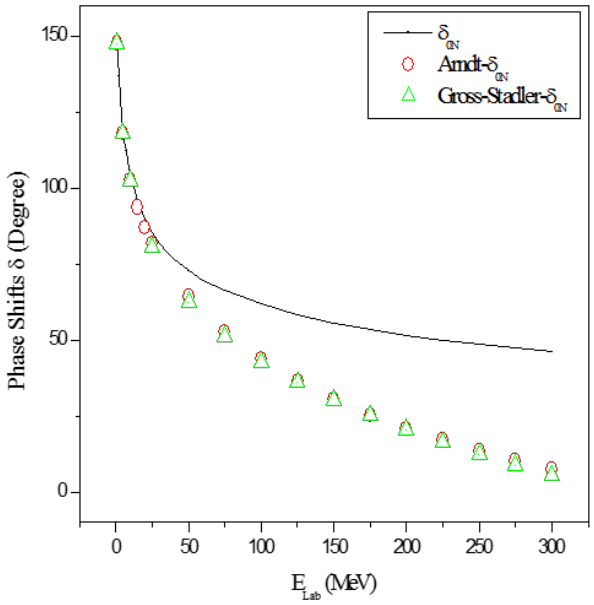

FIGURE-7. $^3s_1$ phase shifts as a function of E_{Lab}.

However, beyond 250 MeV δ_{1NB} compares well with the standard data [8, 9]. This is due to the fact that the r^{-2} term plays a crucial role in the high energy range. Therefore, our supersymmetry generated $^1p_1$ potential is more realistic in the low and intermediate energy scattering compared to the proposed one with direct addition of centrifugal barrier.

In Figure 7, our phase shift values for $^3s_1$ state with the parameters of Arnold and Mackellar [12] agree well with that of-references 8 and 9 up to $E_{Lab} = 25\,MeV$.

In Figure 8, looking closely into this figure we notice that the phase shifts $\delta_{1P}, \delta_{1N}, \delta_{1PB}$ and δ_{1NB} produce correct nature of (p-p) and (n-p) scattering phase shifts for $^3p_1$ state.

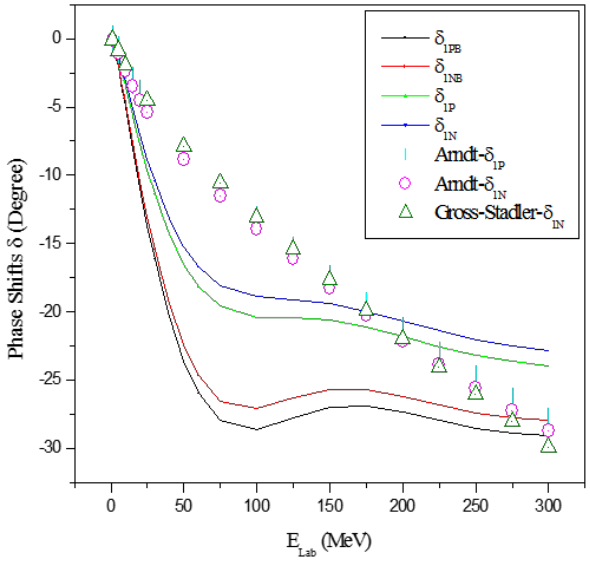

FIGURE 8. $^3p_1$ phase shifts as a function of E_{Lab}.

Among these δ_{1P} and δ_{1N} are superior to their counter parts δ_{1PB} and δ_{1NB} and are quite capable for comparison with Arndt *et al.* [8] and Gross-Stadler [9], up to $E_{Lab} = 250\,MeV$.

Beyond 250 MeV, however, δ_{1PB} and δ_{1NB} are more consistent than their counter parts. It is worthwhile to mention that our higher partial wave potentials are generated from their ground states with the addition of repulsive centrifugal potentials which make the p-p interaction more repulsive than n-p one.

IV. CONCLUSION

From our observation it is reflected that our constructed potentials are quite capable of producing the nature of phase shifts of respective states, but differ in their numerical values.

This is due to the fact that, unlike atomic cases, the nuclear potentials are highly state dependent and cannot be generated in a proper way from any known interaction. The higher partial wave potentials that are generated here belong to Eckart class of potentials; the second terms in them behave as centrifugal barrier.

Our supersymmetry operation for developing p-wave interactions from its s-wave part corresponds to the removal of one bound state and thereby produces a shallow potential with repulsive core from a deep s-wave interaction. Both deep and shallow potentials have been using in nuclear physics calculations, particularly, in nucleus-nucleus interactions.

The bound states of the shallow potentials are related to the actual physical states of the fused nucleus. Michel and Reidemeister [15], nicely explained that it is possible to construct phase equivalent shallow potentials for α + ^{16}O deep potential through SQM. Baye [16] has also found a good agreement between α-α shallow potential of Ali and Bodmer [17] and the deep potential of Buck *et al.* [18].

Our phase shift calculation, through SQM and PFM involves only two parameters, while those of Arndt *et al.* [8] and Gross-Stadler [9], are associated with 52 parameters and 27 parameters interactions respectively.

Therefore, by comparing our phase shifts with those of Arndt *et al.* [8] and Gross-Stadler [9], it can be concluded that this simpleminded combined approach of SQM and PFM to compute nucleon-nucleon scattering phase shifts, will be of quite interesting to a wide variety of physicists and graduate students. Also our method of computing the scattering phase shifts by the use of the variable phase method deserves serious attention.

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