ANALYSIS OF THE UNDERLYING COGNITIVE ACTIVITY IN THE RESOLUTION OF A TASK ON DERIVABILITY OF THE ABSOLUTE-VALUE FUNCTION: TWO THEORETICAL PERSPECTIVES

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This paper presents a study of networking of theories between the theory of registers of semiotic representation (TRSR) and the onto-semiotic approach of mathematical cognition and instruction (OSA). The results obtained show complementarities between these two theoretical perspectives, which might allow more detailed analysis of the students' performance.

Keywords: Absolute value; Cognitive analysis; Derivative; Networking of theories; Onto-semiotic approach; Theory of registers of semiotic representation

Análisis de la actividad cognitiva subyacente en la resolución de una tarea sobre la derivabilidad de la función valor absoluto: dos perspectivas teóricas

En este artículo se presenta un estudio de networking of theories, entre la teoría de los registros de representación semióticos (TRRS) y el enfoque onto-semiótico de la cognición e instrucción matemáticos (OSA). Los resultados obtenidos revelan complementariedades entre estas dos perspectivas teóricas cuya aplicación simultánea permitiría hacer análisis más pormenorizados de las producciones de los estudiantes.

Términos clave: Análisis cognitivo; Conexión de redes de teorías; Derivada; Enfoque onto-semiótico; Teoría de los registros de representación semiótica; Valor absoluto

Pino-Fan, L., Guzmán, I., Font, V., & Duval, R. (2017). Analysis of the underlying cognitive activity in the resolution of a task on derivability of the absolute-value function: Two theoretical perspectives. *PNA*, *11*(2), 97-124.

BACKGROUND

One of the main concerns of the research community on the mathematics education is determining which are the difficulties that learners face on their way to understanding, and therefore, learning, mathematical notions. This interest is reflected in the fact that, over decades, one of the main focuses of research within our scientific discipline has been the characteristics of the learner's cognitive activity. Furthermore, one of the most popular theories in the field is the theory of registers of semiotic representation (TRSR) (Duval, 1995), which focuses on the study of the cognitive activity of students when solving mathematical problems, providing notions that make it possible to analyze and comprehend how subjects use and link the different types of material representations and the role that these representations have in the comprehension of mathematical concepts.

On the other hand, an emerging theoretical model in the mathematics education, which has been gaining more and more importance worldwide, is the onto-semiotic approach (OSA) to mathematical cognition and instruction (Godino, Batanero, & Font, 2007), which is a model that tries to articulate several perspectives of the discipline and dimensions involved in the processes of teaching and learning mathematics. Out of the many dimensions, our main interest in this document is what is known as the *cognitive facet*, which emphasizes the different types of mathematical objects involved in mathematical practices developed in order to solve a certain mathematical task, and also in the connections that subjects establishamong such objects and the meanings that they give them in terms of the context in which these are used.

In this research, we aim at carrying out a comparative study between these two theoretical approaches, the theory of registers of semiotic representation and the onto-semiotic approach, which allows carrying out cognitive analysis from the subjects' performance. In order to conduct this study, and following the proposed methodology for the works within the framework of the networking of theories, we analysed the performance of a future high school teacher in a task related to the differentiability of the absolute-value function.

In the following section, theoretical and methodological notions of the study will be introduced, beginning with the features of the networking of theories that were used, and then, the description of the theoretical and methodological tools proposed by the TRSR and OSA for cognitive analysis. In the third section, the task and the response protocol that will be the basis for the comparative study of the theoretical perspectives is presented. In the fourth section, a detailed development of the cognitive analysis from the TRSR and OSA perspectives is done. In the fifth section, both perspectives of analysis are compared and the approximations and complementarities between the two theoretical methodologies used are performed. Finally, the conclusions of the study are presented.

THEORETICAL AND METHODOLOGICAL NOTIONS

In what follows we summarize some aspects of the two theories involved in this study. But first, we describe the features of the methodology of networking of theories.

Networking of Theories

Currently, there are several theoretical positions that allow conducting cognitive analyses—of students, prospective teachers or teachers—depending on what is desired to observe and which is the concerned mathematical notion (Asiala et al., 1996; Duval, 2006; Godino et al., 2007). However, the complex nature of the subjects' learning phenomena has directed research groups to make efforts to revise and find possible complementarities between theoretical and methodological approaches that allow providing more detailed and precise explanations of such learning processes—e.g., Font, Trigueros, Badillo, and Rubio (2016), carry out a comparative study between the APOS theory and the OSA.

The idea of investigating the networking of theories is not new, as evidenced in the works presented in the working group since the Fifth Congress of European Research in Mathematics Education (CERME 5) to the past CERME 9, on different approaches and theoretical perspectives in mathematics education research. Artigue, Bartolini-Bussi, Dreyfus, Gray, and Prediger (2005) points out that, as a research community, we need to be aware that discussion among researchers from different research communities is insufficient to achieve networking. Collaboration among teams using different theories with different underlying assumptions is called for in order to identify the issues and the questions. In general, research studiesthat have been performed in this area, have explored ways of handling the diversity of theories in order to better grasp the complexity of learning and teaching processes, and understand how theories can or cannot be connected in a manner that respects their underlying assumptions.

In this regards, there are different strategies and methods to deal with this type of studies. For example, Prediger, Bikner-Ahsbahs, and Arzarello (2008), describe different connecting strategies and methods for articulating theories, which range from completely ignoring other theoretical perspectives on the one extreme end, to globally unifying different approaches on the other. As intermediate strategies, the authors mention the need for making one's own perspective understandable and for understanding other perspectives, and the strategies of comparing and contrasting different approaches, coordinating and combining perspectives, and achieving local integration and synthesis. For our study, we have utilized diverse strategies, within the framework of the methods of networking of theories, which will be described below.

Firstly, a team of four researchers was formed, who are also the authors of this document, two of which—the second and fourth author—have both vast experience and deep knowledge in the use of the TRSR. The other two authors—the first and third—have substantial experience in using the OSA. The first and third authors use OSA mainly, but possess knowledge of TRSR too, and the second author uses primarily TRSR but also has sufficient knowledge of OSA. Therefore, an important first phase in networking theories is achieved: the need to really understand the other theory.

Several authors have showed interest in determining the aspects that characterize a theory in order to be able to classify it and compare it (e.g., Bikner-Ahsbahs & Prediger, 2010; Radford, 2008). According to Radford (2008), essential elements of a theory include principles, methods, and paradigmatic research questions. The principles of each theory entail a positioning, explicit or implicit, about the nature of mathematical objects. For this reason, the second step has been set to determine how both theories model mathematical activity and what is their positioning, explicit or implicit, on the nature of mathematical objects. This second step allows seeing the differences and similarities between both theories and also providing a prior general idea on how they can be coordinated.

In order to keep moving forward, once this first comparison between the theories has been made, we have applied one of the basic principles of networking of theories: to ensure that the job of connecting the theories be as precise as possible. According to this principle, the third step was selecting a specific mathematical object—the derivative—as context of reflection. The reason for choosing the derivative over another topic is because it is a mathematical object on which relevant research has been done by both theories, utilizing the basic theoretical notions of each.

Once the mathematical object derivative had been selected, as fourth step we chose a task where such object was used and one answer that, regarding the application of the task to a sample of students, was provided by one of the students. The selection of both the task and the answer has been done, principally, by the first two authors and then agreed with the other two authors; and it was selected due to the complex mathematical activity provided by the student, which we refer to as Juliette. In such activity, Juliette shows difficulties that are taken into account in the analysis conducted with two theoretical perspectives.

As fifth step, we have analyzed the answer given by Juliette, from two theoretical perspectives. On the one hand, we looked at the solution to the task from the perspective of TRSR. This analysis has been basically carried out by the two authors that use TRSR. On the other hand, we looked at the solution to the task from the perspective of OSA. This analysis has been basically carried out by the two authors that use OSA.

As sixth step, in this article we focused on the comparison of the principles used in the cognitive analysis from both theoretical perspectives, and not on the comparison of methods and paradigmatic general research questions between TRSR and OSA. This step has been jointly carried out by the four authors in the section on comparison of analysis.

Theory of Registers of Semiotic Representation (TRSR)

In cognitive psychology, the notion of representation plays an important role regarding the acquisition and the treatment of an individual's knowledge. As Duval (1995) points out, "there's no knowledge that can be mobilized by an individual without a representation activity" (p. 15). However, in mathematics, what matters are the various semiotic systems used to represent numbers, functions, geometrical properties, etc. And unlike other areas of knowledge, any mathematical activity always requires substituting some semiotic representation for another, no matter the semiotic systems that are mobilized. This leads to distinguish two quite different kinds of cognitive operations, which are known as conversion and treatment: to substitute one semiotic representation for another, only by changing the semiotic system mobilized; and to substitute two semiotics representations within the same semiotic system.

Registers of semiotic representation are all the semiotic representations that are used in mathematics for computing, deducing, solving—mathematically—problems. They can be classified into four types according two criteria (Duval, 2006). First, the semiotic representations that are produced are either discursive representations—numerical or algebraic expressions, definitions, descriptions, among others—or non-discursive representation—geometrical figures, graphs, diagrams. Second, their substitution can either be set out in algorithms—numerical or algebraic expressions, graphs—or not be set out in algorithms—language natural, heuristic exploration of geometrical figures. Treatments are always specific to the type of register mobilized.

The cognitive analysis of mathematical activity in terms of registers, and therefore, in terms of conversions and treatments, is based on three ideas that are described below.

- ♦ There are as many different semiotic representations of the same mathematical object, as semiotic registers used in mathematics.
- ♦ Each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented. What is being explicitly stated is the content of the representation.
- ♦ The content of semiotic representations must never be confused with the mathematical objects that these represent.

Take, for example, the mathematical object linear function. First, it can be represented by mobilizing any of the three registers below (see Figure 1).

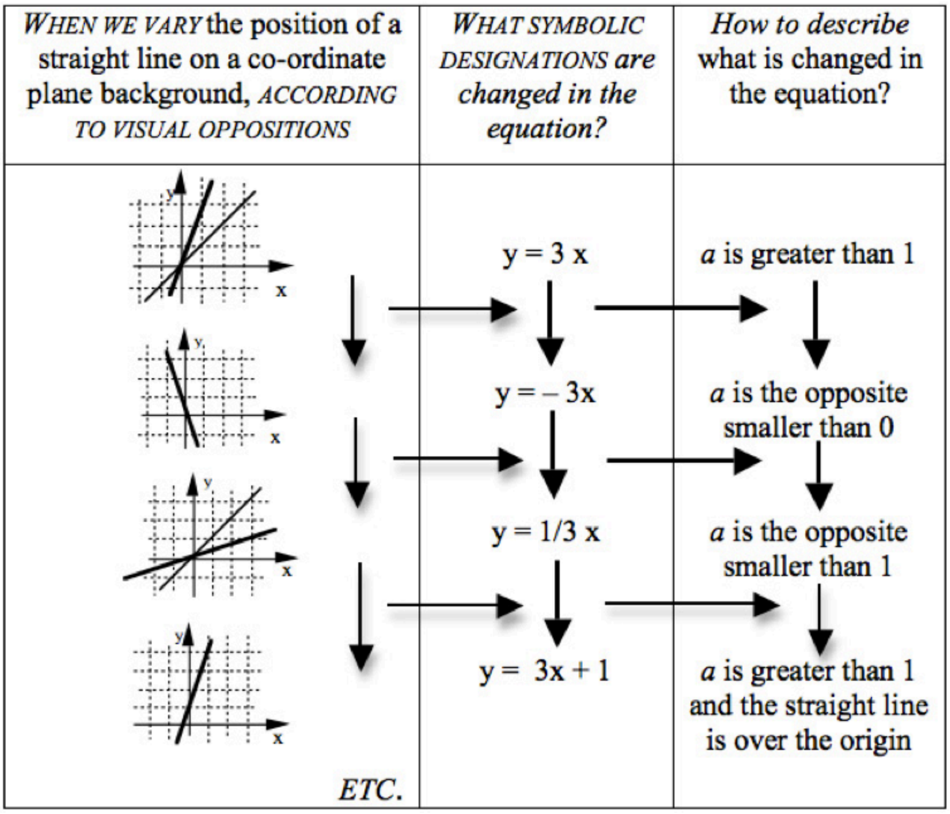


Figure 1. Task on recognition of the object linear function

Second, the content of a semiotic representation changes completely when we jump from a register to another—the horizontal arrows. So, there is a cognitive distance between the respective contents of two semiotic representations of the same object. This cognitive distance varies according the type of the start register chosen, for example, to give the data of a problem, and the register to mobilize in order to solve the problem or to control the relevancy of the answer.

Third, when the student has achievedthe coordination between at least two registers—all horizontal arrows between two columns—they are able to recognize immediately the corresponding semiotic representation in another register. Otherwise they confuse the mathematical object represented with the particular content of the semiotic representation given and they remain blocked, because there are no links at all between the registers.

This recognition is not a question of concept acquisition or definition. It requires becoming aware of what arethe qualitative features from the two respective contents that have to be matched. That is why specific tasks of experimental exploration are needed—all vertical arrows in Figure 1. In these tasks, the relevant variations are the qualitative variations and not the numerical values variations.

From a mathematical point of view, what matters is treatment, not conversion, because computing, deducing, depends on substitutions which are made within the same register by applying properties. Treatment always depends on the semiotic possibilities of internal substitutions or transformations of representations that the register mobilized provides. Therefore, mathematical treatments are cognitively more complex in the use of the register of natural language and those registers that allow visualizing—graphs, geometrical shapes, etc. That is why specific tasks are required in order to make students become aware of the way of reasoning and visualizing in mathematics (Duval, 2008).

Now, in the reality of the mathematical work, conversions and treatments are never separated, because mathematical activity always mobilizes, explicitly or implicitly, two kinds of registers. In other words, the real mathematical activity consists of conversions and treatments performed alternatively or in parallel—in geometry, for example, even if, from a mathematical point of view, only treatments are scientifically relevant. Therefore, the necessary steps in the cognitive analysis of any mathematical activity are the following.

- ♦ Separating all the conversions that are, explicitly or implicitly, required from the treatment.
- ♦ Coding the couple of registers mobilized for each conversion—start register and arrival register.
- ♦ Making explicit the successive substitutions to perform mainly for the treatment in natural language or in visualization registers.

Thus, we get a grid of all the cognitively heterogeneous tasks underlying a mathematical activity that we want to introduce for teaching or for experimental purposes. This grid allows to reach two purpouses that we describe below.

- ♦ To perform an accurate diagnosis of the difficulties and incomprehension points encountered by students which are a barrier to success.
- ♦ To compare the student's production with the information gathered in other similar activities during different periods of time and to assess the capacity of transferring to new situations.

A reliable and controllable interpretation of the students' oral/written/drawn production shall be based on comparison with other data gathered according to the same device of observation and with one variation well-identified. This is what allows determining the scope of the outcomes of a research.

The Onto-Semiotic Approach to Cognition and Mathematical Instruction (OSA)

The OSA is a theoretical and methodological framework that has been developed since 1994 by Godino and colleagues and has been described in several other works (e.g., Font, Godino, & Gallardo, 2013). This theoretical framework arises from the field of research of mathematics education and aims at articulating the diverse dimensions—epistemic, cognitive, affective, interactional, mediational and ecologic—that are implicit in the processes of teaching and learning mathematics (Godino et al., 2007). Relevant for this work are the epistemic facet, which makes reference to the distribution, during the time of teaching, of the components of the institutional meaning (problems, linguistic elements, concepts, propositions, procedures and arguments), and the cognitive facet, which refers to the development of personal meanings—learning.

The notion of *system of practices* plays an important role for the teaching and learning of mathematics within the OSA. Godino and Batanero (1994) refer to the system of practices as "any performance or manifestation—linguistic or not—carried out by someone in order to solve mathematical problems, to communicate the solution to others, to validate the solution and to generalize it to other contexts and problems" (p. 334). Font et al. (2013) point out that mathematical practices can be conceptualized as the combination of an operative practice, through which mathematical texts can be read and produced, and a discursive practice, which allows reflecting on operative practices. These practices can be carried out by one person—system of personal practices—or shared within an institution—system of institutional practices.

Within the OSA, certain pragmatism is adopted since mathematical objects are considered as entities that emerge from the systems of practices carried out in a field of problems (Godino & Batanero, 1994). Font et al. (2013) put it this way: "Our ontological proposal originates from mathematical practices, and these become the basic context from which individuals gain experience and mathematical objects emerge. Consequently, the object gains a status originated from the practices that precede it" (p. 104). Ostensive objects—symbols, graphs, etc.—and non-ostensive objects—concepts, propositions, etc.—intervene in mathematical practices, which we evoke while doing mathematics and are represented in a textual, oral, graphic, symbolic and even gestural way. New objects emerge from the systems of operative and discursive mathematical practices and these show their organization and structure (Godino et al., 2007). If the systems of practices are shared within the core of an institution, then the emerging objects will be considered as institutional objects, while, on the other hand, if such systems correspond to one person, then these will be considered as personal objects. The emergence of a personal object is progressive during the history of a subject, as a consequence of experience and learning, while the emergence of an institutional object is progressive over time.

In order to offer a finer and more pragmatic way to analyze the mathematical practices developed in connection to certain problems, OSA introduces a typology of primary mathematical entities—or primary mathematical objects, that intervene in the systems of practices: (a) situations-problems (extra-mathematical applications, exercises…); (b) linguistic elements (terms, expressions, notations, graphs…) in diverse registers (written, oral, gestural…); (c) concepts/definitions (introduced through definitions or descriptions: line, point, number, function, derivative…); (d) propositions/properties (statements about concepts); (e) procedures (algorithms, operations, calculation techniques…); and (f) arguments (statements used to validate or explain propositions and procedures, deductive or of another type).

Situation-problems are the origin or reason for the activity; language represents the remaining entities and serves as an instrument for the action; arguments justify the procedures and propositions that relate the concepts. These primary mathematical objects are connected with each other, forming intervening networks of objects, emerging from the systems of practices, which in OSA are known as configurations. These configurations can be socio-epistemic (networks of institutional objects) or cognitive (networks of personal objects).

Each one of the primary mathematical objects can be considered from different dual facets or dimensions: personal–institutional, unitary–systemic, expression–content, ostensive–non-ostensive and exemplar–type. Godino, Font, Wilhelmi, and Lurduy (2011) point out that both of these dualities and primary mathematical objects can be analyzed from a process-product perspective, which entails the following processes: (a) institutionalization–personalization, (b) generalization–particularization, (c) decomposition or analysis–composition or reification, (d) materialization–idealization, and (e) representation–signification. The emergence of primary mathematical objects, pointed out before, is linked, respectively, to processes of problematization, communication, definition, algorithmization, enunciation, and argumentation.

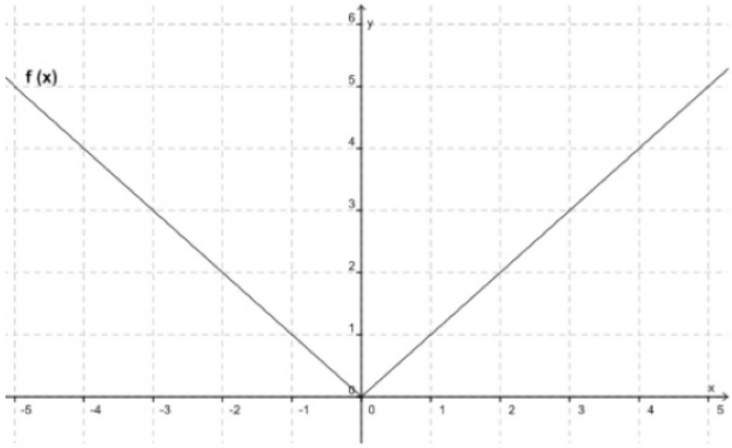
This way, the meaning of mathematical objects in OSA is, basically, conceived in two ways: (a) from a pragmatic-anthropological perspective, which deals with the relativity of the context—or system of mathematical practices—in which these are used, in other words, as emerging and in the sense that is assigned to them in the practices or systems of practices—institutional or personal—in which these are mobilized; and (b) in terms of *semiotic functions*, a notion that allows to make relations among the several entities in a referential and operational way. According to Hjemslev (1943) and Eco (1976), a semiotic functionis the correspondence or dependent relationship (or function) that is established by a subject (person or institution) between an antecedent (expression, signifier) and a consequent—content or meaning, according to a criteria or correspondence code (rules, habits, agreements...). The content—o consequent—of a semiotic function, and therefore, the meaning can be a personal or institutional object, unitary or systemic, ostensive or non-ostensive (Godino et al., 2007); similarly, this object can be a linguistic element, a definition, a proposition, a procedure, an argument or problem (Font et al., 2013). According to Peirce's semiotic (1978), the OSA assumes that both the expression—antecedent of a semiotic function—and the content—consequent of a semiotic function—could be any kind of entity (primary mathematical object or process).

THE TASK AND ITS RESOLUTION: THE CASE OF JULIETTE

The task addressed in this study (Figure 2), has been the object of study in several works (Pino-Fan, 2014; Tsamir, Rasslan, & Dreyfus, 2006), and demands, from who solves it, the mobilization of representations (graphic, symbolic and verbal), and argumentations that justify procedures.

Task

Examine the function $f(x) = |x|$ and its graph.



a) For what values of x is $f(x)$ derivable?

b) If possible, calculate $f'(2)$ and draw a graphic representation of your solution. If not possible, explain why.

c) If possible, calculate $f'(0)$ and draw a graphic representation of your solution. If not possible, explain why.

Figure 2. Task on derivability of the absolute-value function

The selected protocol of resolution is the one given by Juliette to the task above (Figure 3). The case of Juliette arises in connection to a research carried conducted by the first author of this document (Pino-Fan, 2014), which is oriented towards the implementation of a questionnaire to analyze partial aspects of the knowledge of a sample of future teachers of mathematics.

As it can be observed in Figure 3, Juliette had some serious difficulties to solve the task assigned. Her answer was selected from among other 93 answers, due to the complexity of the cognitive activity reflected in her mathematical practice.

2.-

a) In the first instance, the function $f(x) = |x|$ is not differentiable, because the graph has a peak at the point $x = 0$

If it is considered as a function of the type $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the function would be differentiable in the whole domain, in other words, $(-\infty,\infty)$.

b) As mentioned above, the function is not differentiable, therefore $f'(2)$ cannot be calculated and if it was differentiable, its graphic representation would be a point on the Cartesian plane.

Considering this type of function: $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the derivative would be $f'(x) = 1$, and then it would represent a point...

c) As mentioned above, if we consider the function $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$ the derivative would be $f'(x) = 1$ and the graphic representation would be...

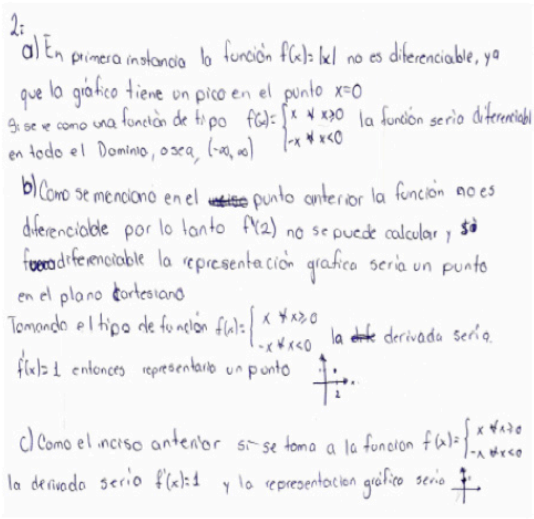

2:
a) En primera instancia la función f(x)= |x| no es diferenciable, ya que la gráfico tiene un pico en el punto x=0
Si se ve como una función de tipo f(x)= {x ∀ x≥0; -x ∀ x<0} la función serio diferenciabl en todo el Dominio, osea, (-∞, ∞)
b) Como se menciono en el ~~inciso~~ punto anterior la función no es diferenciable por lo tanto f'(2) no se puede calcular y si ~~fuera~~ diferenciable la representación grafica seria un punto en el plano cartesiano
Tomando el tipo de función f(x)= {x ∀x≥0; -x ∀x<0} la ~~def~~ derivada sería f'(x)= 1 entonces representario un punto
c) Como el inciso anterior si se toma a la funcion f(x)= {x ∀x≥0; -x ∀x<0} la derivada serio f'(x)=1 y la representacion gráfico serio

Figure 3. Juliette's solution to the task on derivability of the absolute-value function

Analysis of the Protocol from two Theoretical Perspectives

In this section, we develop a detailed cognitive analysis from the TRSR and OSA perspectives.

Analysis from the TRSR Perspective

The formulation of the task contains elements of the verbal, graphic and symbolic registers. The questions related to the symbolic and graphic representation of the absolute-value function, combine the verbal register and symbolic notations. The answers to these questions are expected to be given in the form of any of the registers mentioned before—verbal, graphic, or symbolic.

In general, Juliette's answers for the items (a), (b), and (c), show that she knows the definition of the absolute-value function, and that she can express it in the symbolic register. But regarding the derivative function, she shows deficiencies, because even though she answers that if the graph of the function presents a corner or peak on $x = 0$ then the function is not derivable, in her upcoming arguments some confusions are perceived regarding the domain of the f' function. In the symbolic register, she represents the absolute-value function by parts and for the non-negative values of x she writes the f derivative using symbols. She recognizes $f'(2)$ as image for $x = 2$, but she is not successful regarding $x = 0$. Details per item are presented below.

- For item (a), the domain $f(x)=|x|$ is required. Juliette, being in the graphic register, visualizes the typical edge of the graph of the absolute-value function and she perceptively associates it with the non-differentiability of the function. She does not make the domain of f' explicit, and states that the function is not differentiable. She clearly associates the top point or peak of the graph with the fact that the function can not be derived in its domain. In other words, she does an incorrect generalization. On the other hand, she represents the absolute-value function in the symbolic register, expressing it by parts and stating that if *f*was differentiable, it would be so in its entire domain. In Juliette's answer to item a), an apparent separation between the graphic and symbolic registers is perceived because she answers through the visualization of the graph first, and then, she performs a treatment of the algebraic expression of the absolute-value function through the symbolic register.
- For item (b) it is required to calculate $f'(2)$. Here, we can appreciate a new contradiction in her answer. First, she answers that "$f'(2)$ cannot be calculated", and justifies her answer by citing her answer to item (a). Then, she adds a hypothesis, "if it was differentiable, the graphic representation would be a point on the cartesian plane". It is not clear whether the graphic representation of the function f' is a point, or she is only referring to $x=2$. In any case, we appreciate that she recognizes, in isolation, that the notation $f'(2)$ represents the ordinate of a point in the cartesian plane, since she draws it but does not justifies it. It could be stated that Juliette does a simple passage from symbolic to graphic.
- For item (c) the question is if it is possible to calculate $f'(0)$. Juliette is in the symbolic register and suggests a new hypothesis, "if we take a function of the type $f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$ the derivative would be $f'(x)=1$ [considering $x\geq 0$]", and then draws the graphic representation of the image of zero, $f'(0)=1$. Although her treatment is coherent because she justifies her answer deriving for the values $x\geq 0$, obtains $f'(0)=1$ and graphs the corresponding point, she clearly does not master the absolute-value function and does not articulate its algebraic representation, by parts, with its graphic representation.

In general, the symbolic register has prevailed over the graphic register in Juliette's answers. She only used the graphic register to graph points, but she failed to correctly read the graph of the absolute-value function. The question of item (c) was a key question and required the articulation of the symbolic and graphic registers, but despite the treatment and passages that can be observed in her answers, the lack of mastering of her mathematical knowledge did not allow her coordinate the registers at play—graphic and symbolic.

Analysis from the OSA Perspective

We have organized the analysis in two levels. A first level where the mathematical practice carried out by Juliette is described in general terms—a fairly more global analysis; and a second level where, in a meticulous way, detailed information about primary mathematical objects, their meanings and the processes mobilized in her practice, as well as the way in which she relates to them—cognitive configuration of objects and processes—is provided.

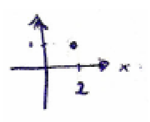
Mathematical Practice

Firstly, we observe that Juliette begins her practice based on a visual justification to answer, although wrongly, subtask (a), pointing out the existence of a peak at the point of domain of the function $x = 0$. From the beginning of her practice, we can observe that Juliette confuses the non-derivability—local—at a point of domain of the absolute-value function with, her misconception of, non-derivability of the function—global. Later, Juliette writes the symbolic definition, by parts, of the absolute-value function. We could say that, in a certain way, such definition is correct, however, she does not make crucial considerations, like for example, that the point of domain of the function $x = 0$ belongs to both $f(x) = x$ and $f(x) = -x$. This fact leads her to a cognitive conflict that is shown in her sentence "If it is considered as a function of the type... the function would be differentiable in the whole domain, in other words, $(\infty, -\infty)$". This cognitive conflict generated from her visual interpretation of the graph of the function—the function is not derivable since it has a peak in $x = 0$—in contraposition to her interpretation of the symbolic definition, by parts, of the function—she considers that $f(x) = x$ exclusively for $x \geq 0$, is what leads her to give incorrect answers to the other subtasks.

Cognitive Configuration of Objects and Processes

Juliette mobilizes in her practice a series of primary mathematical objects—linguistic elements, concepts/definitions, properties/propositions, procedures and arguments—and processes—emerging—which are detailed below (see Table 1, Table 2, Table 3, Table 4, and Table 5). Similarly, as part of the configuration of objects and processes, the previous objects and processes that Juliette must interpret and understand before starting her practice are identified.

Table 1
Linguistic Elements

Mathematical objects	Meanings
	Previous
LE1: $f(x) = \lvert x \rvert$	Symbolic representation of the absolute-value function.
LE2: "Graph of the function" (Figure 1)	Graphic-cartesian representation of the absolute-value function with domain (-5, 5). Also, all the real numbers can be inferred as the domain of the absolute-value function, just like Juliette does.
LE3: $f'(2)$	Symbolic representation that denotes the derivative of the absolute-value function at the point of domain $x = 2$.
LE4: $f'(0)$	Symbolic representation that denotes the derivative of the absolute-value function at the point of domain $x = 0$.
	Emerging
LE5: $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$	Symbolic expression that determines the definition, by parts, of the absolute-value function.
LE6: $(-\infty, \infty)$	Symbolic-notational expression that refers, according to Juliette, to the domain of the derived function.
LE7: $f'(x) = 1$	Symbolic expression that Juliette uses to denote the derivative of the function at the point of domain $x = 2$. This expression is also used by Juliette to represent the derivative of the function at the point $x = 0$.
LE8: 	Graphic representation that denotes the derivative of the function at the point of domain $x = 2$.
LE9: 	Graphic representation that Juliette uses to refer to the derivative of the function at the point of domain $x = 0$, a point at (0,1) of the cartesian plane.

Note. LE=Linguistic element.

Table 2 shows concepts and definitions related to mathematical practice carried out by Juliette.

Table 2
Concepts/Definitions

Mathematical Objects	Meanings
	Previous
CD1: Function	Particularized as absolute-value function and referred to graphically as well as symbolically.
CD2: Derivative of a function and its domain	Particularized, with the sub-task (a), as the derivative of the absolute-value function.
CD3: Derivative at a point	Specifically, at the points of domain of the function $x=2$ and $x=0$.
	Emerging
CD4: Absolute value	Defined symbolically as: $\lvert x\rvert=\begin{cases}x, & x\geq 0\\ -x, & x<0\end{cases}$ Juliette extrapolated this implicit definition of absolute-value to her definition of absolute-value function.
CD5: Domain	Of the derivative of the function, referred by Juliette as $(-\infty,\infty)$.

Note. CD=Concept/definition.

Table 3 shows properties/propositions that Juliette mobilizes in her practice.

Table 3
Properties/Propositions

Mathematical Objects	Meanings
	Previous
PP1: “Examine the function $f(x)=\lvert x\rvert$ and its graph”	Proposition that establishes a previous process of enunciation-representation, which allows the linking of the symbolic representation of the function with its graphic representation.

Table 3
Properties/Propositions

Mathematical Objects	Meanings
	Emerging
PP2: "...the function $f(x)=\|x\|$ is not differentiable..."	This proposition supposes the partial answer that Juliette provides to sub-task (a).
PP3: "If it is considered as a function of the type $f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$ the function would be differentiable in the whole domain, in other words, $(-\infty,\infty)$".	Proposition that accounts for a definition process—definition of the function by parts. This proposition also refers to: (a) a process of algorithmization, which allows to derive, implicitly, the function by parts (this process of algorithmization refers to a procedure); and (b) a process of argumentation, in which a special treatment for the derivative of the function at the point of domain $x=0$ is not considered. This proposition also supposes a partial answer by Juliette to subtask (a).
PP4: "... $f'(2)$ can not be calculated..."	Partial answer to sub-task (b). Proposition that accounts for a process of argumentation of a graphic-visual type, in which Juliette presupposes that, since the value function is not differentiable because it has peaks, then the derivative at the point of domain $x=2$, cannot be calculated.
PP5: "... and if it was differentiable [the absolute-value function] then, the graphic representation would be a point on the cartesian plane..."	Partial answer to sub-task (b). Proposition that shows evidence of the procedure of deriving by parts the function. This proposition is reaffirmed with PP6.

Table 3
Properties/Propositions

Mathematical Objects	Meanings
PP6: "Considering the type of function 'LE5', the derivative would be 'LE7' then it would represent a point 'LE8'"	This proposition could be decomposed into two. The first one would make reference to the processes of algorithmization and argumentation of PP3 and PP4. A second one would refer to a process of representation in which Juliette links LE7 and LE8.
PP7: "...if the function 'LE5' is considered, the derivative would be 'LE7' and the graphic representation would be 'LE9'"	Answer to sub-task c). This proposition refers to a process of algorithmization that leads to the procedure of deriving the function by parts, and to a process of argumentation, based on LE5 and the process of algorithmization, that allows Juliette to find the derivative for $x=0$. Likewise, this proposition also makes reference to a process of representation in which Juliette links LE7 and LE9.

Note. PP=Property/proposition.

We could also analyze the meaning that Juliette gives to the sentence that lies before PP7, "As mentioned before [her answer to sub-task (b)]...", as a proposition that suggests as a first partial answer to sub-task c), that $f'(0)$ cannot be calculated because the graph of the function has a peak—as in the case of sub-task (b). This is relevant because the two propositions (PP4 and PP5) that make reference to partial answers to sub-task (b), show contradictions in the answers that Juliette provides to sub-task (b).Thus, a cognitive conflict arises within her that leads her to argue that the derivative in $x=2$ cannot be calculated if based on the graphic-visual aspects only, but can indeed be calculated if based on purely symbolic-algorithmic aspects.The same occurs with her answer to sub-task (c).

Table 4
Procedures

Mathematical Objects	Meanings
	Emerging
P1: Graphic-Visual	Procedures that allow Juliette to provide answers to sub-tasks (a), (b) and (c), from her interpretation of the graphof the absolute-value function. In the case of Juliette's answer, these kinds of procedures are connected with the type of arguments A1. By means of these types of procedures Juliette achieves propositions such as PP2 and PP4.
P2: Calculation of the derivative from LE5	Procedures that Juliette utilizes to provide answers, through propositions PP3, PP5 and PP6, to sub-tasks (a), (b) and (c) respectively. Basically, this procedure consists of the derivative by parts of the function, from the definition CD4 that is also mentioned in LE5.

Note. P=Procedure.

The description of the arguments that Juliette mobilizes in her practice are displayed in Table 5.

Table 5
Arguments

Mathematical Objects	Meanings
	Emerging
A1: Graphic-Visual "…because the graph has a peak at the point $x=0$"	Argument centered on the visual consideration of the peak in the graph of the function. This argument is the one Juliette uses to point out that the function is not differentiable and therefore, it is not possible to calculate the derivative at the points of domain $x=2$ and $x=0$.
A2: Symbolic-Algorithmic	Argument presented by Juliette in which the process of definition of the absolute-value function (symbolically: CD4, LE5) and the process of algorithmization that allows obtaining the procedure of calculation of the derivative by parts (P2), are considered.

Note. A=Argument.

As explained in a preliminary section (on the OSA to cognition and mathematical instruction), the notion of meaning is a key notion within the OSA and it can basically be conceived in two ways. In this sense and complementary to the description of primary mathematical objects and processes that Juliette mobilizes in her practice, we can identify the relevant semiotic functions that she establishes to each one of the items of the task.

For item (a) Juliette establishes three semiotic functions. The first one has a graphic-visual nature because, from the absolute-value graph—LE2, antecedent, the proposition PP2—consequent—is enunciated, arguing such proposition in a visual way through the peak in the graph—argument A1, correspondence code. The content—or consequent—of this first semiotic function constitutes, according to Juliette's practice, a partial answer to item (a) "...the function is not differentiable..."

The second partial answer to item (a) reflects the establishment of two semiotic functions. The first of them accounts for a manipulation of the symbolic representation of the function, that allows moving from the antecedent LE1 to the consequent LE5 by means of the implicit definition of absolute-value—CD4, correspondence code. The second one, allows us perceiving the symbolic-algorithmic nature of Juliette's second partial answer because, from the way she defines the absolute-value function—CD4, antecedent, and by procedure P2—correspondence code, she achieves the proposition PP3—consequent—that constitutes her partial answer: "...the function is differentiable in the whole domain...".

With the two partial solutions that Juliette provides as answers to item (a), it becomes evident that there is an apparent disconnection between the graphic-visual interpretation and treatment that she performs in order to give her first partial answer, and the symbolic-algorithmic manipulation and interpretation that leads her to give her second partial answer.

In the same way, for item (b) Juliette provides two partial solutions, one of graphic-visual nature and another one of symbolic-algorithmic nature. In the first partial solution, the establishment of a semiotic function can be identified. Here, the antecedent is the proposition PP2 given in her solution to item (a). Thus, since Juliette considers that the function is not differentiable at any point of the domain—because there is a peak in the graph—argument A1, correspondence code—then "... $f'(2)$ can not be calculated..."—PP4, consequent.

In the second partial solution to item (b), the establishment of two concatenated semiotic functions can be identified. For the first one, Juliette starts in the way she defines the absolute-value function—CD4, antecedent), and through a procedure of deriving the positive part of the function $f(x)=x, \forall x, x \geq 0$—P2, correspondence code, she concludes that the derivative at the point of domain is 1, which she represents with LE7—consequent. For the second semiotic function, LE7 is established as antecedent, and through the proposition PP5—correspondence code, provides the graphic representation—LE8, consequent—of LE7.

Finally, for item (c) it is possible to identify two concatenated semiotic functions. For the first one, Juliette once again starts from the definition CD4—antecedent, and through the procedure P2—correspondence code, of deriving $f(x)=x$ since $0 \geq 0$, she concludes that the derivative at the point of domain is 1, which she represents with LE7—consequent. The second semiotic function has the LE7 as antecedent and LE9 as consequent.

In general, it is possible that Juliette does not make a connection between the graphic and symbolic representations of the function, even though such connection is established in the formulation of the task by means of the a priori, or institutional, semiotic function in which mathematical objects that had been previously enunciated in Tables 1, 2 and 3 are mobilized. This a priori semiotic function connects symbolic—LE1, antecedent—and graphic—LE2, consequent—representations of the function, through a proposition PP1—correspondence code—imposed as if it was a norm.

Comparison of the Analysis

As seen in the previous section, the analysis conducted with both TRSR and OSA, show deficiencies in Juliette's mathematical activity, related to the lack of connection of the interpretations and treatments that she makes in the graphic and symbolic representations of the absolute-value function. However, there are some qualifications regarding the analysis carried out and the results obtained that are intrinsic to the principles and notions of each theoretical perspective. The Table 6 presents in detail the general methodologyof analysis used by both theoretical perspectives.

Table 6
Methodology for Cognitive Analysis According to TRSR and OSA

Theory of Register of Semiotic Representation (TRSR)	Onto-Semiotic Approach (OSA)
Identification of the registers involved in the formulation of the task.	The proposed task is decomposed into basic units of analysis, and based on this decomposition, primary mathematical objects, their meanings and previous processes are identified. Semiotic functions established a priori are identified, if there are any.

Table 6
Methodology for Cognitive Analysis According to TRSR and OSA

Theory of Register of Semiotic Representation (TRSR)	Onto-Semiotic Approach (OSA)
It focuses on the analysis of the mathematical activity of the individual, focusing on the registers that are at play when solving a task.	It focuses the analysis on the subject's mathematical practice.
Identification of significant units within the representations put into play in each register.	Describing the subject's mathematical practice in general terms, identifying basic units of analysis.
Identification and characterization of the subject's cognitive operations and characterization of the cognitive operations of treatments and conversions/passages in the subject's activities. In the treatments, specify the type of register utilized from the subject's production, determining: if it is a visual recognition, a calculation, or a utilization of a discursive or algebraic definition. In the conversions, specify the sense of these in terms of congruence or non-congruence of the representations at stake. In this stage, the pair of registers mobilized and the sense of the conversion should be indicated. Furthermore, from a mathematical point of view, it should be pointed out if the conversion is explicit or implicit.	Identifying and describing the configuration—or sub configurations—of primary mathematical objects—linguistic elements, concepts, propositions, procedures and arguments, processes and their meanings, involved in the subject's mathematical practice. Identifying and reconstructing the semiotic functions established by the subject, which provide meaning to the mathematical objects mobilized during his/her practice and give sense to the elements that form the configuration described in the previous step.
Studying conversions in terms of congruence—or incongruence—of the articulation—or no articulation—that a subject performs between the different registers of representation involved.	Studying coherence, from a mathematical point of view, regarding the connection—or lack of—that a subject makes between diverse sub-configurations and, in consequence, between the different semiotic functions that the subject establishes within his/her practice.

A first key aspect that is important to distinguish is that, while the analysis from the OSA perspective focused on the subjects' mathematical practices, and mathematical objects, processes and their meanings, that emerge from such practices, the TRSR focused its analysis primarily on the registers of representation that the subject mobilizes in his/her productions. In this way, the methodology proposed by TRSR can be considered as more "global", in the sense that the subjects' cognitive activity is analyzed without performing valuations from a mathematical point of view, as it is done with the tools of OSA. For example, with TRSR, it is possible to draw conclusions such as there are deficiencies in the subject's knowledge because he/she does not carry out conversion, passage or treatment. However, contradictions can indeed be detected in the subject's cognitive activity and whether the task was successfully solved or not can also be determined. On the other hand, with OSA it is possible to provide a detailed explanation, from a mathematical point of view, of what are the knowledge deficiencies the subject has in relation to a certain problem, by identifying and describing in a detailed way, the mathematical objects, processes and their meanings, involved in his/her mathematical practice.

Yet, it must be pointed out that in OSA there is not systematization for the analysis of linguistic elements. As a part of the methodology proposed by OSA, language signs—linguistic elements—can be identified, which the subject uses to express cognitive activity. However, whether such linguistic elements make reference to representations of objects in a same register or in different ones is not made explicit in the analysis; in other words, the linguistic elements of OSA would correspond to representations of objects in different registers. In this way, a linguistic element encompasses different languages—verbal, figurative, symbolic... These different languages could make reference both to registers of semiotic representation and semiotic systems.

In this regard, TRSR makes a clear distinction between register of semiotic representation and semiotic system. As discussed before, a register is a semiotic system that fulfills three conditions: (a) traces or significant units of representation that belong to a system must be clearly identifiable and should not allow contradictions; (b) it allows internal treatments in each register; and (c) it allows passages or conversions from one register to another. Thus, we can observe how the notion of register of semiotic representation of TRSR, complements and enriches the notion of linguistic elements of OSA, by making a very clear distinction between register and semiotic system, and systematizing the analysis of such registers. Having said that, it is quite undeniable that in the methodologies of analysis proposed by both theoretical approaches, it is agreed that language is the most relevant semiotic system of them all. In this sense, we could add that language is a multifunctional system. Another example of multifunctional system is the figural register, since geometries—flat, spherical…—do not constitute a register, but a mathematical framework—or a framework for other disciplines, in which several registers can work. Therefore, it is important not to confuse register with framework.

Despite the systematization that TRSR provides for the analysis of registers of representation, by means of the study of treatments and conversions, some relevant aspect must be taken into account for the improvement of cognitive analysis and provide more detailed explanations, from a mathematical point of view, about the subject's cognitive activity to which we have access to through his/her mathematical practice. As mentioned above, since it is a global theory and it focused on a subject's cognitive activity—and in his/her ability to mobilize diverse registers of representation, the TRSR focuses its attention on the study of registers of representation mobilized by such subject, leaving the content and the meaning of the mathematical objects being represented implicit. For example, if we consider the answer given by Juliette to item (c), with the TRSR methodology of analysis it can be concludedthat she performs a treatment in the symbolic register that allows passing from the expression $f(x)=\begin{cases} x, x\geq 0 \\ -x, x<0 \end{cases}$ to the expression $f'(x)=1$, then she performs a direct passage that allows her to move from the latter expression—given in the symbolic register—to the graph of a point on the cartesian plane—graphic register. In general, we can observe how often conversions/passages always have as starting point and as point of arrival a register of representation, that could as well be the same—in the case of treatments—or different—in the case of conversions/passages. But, how does Juliette's answer relate to the way she conceives the definition of absolute-value function? What explanation can be provided to the fact that Juliette had expressed the derivative of the absolute-value function at $x=0$ as $f'(x)=1$ instead of $f'(0)=1$—although we know that such answer is mathematically incorrect?

The notion of semiotic function, introduced by OSA, allows giving a better explanation about what is Juliette's cognitive activity to give her answer and allows responding to the two previous questions. As evidenced with the analysis performed with the OSA perspective, Juliette establishes two semiotic functions as part of her answer to item (c). As described in the previous section, the first semiotic function is established from the way Juliette defines the absolute-value function (antecedent), considering $x=0$ as part of the domain of the line segment $f(x)=x$. Afterwards, through the implicit procedure of derivation of $f(x)=x$ —correspondence rule, she finally obtains $f'(0)=1$—consequent. It is possible to observe that, in order to establish this first semiotic function, the meaning that Juliette gives to the mathematical object definition of absolute-value function is emphasized, leaving the representation of such mathematical object implicit. The semiotic function that we have just discussed has a definition as antecedent, an implicit procedure as correspondence code, and a representation as consequent—that lies in the same register than the antecedent. Something similar can be said about the second semiotic function, but the difference is that it has a representation in the symbolic register as antecedent, a proposition in the verbal register as correspondence code, and a representation in the graphic register as consequent. Thus, we can notice how the semiotic function of OSA, reinterprets and enriches the notions of treatments and conversions/passages of TRSR, by emphasizing the meaning—personal or institutional—given to the mathematical content of the objects represented and contemplating that both antecedent and consequent of the function can be languages—representations, concepts/definitions, properties/propositions, procedures or arguments.

Finally, it is important to point out that similar analyses can be carried out for the expected solutions—from an expert or institutional point of view—from the two theoretical perspectives—cognitive analysis of the answer of the expert in TRSR and epistemic analysis in OSA—analysis of expected institutional meanings. However, for length reasons and because it would be redundant with what has already been described in this document, we do not present such a priori analysis in this document. In any case, the last stage in the methodological approach of both perspectives (Table 6) makes reference to the study of the proximity of the subject's knowledge in relation to the knowledge that was expected to be mobilized in his/her practice.

Final Reflections

In this paper, we discussed the relation between representations and the underlying mathematical activity during the development of a task—about derivative. In order to examine such a relation, it is necessary to rely on models which analyze students' mathematical activity. In the literature, most models are cognitive-based. They consider that learning a mathematical concept and its application occurs if various appropriate internal representations are developed and integrated together with functional relations among them. Other few models are semiotic and pragmatic-based. The pragmatic approach also gives importance to the use of diverse representations; however, different reasons are provided in comparison to those from the cognitive approach. On the one hand, the cognitive approach primarily explores representations from a representational perspective. On the other hand, the pragmatic approach emphasizes the instrumental dimension, namely, what can be done with a representation.

We investigate this relation, first, from a cognitive perspective (TRSR) and second, from a pragmatic and semiotic perspective (OSA). The notions of semiotic system and register of semiotic representation of the TRSR are essential for the comprehension of the cognitive activity needed to solve a task, while that OSA provides a level of analysis of the subject's cognitive activity that shows mathematical objects that are involved in the processes of treatment and conversion/passages between registers of semiotic representation. This level of analysis complements the analysis carried out using the tools of TRSR, because with the tools of configuration of objects and processes and semiotic function, the contents of representations become explicit and are used as part of such cognitive activity. The registers of representation are implicitly involved in semiotic functions; however, these emphasize the mathematical content of the representation. The relationship between notions of mathematical objects and its meanings—as considered in OSA—and semiotic representation—as considered in TRSR, are essential for the analysis and characterization of mathematical knowledge.

The results of the comparison of analysis presented in comparison of the analysis section show that between these two theoretical perspectives there are complementarities that would allow performing more precise and finer cognitive analysis, from the subjects' production. As a result of these complementarities, we thought it was pertinent to propose the following methodology to develop cognitive analysis, integrating the contributions of both perspectives:

- To center the analysis on subjects' production, since it is through it that we can get closer to their cognitive activity.
- To decompose the subject's production in significant units of analysis.
- To identify in each significant unit, the language signs that the subject utilizes to express his/her knowledge. It is important to specify if these signs make reference to a semiotic system or to a register of representation, and also, to which type does it refer to.
- To study and describe if the representations—language signs—that the subject utilizes to show his/her knowledge make reference to concepts/definitions, properties/propositions, procedures or arguments—identification of configurations or sub configurations of objects and processes. The meaning that the subject gives to each of such mathematical objects (concepts, properties…) must be described.
- To identify and reconstruct the semiotic functions that the subject establishes as part of his/her practice. Determine if such semiotic functions make reference to cognitive operations of treatments or conversions/passages.
- To study coherence, from a mathematical point of view, in relation to the connections—or lack of—that the subject makes among the diverse sub-configurations and, in consequence, among the diverse semiotic functions that he/she establishes during his/her practice. Similarly, to study the semiotic functions that make reference to the conversions in terms of congruence—or non-congruence—of the articulation—or no articulation—that a subject performs among the different registers of representation involved.

The above-mentioned methodology can also be utilized for the performance of respective a priori analysis, and which will unveil expected mathematical knowledge, which is indispensable to determine the proximity of the knowledge shown by subjects—based on the latter point of the previously mentioned methodology—in relation to the expected knowledge. Besides, the methodology is flexible since it can be adapted both to the productions of a subject regarding diverse types of tasks, and the mathematical objects that such tasks demand. Thus, the combination of the two frameworks contributes to literature by extending our understanding of the relationship between representations and the underlying mathematical activity during the development of tasks that provide better explanations about the aspects that make it possible or impossible to comprehend mathematical notions.

Acknowledgements

This study has been performed in the framework of research projects FONDECYT Nº 11150014 (Funded by CONICYT, Chile), and EDU2015-64646-P (MINECO, Spain).

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Received: December 2015. Accepted: April 2016.
Handle: http://hdl.handle.net/10481/44148