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Establishing Profiles on the Use of Number Sense

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Establishing Profiles on the Use of Number Sense

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Abstract

Number sense includes the ability to use numbers and operations in a flexible and reasonable way. This work presents a case study on the use of number sense by eighth grade students (13-14 years old). We analyze individual interviews of eleven students that include number tasks that can be solved using strategies associated with number sense. By studying the strategies used by the students to answer the questions, we establish four profiles based on their preference for using number sense strategies, rules or algorithms, or on their lack of knowledge of basic number concepts.

Keywords: Number sense, student profiles, strategies

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Estableciendo Perfiles en el Uso del Sentido Numérico

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Resumen

El sentido numérico incluye la habilidad para usar los números y las operaciones de una forma flexible y razonable. Este trabajo presenta un estudio de casos sobre el uso del sentido numérico en estudiantes de secundaria (13-14 años). Se analizan entrevistas individuales realizadas a once estudiantes sobre tareas que pueden resolverse utilizando estrategias asociadas al sentido numérico. Analizando las respuestas de los estudiantes al responder las cuestiones hemos establecido cuatro perfiles de en función de sus estrategias basadas en: sentido numérico, reglas o algoritmos o dificultades con conceptos numéricos básicos.

Palabras clave: Sentido numérico, perfiles de estudiantes, estrategias.

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The educational community around the world has focused recently on providing those skills that allow students to manage in both academic and real-world situations. This has led to having the acquisition of these skills included in many national mathematical curricula (Australian Education Council, 1990; Boletín Oficial del Estado, 2007; National Council of Teachers of Mathematics [NCTM], 2000). This kind of skill in mathematics includes, among other aspects, an understanding of numbers and operations, the ability to use number knowledge in a flexible way, exhibiting different strategies for handling numbers and operations, and the ability to assess the validity of results, commonly referred to by researchers as number sense (McIntosh, Reys, & Reys, 1992). The term “number sense” is frequently used as a general synonym for “numerical knowledge”, understood as the general knowledge that a person has of numbers and operations. Different authors note that number sense is difficult to describe, despite being recognizable in the action of resolving number problems (McIntosh et al., 1992; Sowder, 1992). Number sense is viewed as a well-organised conceptual network that allows relating numbers and operations, their properties and solving number problems in a creative and flexible way (Sowder, 1992).

The acquisition of number sense is gradual and must start from the earliest years of schooling. Curriculum documents propose to develop number sense with an approach away from algorithms and strive to have students acquire certain number handling skills that will be useful to them in academic and real situations. McIntosh et al. (1992) note that when first learning numbers, there are children who exhibit creative and efficient strategies, but the emphasis placed on formal algorithms halts the use of individual and informal methods and rule based procedures become the preferred solving methods.

Different studies have shown that despite the importance given to number sense in the curricula of various countries, when students are given activities that can be solved using different strategies (such as using properties, relating operations, estimating, etc.), they prefer to follow algorithms and rule-based methods to arrive at the exact answer (Alsawaie, 2011; Markovits & Sowder, 1994; Reys, & Yang, 1998; Veloo, 2010; Yang, 2005; Yang, Li, & Lin, 2008; Yang & Tsai, 2010). Even if curricula specify the development of number sense, teachers continue to promote learning that is more based on numerical rules and algorithms. In fact, Alajmi and Reys (2007) indicate that in-service teachers tend to use rules and calculations to check whether a number answer is reasonable and accept as correct those that involve an exact calculation and are close to the precise answer. In said work, even though teachers recognized that estimation is a useful tool in everyday life, they believed that learning in school must focus on acquiring number knowledge through the application of rules. We regard this view of learning as halting the development of number sense in the classroom.

Most of the work done in this area reaches these conclusions based on written tests in which the students give a single answer, sometimes with no explanation or justification. In this study, we consider whether students who rely on formal, rule-based procedures know of other strategies associated with a proper number sense but that are not shown in their *first answer*. This serves to expand the view of the work done involving number sense in secondary school students by considering whether, or not, they know of different strategies to approach the same task. Perhaps students who offer a formal answer feel more comfortable with those methods, feel bound by the rules (Sengul & Gulbagci, 2012) or think that their teachers expect rule-based answers as part of the didactic contract between student and teacher (Brousseau, 1998).

We also observe if there exist a series of common features based on student groups that can be used to identify them through behavioral profiles when facing tasks involving number sense. This was all undertaken using a qualitative analysis methodology that relies on individual interviews aimed at finding the various methods known to students for solving problems.

Background

Sowder (1992) noted that the lack of an operational definition of number sense is a significant impediment to its evaluation, which is why various components have been used to characterize it. Proposals for said characterization have been put forth using different numbers of components, all of them being grouped or divided depending on their practicality to the research (McIntosh et al., 1992; Reys & Yang, 1998; Almeida, Bruno y Perdomo-Díaz, 2014, 2016; Berch, 2005; Methe et al., 2001; NCTM, 2000). Table 1 shows a component description, framework of our research, which considers the components of the different proposals made to date. This framework features two groups of components, the first directly related to knowledge and facility with numbers and the second to the application of said knowledge and facility with solving number problems.

Table 1.
Components of number sense

Knowledge of and facility with numbers and operations
Component 1. Understand the meaning of numbers.
Component 2. Recognize the relative and absolute size of numbers and magnitudes using estimates or numerical properties to make comparisons.
Component 3. Use benchmarks to estimate a number or magnitude when comparing or doing calculations.
Component 4. Use graphical, manipulative or pictorial representations of numbers and operations.
Component 5. Understand operations and their properties.
Apply knowledge and ease of use of numbers and operations to solve number problems
Component 6. Understand the relationship between the problem's context and the operation required.
Component 7. Realize that there are multiple strategies.
Component 8. Recognize the reasonableness of the problem.

We must bear in mind that even though number sense is arranged along these primary and seemingly independent components, they are strongly correlated.

Most research on number sense has been carried out on primary school students (Alsawaie, 2011; Veloo, 2010; Yang, 2005; Yang et al., 2008; Yang & Tsai, 2010), with less emphasis being placed on secondary education (Markovits & Sowder, 1994; Veloo, 2010; Yang et al., 2008). The researchers who have evaluated number sense in students from both educational stages have concluded that most students tend to use rules and algorithms when solving number problems (Veloo, 2010; Yang, 2005; Yang et al., 2008). Inadequate number sense skills have also been found in research involving in-service and pre-service primary and secondary school teachers, which highlights the need to address this deficiency as part of their university (Veloo, 2010; Almeida, Bruno y Perdomo-Díaz, 2014, 2016; Tsao, 2004; Sengul, 2013; Yang, Reys, & Reys, 2009). These findings have serious implications for classroom instruction, especially if we consider that a limited knowledge of a subject restricts teachers' ability to promote conceptual learning in their students.

A person's number sense plays an important role when deciding on a calculation method for a given situation: strict calculation, mental calculation or estimate. Sowder (1992) notes that instruction on estimating and mental calculation is one pathway to developing number sense. In this regard, researchers have analyzed estimating and mental computation skills in comparison to the general level of number sense, the conclusion being that the flexible use of numbers when estimating and recognizing a suitable estimate is a good indicator of number sense (NCTM, 2000; Sowder, 1992). In general, students who exhibit good results with strict calculations do not obtain the same results in those tasks where they need to make use of number sense. Findings from these studies confirm the idea that strict calculation skills without understanding are of little use in contexts that require more than just algorithms (NCTM, 2000; Veloo, 2010; Mohamed & Johnny, 2010). In contrast to the foregoing, a significant correlation has been observed between high academic grades in mathematics and a good level of number sense in students (Yang et al., 2008). Number misconceptions are often a hindrance to the development of proper number sense strategies, as observed by Sengul and Gulbagci (2012) regarding decimal numbers.

Among the research that analyzed the most complex components for students, Yang, et al. (2008) and Mohamed and Johnny (2010) found that understanding the relative effect of operations (Component 5, Table 1) and checking the data and recognizing when a result is reasonable (Component 8, Table 1) are the components that present the most difficulties.

Helping students develop number sense is an important objective of mathematical instruction (Anghileri, 2006; Reys et al., 1999). Thus, research has been conducted to analyze classroom methodologies to develop it. Markovits and Sowder (1994) examined the effect of a method used in seventh grade whose purpose was to develop number sense through activities rich in number exploration, a search for relationships between numbers and operations and other activities in which the students had to invent or discover rules. Their findings indicate that number sense is developed over time and that in addition to acquiring new knowledge, the students could reorganize existing knowledge.

Similar results were obtained in more recent research in different countries (Veloo, 2010; Yang, et al., 2008). These studies reveal that proper instruction that develops number sense yields more significant learning than traditional methodologies.

By using technological tools, Yang and Tsai (2010) achieved better results in developing number sense in sixth grade students in Taiwan than other groups that adhered to traditional methods. The same did not occur, however, with high performing sixth-grade mathematics students in Abu Dhabi (United Arab Emirates) who had been taught using reformed textbooks since the first grade is an effort to develop number sense (Alsawaie, 2011). The author concludes that reforming textbooks is not sufficient to improve number sense, and that the role of the teacher and the use of the proper method are key to developing number sense.

As noted earlier, the studies conducted do little to reveal how the students think and what possible alternatives they consider when answering a question. This led us to engage in our own research, where the students can explain their reasoning or to give other possible answers.

Objectives

The main goal of our research is to study how flexible secondary school students are when solving number tasks by considering their use of different strategies and identifying those that demonstrate the proper use of number sense. This study will serve to recognize those students who have a number sense that is not reflected in written exercises where they only provide one possible solution. This study will also allow us to establish student profiles based on the type of strategy they use to solve the problems. Said profiles are established for creating a tool that will facilitate the identification of students and how they solve number sense problems in future research. This objective is broken down as follows:

1. Analyze and categorize the different strategies used by secondary school students for the same number problem, noting whether they can employ more than one strategy and whether the use of number sense is present in any of them.

2. Establish student profiles based on the strategy type (by category) they use.

Since number sense is a very broad concept, our research focused on tasks that were designed to be solved primarily by using some of the components listed in Table 1, such as: Component 3. *Use benchmarks to estimate a number or magnitude when comparing or doing calculations*; Component 4. *Use graphical, manipulative or pictorial representations* (we will focus on the use of graphical representations). Although we expect to see certain number sense strategies that involve the above components, other strategies may appear that will also be analyzed.

Method

To address the stated objectives, we engaged in a qualitative study with the characteristics described below.

Sample

Eleven eighth-grade students ages 13 and 14 were interviewed at a public school in Spain (Student 1- 11). To select the students a written test was given to two groups of forty-seven eighth graders. The test contained items associated with the three number sense components described earlier (Bruno y Perdomo-Díaz, 2014, 2016). Based on the results of the test, the eleven students were selected. We ensured that the full group was well represented in the sample by including students who resorted to different strategies and who had different academic levels.

Instrument

The questionnaire used in the interview featured six items, five of them designed or adapted by the authors based on a review of the existing literature and other materials on number sense, and the sixth item adapted from Reys and Barger (1991). The purpose of the questionnaire was to encourage the use of number sense strategies that include the use of the specified components. Even though three components comprise the objective, the possibility exists that other strategies will appear.

Table 2.
Distribution of items and components

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
Component 3. Use benchmarks to estimate a number or magnitude when comparing or doing calculations.	x	x	-	x	-	x
Component 4. Use graphical, manipulative or pictorial representations of numbers and operations.	x	x	x	x	-	-
Component 8. Recognize the reasonableness of the problem	x	x	x	x	x	x

Procedure

The interview was divided into two phases. In the first part, the students answered five items without any involvement by the interviewer to obtain what we will call the *first answer*. At the end of this phase, the interviewer asked the students to explain the reasoning used with each item. In those cases, where the students did not make use of number sense, they were asked about another type of reasoning to gain some insight into possible knowledge of number sense strategies that the students were not exhibiting. When the students made use of number sense and a strategy that relied on another of the study's components was observed, they were asked about another possible justification. In general, an effort was made to have each student answer the questionnaire by making use of all possible strategies involving the use of the components contained in this study, thus arriving at a *second answer* or even at a *third answer*.

The interviews were video recorded and lasted from 25 to 45 minutes, depending on the range of strategies used by each student.

The interviews were analyzed by classifying the answers into the following categories, which were adapted from the classification proposed by Yang et al. (2009):

- Number sense based (NS), when using one or several components of the number sense framework;
- Non-number sense based (NNS), if they only made use of algorithms or memorized rules;
- Partially number sense based (PNS), if they combine the use of components of number sense by using memorized rules and / or algorithms;
- Other (Oth), students do not provide sufficient grounds to identify what reasons led them to the answer(s); or if there is no justification;
- Blank (B), if there is no answer to the question.

In addition, the codes 1 and 0 were used to track correct and incorrect answers.

Thus, an answer in Table 3, 4, 5 and 6 classified as 1NS indicates a correct answer obtained using a number sense strategy, while 0NS indicates that while the strategy used relied on number sense, the final solution is incorrect.

Results

The results are shown in two sections. In the first we analyze the different strategies encountered in each item and we show sample responses. In the second we describe the student profiles based on their answers.

Strategies Used by the Students for Each Item

Following a qualitative analysis of the different strategies used to solve each question, the answers to the six problems were classified (Table 3, 4, 5 and 6). The classification shown is for the *first answer* and, following the interviewer's request to consider solving the task using a different method, for the *second* and *third answers* (if applicable).

Strategies for item 1

Item1. Raquel has 9 glasses with 0.45 liters of water that she wants to move. She has a 5-litre jug and states she should make two trips. What do you think?

The goal was to see if they were capable of estimating by making use of benchmarks such as, for example, 0.5 instead of 0.45, or 10 glasses instead of 9, which would allow solving the problem without the need for an exact calculation. Another strategy considered beforehand was to draw the situation (graphical representation) and partition the jug such that the representation could be used to solve the problem. Only one student (Student 1) solved this task using number sense in his *first answer*, and only four used number sense for their *second answer* (Students: 2, 3, 4 and 8), all of them correctly. The remaining students relied on algorithmic multiplication to obtain an exact result, except for Students 6 and 11, who made calculation mistakes or misinterpreted the situation. The two examples of answers obtained using number sense and one example of a non-number sense strategy are shown.

Number sense

- Component 3: “If we had 10 glasses that would be 4.5 l, which the jug can hold and we have less” (Student 1’s *first answer*).
- Component 4: “Each little square could hold half a liter of water, so with each glass I’ll fill a little less. In all I’ll fill 4 liters and a little bit” (Student 4’s *second answer*).

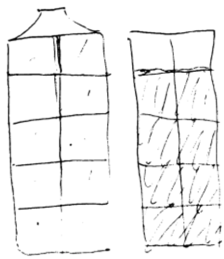

Figure 1. Student 4’s second answer to item 1.

Non-number sense

- Multiplication algorithm: “No, because she only has to carry 4.05 liters of water” (Student 4’s *first answer*).

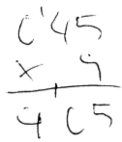

Figure 2. Student 4’s first answer to item 1.

These two examples show how Student 4 solved the problem first with an algorithm (Figure 2) and then, when asked to solve it another way, resorted to a graphical representation (Figure 1).

Strategies for item 2

Item 2. Sort the fractions 2/5, 7/8 and 4/3 from smallest to largest.

The use of benchmarks or graphical representations to compare fractions was the main goal of this exercise. Only two students (Students 1 and 2), provided an initial answer based on number sense, with seven students employing number sense to arrive at a *second* or *third answer* (Table 3). Student 9 and Student 10 did not use a proper strategy based on number sense. In contrast, it is interesting to note that six students obtained an incorrect *first answer* using rules or other procedures, with three of them subsequently obtaining the correct answer using number sense. Let us look at an example of each of these strategies.

Number sense

- Component 1: (Student 1’s *first answer*).

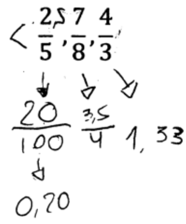

Figure 3. Student 1's first answer to item 2.

The strategy used by Student 1 in his *first answer* is to express the fractions as decimals or as simpler fractions that can then be more easily compared (Figure 3). This type of strategy we classify in the *non-number sense* based category when carried out using the division algorithm. In this case, however, the student did it mentally, justifying the result using number properties by looking for equivalent fractions in those cases in which the mental calculation of the division proved challenging. First he mentally calculated 4/3 as equivalent to 1.33. He then looked for an equivalent fraction for 2/5 (though he made a mistake, resulting in an answer that we code as incorrect and based on number sense) that allowed him to express it more simply as a decimal. Lastly he looked for an equivalent fraction for 7/8 that allowed him to decide where to situate this last fraction without having to convert it to a decimal. This student demonstrated a knowledge and understanding of the number system, which is directly related to component 1 in the number sense framework (Table 1).

- Component 3: "4/3 is more than one because one would be 3/3. 2/5 is less than one half and 7/8 is almost one" (Student 4's *second answer*).

Thus, number sense strategy compares the fractions given by using benchmarks without relying on any calculations or graphical representations. Specifically, it compares the fractions to one and to the fraction ½. Other students only compared them to one, this being sufficient to obtain the correct answer knowing the distance that exists between the fractions given.

- Component 4: (Student 3's *third answer*, Figure 4).

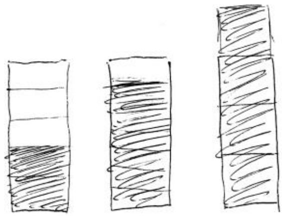

Figure 4. Student 3's third answer to item 2.

Although most of the students attempted a graphical representation in their initial or subsequent answers, these were not always correct and sometimes contained conceptual errors involving fractions.

Non-number sense

- Expressed the fractions with a common denominator by using an algorithm: (Student 4's *first answer*).

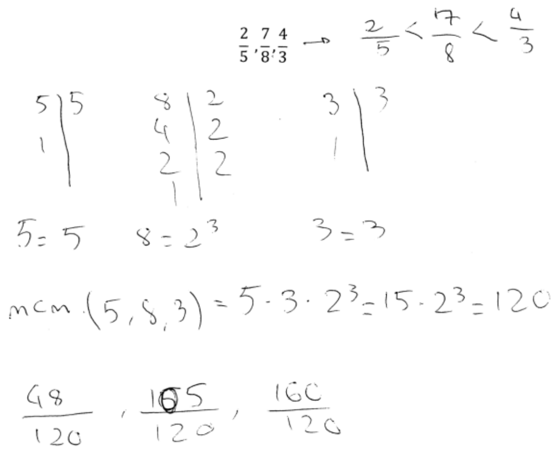

Figure 5. Student 4's first answer to item 2.

The reasoning demonstrated, such as expressing the fractions with a common denominator by using an algorithm (calculating the least common multiple, Figure 5), was included in the non-number sense category.

- Expressed the fractions as decimals using the division algorithm: (Student 8's *first answer*).

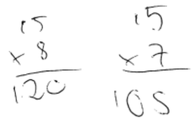

Figure 6. Student 8's first answer to item 2.

This reasoning only appeared once for this item. As Figure 6 shows, the Student 8 used the division algorithm, but did so incorrectly with two fractions, resulting in an incorrect answer being obtained.

Strategies for item 3

Item 3. Imagine you're celebrating your birthday and your mom has baked two cakes. It's time to blow out the candles and then you, as the birthday boy/girl, must cut the cake. For yourself you cut 1/4 of a cake, you give your family 2/3 of a cake and your friends eat 6/8 of a cake. Will you have more than half a cake left over?

The cake problem was included on purpose, the goal being to have the students use a graphical representation to answer the question. For this item, only three students (Students: 8, 10 and 11) used number sense in their *first answer*, and only Student 8 did so correctly. Of those students, whose *first answer* was based on the use of a rule or algorithm, all except Student 1 obtained their *second answer* using number sense.

As in previous items, we found different answers based on number sense.

Number sense

- Components 1, 3 and 5: "(…) ¼ + 2/3 is almost one cake, you would have some left over, 6/8 is equivalent to 3/4 … (thinks). Oh ok! ¾ + ¼ is one cake and 2/3 is more than half a cake, so you would have less than half left over" (Student 4's *second answer*).

This student resorted to the concept of fractions when simplifying (component 1) and realized that by adding two of them he would get a whole and that the third fraction is greater than one half; in other words, he selected the unit as his benchmark (component 3), associating the fractions (using the associative and commutative properties of addition, component 5) to simplify the comparison of the sum.

- Component 4: "If I represent the pieces in three cakes and then combine them as if I had two, I see that there is less than half a cake left over" (Student 3's *second answer*, Figure 7)

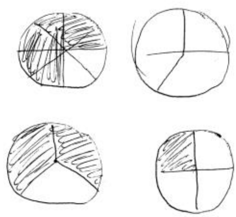

Figure 7. Student 3's second answer to item 3.

As with item 2, some of the students who tried to answer graphically exhibited deficits and conceptual errors involving the addition of fractions, which we classified as incorrect number sense.

Partial number sense

- Component 3 and use of the least common multiple: "I calculate the least common multiple and divide the two cakes" (Student 2's *second answer*, Figure 8).

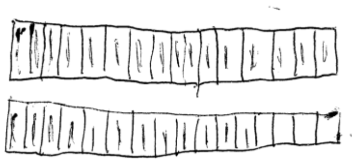

Figure 8. Student 2’s second answer to item 3.

The above strategy was classified as using *partial number sense* since even though Student 2 used a graphical representation, to do so he calculated the least common multiple and used an algorithm to express the fractions with a common denominator. In this case we should note that the student made a mistake in calculating the least common multiple as 16, though the process used was correct.

Non-number sense

Expressed the fractions with a common denominator using an algorithm. “I would have 8/24 left over, less than half a cake” (Student 6’s *first answer*, Figure 9).

$$\frac{6}{24} + \frac{16}{24} + \frac{18}{24} = \frac{40}{24}$$

Figure 9. Student 6’s first answer to item 3.

This strategy was very common among the students (Students 1, 2, 3, 5, 6 & 7), though it was not always used correctly, and was the only one classified in this category.

Strategies for item 4

Item 4. Given that ¼ + 6/8 = 1, what can we say about the following sums? a) Is 2/7 + 6/8 greater or less than 1? b) Is ¼ + 3/5 greater or less than 1? c) Is 4/9 + 3/10 greater or less than 1?

Item 4 was proposed to have students use a reference to estimate the sums of two fractions, since they were given one, or to have them use a graphical representation. Students 1, 2, 3, 4 and 5 replied correctly using number sense for at least two of the questions. The remaining students used algorithms, none of them correctly. They also failed to obtain a *second answer* using number sense.

Number sense

- Component 3: "One fraction equivalent to 3/5 would be 6/10, which is less than 6/8, meaning the sum would be smaller" (Student 4's *first answer* to part b); "4/9 is almost one half and 3/10 is much less than one half, so the sum is less than 1" (Student 4's *first answer* to part c).

We encounter two different strategies that use one reference exclusively. In Student 4's answer to part b, he compares the second summand of the sums to estimate which is greater. In contrast, in part c he uses ½ as a reference to compare both factors and determine whether the result is greater or less than unity. In this case, he does not use the reference given, which demonstrates flexibility in his way of thinking.

- Component 4: "1/4 + 3/5 is less than 1" (Student 2's *first answer* to part a, Figure 10).

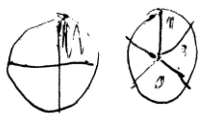

Figure 10. Student 2's first answer to item 4.

In the above strategy, Student 2 resorts exclusively to the use of a graphical representation without references. He visually estimates that the result will be smaller than one by graphically representing the two factors being added.

- Components 3 & 4: "¼ + 3/5 is less than 1 because if we compare it to the sum ¼ + 6/8 we see that 3/5 is less than 6/8" (Student 3's *first answer* to part b, Figure 11).

Figure 11. Student 3's *first answer* to item 3.

In Student 3's last correct answer using number sense, we find the combined use of graphical representations (component 4) and benchmarks (component 3). As with Student 4, Student 3 compares the terms that are different in the sums by graphically representing both.

Non-number sense

- Memorized rule: "I compare the fractions by subtracting the numerator and denominator. If the difference is greater the fraction is smaller" (Student 9's *first answer*).

This strategy was classified as an incorrect memorized rule in which the student applies a rule without making sense of its meaning or attempting to justify it. Moreover, as happened in item 3, we find the use of algorithms for adding fractions involving the calculation of the lowest common factor (similar to Figures 5 and 9).

Strategies for item 5

Item 5. Suppose the group from 2A takes three hours to paint a room, while group 2B takes six hours to paint the same room. If the two groups work together, how long will it take them to finish the job? Select the most reasonable answer. a. 18 hours. b. 9 hours. c. 4 hours d. 3 hourse. 2 hours f. 1 hour g. ½ hour

Item 5 poses a situation that can be solved in different ways, far removed from rule-based procedures. In this item, we see not only how students approach the solution, but whether they evaluate their answers against the other options and if they can find arguments to justify them.

In the resulting answers, we only found one correct strategy that used number sense, given by two students (Students 3 and 6) as the *first answer*, and by two others (Student 1 and Student 5) as their *second answer*. It was thus a complex problem to solve. In fact, seven students did not find the right answer and did not give a correct answer on their second attempts to solve it.

Number sense

- Component 8: “It would take roughly 2 hours because if the two groups took 3 hours separately, in 1.5 hours they would paint the room together, but since 2B takes longer, it would take them 2 hours” (Student 1’s *second answer*).

Non-number sense

- Use calculation algorithms: The answers that do not use number sense are those that apply some rule or unjustified computation, like Student 5, who in his *first answer* added the hours and said it would take them nine hours. In this case the student also showed a lack of number sense by not realizing that his answer was not reasonable.

We found with this group of students that they calculated the arithmetic average of the hours it took each group, while others decided to apply a division algorithm since they knew that the result had to be smaller, and division, in their opinion, is the operation that can lead them to the correct answer. But they did not give sufficient justification for why they carried out the computation.

There were also students who did not offer any type of basis for selecting an option, even when the interviewer asked about their reasoning. These were classified as *Other* if they at least provided an answer, or *Blank* otherwise.

Strategies for item 6

Item 6. Your school institutes a new rule whereby all students must hand in their cell phones at the office when entering. The principal keeps them all in a box. If all students hand in their phones, approximately how much would the box weigh if all the students handed in their phones?

Item 6 was included to see if students could make use of personal benchmarks to estimate magnitudes. Given the nature of the problem, all the strategies encountered in this item resorted to estimates. Thus, the only categories possible include the use of number sense to some extent.

In the number sense answers, the students estimated the total weight of the box by estimating the magnitudes (weight of a phone and number of students) and then doing a mental calculation of the factors involved. In the answers that used number sense partially, the students carried out the multiplication to calculate the exact weight based on their benchmarks.

The students' answers yielded estimates for the weight of a cell phone of between 50 and 400 grams, and the number of students between 136 and 300. As concerns the number of students, the school has four grades with two groups each. The number of students in each group ranges from 15 to 33, so a reasonable answer would be in the range of 120 to 264 students. As for the weight of a cell phone, this parameter is more variable given the diversity of phones available on the market. After conducting a study of the devices currently available, we found that most phones weigh between 100 and 250 grams, though we also found that latest generation phones are increasing in size and occasionally weigh more than 250 grams. We also noticed that the phones before the current smartphones were lighter than those sold today. Thus, and since it was only an estimate, we accepted as correct those answers that assumed a weight between 50 and 300 grams.

We will now present some of the answers.

Number sense

- Component 3: "A cell phone weighs about 200 grams and there are approximately 200 students in the school, which gives 40000 grams, or 40 kilos" (Student 11's *first answer*, Figure 12).

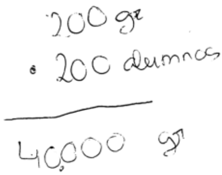

Figure 12. Student 11's *first answer* to item 6.

Student 11 used two benchmarks, the weight of a cell phone (200 gr) and the number of students in the school (200), to estimate the total weight of the box.

Partial number sense

- Component 3 and multiplication algorithm: "24 Students in each class times 8 groups. Each phone weighs 400 grams. The total is 86.8 kilos" (Student 8's *first answer,* Figure 13).

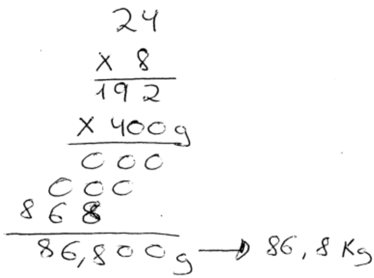

Figure 13. student 8's *first answer* to item 6.

This student wrongly estimated the weight of a cell phone and although the procedure was correct, he did not obtain a reasonable answer.

In both categories, we found students who made good estimates for the parameters involved but who incorrectly calculated the final weight, and vice versa, students who used incorrect benchmarks but whose calculations were correct, which obviously yielded an incorrect final answer. In this latter case, there were students who regarded their final answer as valid, meaning they did not evaluate its correctness (component 8).

To estimate the number of students at the school, some students used the number of students in their class or in another class at the school as a benchmark. They made no obvious use, however, of a personal benchmark for the weight of a cell phone.

Student Profile Based on the Use of Number Sense

Following a qualitative analysis of the different strategies used to solve the problems, the answers of the 11 students were classified (Tables 3, 4, 5, and 6). Shown are the classifications for the *first*, *second* and *third answers* (depending on the case). Component of number sense used for each answer are in brackets (e.g., 1NS(3) indicates correct answer with number sense, using Component 3).

This analysis shows that while the students respond differently, some share common characteristics that allow us to establish four profiles based on their initial answer and on the subsequent alternatives involving their use of number sense. These are not pure profiles in the sense that one student may better fit the definition of one profile than another; rather, the profiles are grouped because they share certain characteristics in common, despite also sharing certain minor differences.

Profile 1: Tendency to use number sense strategies. We placed two students (Student 1 and Student 2) in this profile. These students are generally characterized by their mostly correct use of number sense strategies. There was a tendency to look for number sense strategies not only for the *first answer*, but when asked for alternative strategies. They exhibited flexibility when looking for different strategies and despite the occasional use of rules, they showed an ability to avoid them when a number sense strategy was available.

Profile 2: Tendency to use rules and algorithms. Aware of number sense strategies. The three students we placed in this profile (Student 3, Student 4 and Student 5) are characterised by the fact that they answered over half of the problems using rules and algorithms, though they demonstrated an ability to use number sense strategies correctly when asked for a *second answer*. Most of these students' answers were correct, though Student 5 gave incorrect answers to four problems when not using number sense and correct answers when using number sense strategies. They exhibited a knowledge of different strategies when asked for alternatives, but they preferred rules and algorithms, stating that to justify their reasoning they needed to write down some kind of calculation.

Table 3.
Student answers and classification: Profile 1

Student	Answer	Item 1	Item 2	Item 3		Item 4	Item 5	Item 6
1	1st	1NS(3)	0NS(1)	1NNS	a	1NS(1)	0NNS	0NS(3)
					b	1NS(1)	0NNS	0NS(3)
					c	0NS(1)	0NNS	0NS(3)
	2nd	-	1NS(3)	-		-	1NS(8)	-
	3rd	-	0NS(4)	-		-	-	-
2	1st	1NNS	1NS(3)	0NNS	a	1NS(3)	0Oth	0NS(3)
					b	1NS(4)	0Oth	0NS(3)
					c	1NS(3)	0Oth	0NS(3)
	2nd	1NS(3)	1NS(4)	0PNS(4)		-	-	-
	3rd	-	-	-		-	-	-

*Profile 3: Tendency to use rules and algorithms. Unaware of number sense strategie*s. The students in this group (Student 6, Student 7, Student 8 and Student 9) answered more than half of the problems by using rules and algorithms and when asked for an alternative method, they exhibited a lack of number sense in their efforts. Moreover, in their rule-based answers they also differed from profile 2 by having a smaller percentage of right answers. In addition, when trying to make use of number sense, they exhibited conceptual difficulties that led them to select the wrong answers and faulty reasoning.

In this profile the justification for choosing rule-based reasoning as their first option stemmed from a lack of confidence brought on by conceptual obstacles. This led them to resort solely to memorized rules that require no knowledge of the justification that makes them valid.

Table 4.
Student answers and classification: Profile 2

Student	Answer	Item 1	Item 2	Item 3		Item 4	Item 5	Item 6
3	1st	1NNS	1NNS	1NNS	a	0NNS	1NS(8)	1NS(3)
					b	1NS(3 &4)		
					c	1NS(4)		
	2nd	1NS(3)	1NS(3)	1NS(4)		-	-	-
	3rd	-	1NS(4)	-		-	-	-
4	1st	1NNS	1NNS	1PNS	a	1NS(3)	0B	1PNS(3)
					b	1NS(4)		
					c	1NS(4)		
	2nd	1NS(4)	1NS(3)	1NS(4)		-	-	-
	3rd	-	1NS(4)	-		-	-	-
5	1st	INNS	0NNS	0NNS	a	1NS(3)		
					b	1NS(3)	0NNS	1PNS(3)
					c	0NNS		
	2nd	-	1NS(3)	1NS(4)		-	1NS(8)	-
	3rd		0NS(4)	-		-	-	-

Profile 4: Problems with mathematical content. Unaware of number sense strategies. The students in this group (Student 10 and Student 11) exhibited no clear tendency to use rules or algorithms, but also tended not to use number sense, as their first strategy. Their answers to the six problems were varied and characterized by a lack of mastery of the mathematical principles involved. This resulted in their use of inconsistent reasoning that was erroneous more than half the time. When asked to use other methods, they failed to obtain the correct answer. For example, they incorrectly used graphical representations of fractions due to their misconceptions of the construct.

Oftentimes their efforts to make use of number sense, along with the conceptual deficits they exhibited, caused them to give incorrect answers,

Table 5.
Student answers and classification: Profile 3

Student	Answer	Item 1	Item 2	Item 3		Item 4	Item 5	Item 6
6	1st	0NNS	0Oth	1NNS	a	0NNS	1NS(8)	0PNS(3)
					b	0NNS		
					c	0NNS		
	2nd	-	1NS(3)	0NS(4)	a	0NS(4)	-	-
					b	0NS(4)		
					c	0NS(4)		
	3rd		INS(4)					
7	1st	1NNS	1NNS	0NNS	a	1NNS	0NNS	1PNS(3)
					b	0NNS		
					c	0B		
	2nd	-	1NS(3)	1NS(4)	a	0NS(4)		
					b	0B	-	-
					c	0B		
	3rd	-	0NS(4)	-		-	-	-
8	1st	1NNS	0NNS	1NS(4)	a	0NNS		
					b	0NNS		
					c	0NS(1)		
	2nd	1NS(4)	0NS(3)	-	a	0NS(4)	-	-
					b	0B		
					c	0B		
	3rd	-	1NS(4)	-		-	-	-
9	1st	1NNS	0NNS	0Oth	a	0NNS	0NNS	1NS(3)
					b	0NNS		
					c	0NNS		
	2nd		0NS(4)	0NS(4)	a	0NS(4)	-	-
					b	0NS(4)		
					c	0NS(4)	-	-
	3rd	-	-	-		-	-	-

resulting in their reasoning being classified as *Other*, since they did not offer sufficient arguments for classification in another category.

Table 6.
Student answers and classification: Profile 4

Student	Answer	Item 1	Item 2	Item 3		Item 4	Item 5	Item 6
10	1^{st}	0B	0NNS	0NS(4)	a	0B	0NNS	0NS(3)
					b	0B		
					c	0B		
	2^{nd}	INNS	0NS(4)	-	a	0NS(4)	-	-
					b	0NS(4)		
					c	0NS(4)		
	3^{rd}	-	-	-		-	-	-
11	1^{st}	0NNS	0B	0NS(4)	a	0NS(4)	0NNS	1NS(3)
					b	0NS(4)		
					c	0NS(4)		
	2^{nd}	-	1NS(3)	-		-	-	-
	3^{rd}	-	0NS(4)	-		-	-	-

Conclusions

In this research, we consider a case study involving secondary school students, the purpose of which is to analyze how they approach tasks that can be solved using number sense. On the one hand, we analyze whether students offer a single answer to the same number problem. In particular, if they know of any number sense strategies after initially approaching the problem using algorithms or rules. On the other hand, we establish behavior patterns or profiles for the students' answers regarding the use of number sense. Given the broad range of aspects encompassed by number sense, we limited these to tasks associated with three components, specifically, Component 3, 4 and 8 (Table 1).

The results of the students analyzed indicate that the use of different strategies depends on their mathematical knowledge. We see that even though the students' *first answers* were sometimes based on the use of rules or algorithms, some were also able to resort to strategies associated with number sense. This could be indicative of what students think their teachers expect of them, meaning it could be an obstacle present in the didactic contract (Brousseau, 1998).

Of the students analyzed, those who gave the most wrong answers to the tasks were those whose answers were least associated with number sense, even though all the tasks were from an academic level below their current grade level. Similarly, the students with the rightest answers were also able to find alternative answers. This indicates the difficulty involved in using number sense when a mastery of number knowledge (conceptual or procedural) is lacking.

Our findings allowed us to establish four Student profiles regarding the use of number sense: the purest number sense profile for the student who generally seeks out correct number sense procedures and avoids algorithms; a profile that tends toward rules, despite knowing number sense strategies, and using both more or less correctly; a profile that tends toward rules and makes mistakes when using number sense; and lastly a profile for students who do not use number sense and who also do not know rule-based strategies, probably due to conceptual and procedural obstacles.

Establishing student profiles to represent the use of number sense is a very useful research tool in this area as it allows for a more in-depth identification of the students, one that is based not only on their *first answers*. It also has a bearing on learning since it can be used to develop learning sequences that consider these characteristics such that students can build a proper concept of number sense. The learning environment must also allow students who tend to rely on rules to use and develop alternative strategies they already know.

This study presents certain limitations; specifically, we focused on three components of the framework, though it could be expanded to analyze other components and other levels, or even to contrast the profiles based on educational level (primary and secondary) or to differentiate between the various educational levels. As a tool, these profiles can be used to compare how a student might improve after taking part in a program designed to improve number sense, or simply to compare one student with another.

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