

## Self-Similarity of Space Filling Curves

## Auto-similaridad de las Space Filling Curves

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### Abstract

We define exact self-similarity of Space Filling Curves on the plane. For that purpose, we adapt the general definition of exact self-similarity on sets, a typical property of fractals, to the specific characteristics of discrete approximations of Space Filling Curves. We also develop an algorithm to test exact self-similarity of discrete approximations of Space Filling Curves on the plane. In addition, we use our algorithm to determine exact self-similarity of discrete approximations of four of the most representative Space Filling Curves. We found that SFCs like Moore's based on recursive structure are actually not self-similar, highlighting the need to establish a formal definition of the concept for SFCs.

**Keywords:** *Fractals, self-similarity, Space-Filling Curves.*

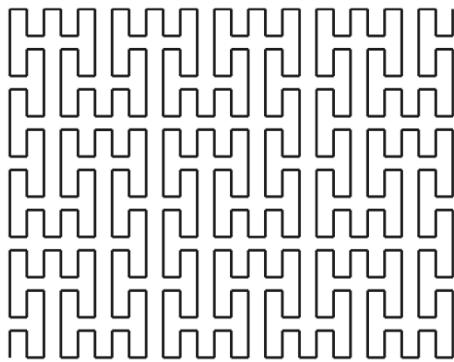
### Resumen

El propósito de este artículo es desarrollar un test que permita determinar la auto-similaridad de una Space Filling Curve (SFC), estudiándolas desde el punto de vista de la teoría fractal y concentrándonos en la propiedad de auto-similaridad. El test consiste de dos fases, en la primera se identifica una partición especial de la curva denominada partición *CM* y luego se muestra que la curva es auto-similar si y sólo si es auto-similar bajo dicha partición. Adicionalmente, el test es aplicado a cuatro famosas SFC (Peano, Moore, Meander y Lebesgue) para determinar su auto-similaridad. Se encuentra que algunas SFC como la de Moore con estructura recursiva y aparente auto-similaridad son en realidad no auto-similares, resaltando la necesidad de formalizar el concepto.

**Palabras Clave:** *Auto-similaridad, fractales, Space Filling Curves.*

## 1. Introduction

In 1878, Cantor established the equipotency between  $\mathbf{R}$  and any  $n$ -dimensional euclidean space  $\mathbf{R}^n$  (Sagan, 1994a). In the plane, it implies the existence of a bijective mapping from  $\mathbf{R}$  to  $\mathbf{R}^2$  (an interval can be mapped into the plane). A year later, Netto proved that such mapping cannot be continuous (Bader, 2013). A decade later, Peano proved the existence of a continuous surjective mapping from  $\mathbf{R}$  to  $\mathbf{R}^2$  (Sagan, 1994c). He proposed a continuous curve that goes through every point of the plane (Figure 1), and those curves were called Space Filling Curves (SFCs). SFCs has fascinated mathematicians since then, and brilliant mathematicians as Hilbert, Moore, Lebesgue, Meander, Poyla and Sierpinski has proposed their own SFCs exploring different properties and perspectives (Sagan, 1994a).



*Figure 1. A discrete approximation of the Peano Space Filling Curve.*

But mathematicians are not the only ones who have been fascinated with SFCs, they also are extensively used in engineering (Bader, 2013). SFCs are used to solve combinatorial optimization problems in industrial contexts (Hu & Wang, 2004; Hua & Zhou, 2008). They also have been used as a scanning technique for image processing and compression (Dafner et al., 2000). Anotaipaiboom & Makhanov (2005) used SFCs to plan movement paths of robotic arms with numeric control. Additionally, Valle & Ortiz (2011), Bahi et al. (2008), and Chen & Chang (2005) have explored applications of SFCs for clustering. Furthermore, SFCs have been used for

pattern recognition in several contexts (Rosenthal & Kouri, 1973; Lam & Liu, 1996).

In the literature of Space Filling Curves, mathematicians have stated a close relation between SFCs and fractals. SFCs have patrons and regularities that also are commonly found in fractals and that relation has been linked to the self-similarity property of fractals (Plaza et al., 2005; Haverkort & Walderveen, 2010; Skubalska-Rafajowicz, 2005). Wunderlich (1954) linked fractals and SFCs by introducing Iterated Functions System (IFS) in the generation of discrete approximation of Space Filling Curves. IFS is a method to generate discrete approximation of fractals by means of their self-similarity property and recursive algorithms. In this way, a discrete approximation of a fractal of  $n^{th}$  order is composed of a finite number of copies of itself that can only differ from the original in size and orientation and are integrated under a regular patron consistent through iterations.

McClure (2003) studied the self-similarity of the Hilbert curve, and lays out the advantages derived from this property by utilizing Iterated Functions System (IFS) as Wunderlich (Hutchinson, 1981). But, since not all SFCs are exact self-similar, in this paper, we propose a method to determine whether a discrete approximation of Space Filling Curve is exact self-similar.

Despite the extensive attention that SFCs have received in the literature and their evident link with fractals, there is no formal definition for their self-similarity. This paper presents a formal conceptualization of the self-similarity adapted to the context SFCs and an efficient computational test to determine whether a specific SFC is self-similar.

The paper is organized as follows: Section 2 presents an introduction to the theory of self-similarity and to SFCs theory, also a definition of exact self-similarity of discrete approximations of SFC. Section 3 presents the test that determine whether a discrete approximation of a Space Filling Curve on the plane is exact self-similar. In Section 4, we determine self-similarity of four

of the most representative SFCs using our test. Finally, Section 5 provides conclusions and future research.

## 2. Context

By definition, every fractal has the property of self-similarity (Barnsley, 1993). A set is self-similar if when we examine small portions of it, we only see reproductions of our initial set (Rubiano, 2002), as shown in Figure 2.

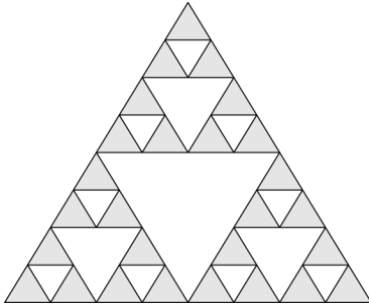


Figure 2. Sierpinski triangle.

Ferrari (2007) establishes a formal definition of self-similarity, as follows:

### Definition 1:

A compact set  $X \subset \mathbb{R}^2$  is self-similar if there is a finite set of contractive similarities  $F_N = \{F_1, F_2, \dots, F_n\}$  such that  $X = \bigcup_{i=1}^n F_i(X)$  and the subsets of  $X$  of the form  $F_i(X)$  overlap only in the boundaries.

Contractive similarities are affine transformations of the form  $F_i = r_i D_i x + b_i$ , where  $r_i$  is a scalar such that  $0 < r_i < 1$ ,  $D_i$  is the matrix of rotations and  $b_i$  is a vector in  $\mathbb{R}^2$  (Barnsley 1993). Note that  $\bigcup_{i=1}^n F_i(X)$  is a cover of  $X$ . In addition, the intersection of subsets  $F_i(X)$  is only allowed on the borders, a property which makes it a quasi-partition. In SFCs context, the latter restriction is even stronger, since these curves go through every point of the plane exactly once. Thus, in SFCs context,  $\bigcup_{i=1}^n F_i(X)$  is a partition of  $X$  and every  $F_i(X)$  is an equivalence class.

In this paper, we analyze SFCs from a combinatorial point of view, just as Dai & Su (2003) do. As well

as in geometry a triangle is represented by three non-collinear points, we represent a SFC by means of an ordered finite list of points,  $SFC = \{p_1, p_2, \dots, p_n\}$ . Figure 3a shows the Hilbert curve and the circles indicate the set of points by which it is represented. In this case, the Hilbert curve is found in the 4x4 lattice, hence 16 points are used to represent it. The order in which these points are found in the list, is the same order in which these points are found in the curve.

Figures 3b, 3c, 3d and 3e show partitions of the curve of Figure 3a. Figure 3b is a partition with four equivalence classes, and each equivalence class has four points. Note that the union of the four equivalence classes equals the 16 points in Figure 3a and no point belongs to more than one equivalence class.

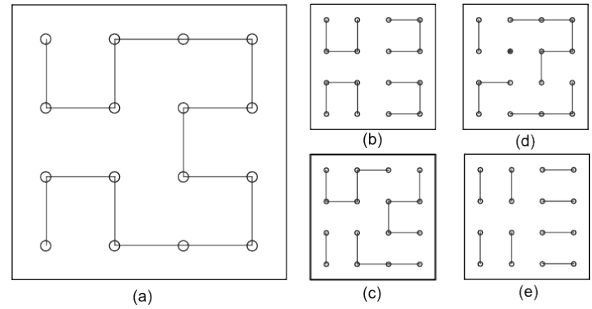


Figure 3. Different cardinality partitions of Hilbert curve.

## 3. Self-Similarity Test

It is seen in Figure 2 that the largest triangle is made up of three smaller triangles which in turn are made up of three smaller ones. To prove self-similarity of this set, it is sufficient to show its IFS, which is easily found by inspection. However, it is not always evident to prove self-similarity of a set.

According to the definition of self-similarity, in order to prove that a set is not self-similar, it is necessary to show the non-existence of a partition  $\mathcal{P}$  such that all equivalence classes  $p_i$  of  $\mathcal{P}$  be contractive similarities of  $X$ , that is,

$$\neg \exists \mathcal{P} \forall_i \exists_j \{p_i = F_j(X)\}$$

Thus the formula may be written as,

$$\forall_{\mathcal{P}} \exists_i \forall_j \{p_i \neq F_j(X)\}$$

This would imply that to prove the self-similarity of a compact set  $X$ , it is necessary to evaluate all partitions of  $X$  and find in everyone of them an equivalence class which is not an affine transformation of  $X$ . This is a very expensive computational task, mainly because the amount of partitions of  $X$  is very large (Rota 1964).

In this paper, we present a self-similarity test that is useful to determine whether a *SFC* is self-similar or not. For a given curve  $X$ , the test identifies a special partition of  $X$  that we denote  $CM$ , such that  $X$  is self-similar if and only if  $X$  is self-similar under  $CM$ . Lastly, it determines if  $X$  is self-similar under  $CM$ , or not. It is thus necessary to prove self-similarity of  $X$  under one partition only, which substantially lowers the computational cost.

The reader will next find section 3.1 where the procedure to identify the  $CM$  partition of  $X$  is described, and section 3.2 where it is shown how to determine the self-similarity of  $X$  under a partition  $\mathcal{P}$ .

Let the reader be reminded that the procedures and propositions from here on must be considered in the context of the *SFCs* and do not apply directly to every type of set.

### 3.1 $CM$ Partition of $X$

Let's call  $\mathbf{P}(X)$  the set of all partitions of  $X$ . Since  $CM \subset \mathbf{P}(X)$ , what we will do to find  $CM$  is to progressively eliminate all partitions  $\mathcal{P}$  of  $\mathbf{P}(X)$  under which we know beforehand that  $X$  is not self-similar.

To do this, we will resort to the properties of self-similar sets, in such manner that each property will help us discard those partitions of  $\mathbf{P}(X)$  that do not comply with it. Let's denote by  $H$  the subset of  $\mathbf{P}(X)$  in which we consider that  $CM$  is found. We start by equating  $H$  to  $\mathbf{P}(X)$ , and then, by applying each property we will gradually reduce the amount

of partitions included in  $H$  until the only one left is  $CM$ . The basic foundations for the process are the following two propositions:

*Proposition 1:* It is necessary that all equivalence classes  $p_i$  of  $\mathcal{P}$  be affine transformations of  $X$ , for  $X$  to be self-similar under  $\mathcal{P}$ .

*Proposition 2:* It is necessary that all equivalence classes  $p_i$  of  $\mathcal{P}$  be affine transformations of each other, for  $X$  to be self-similar under  $\mathcal{P}$ .

*Proposition 1* follows from *Definition 1*. For *proposition 2*, it has to be shown that:

If  $X$  is self-similar under  $\mathcal{P}$ , then,  $\forall_{i,k} \exists_G \{G(p_i) = p_k\}$

Where  $p_i$  is the equivalence class  $i$  of  $\mathcal{P}$  and  $G$  is an affine transformation. In this case,  $\forall_{i,k} \exists_G \{G(p_i) = p_k\}$  is a necessary condition for  $X$  to be self-similar under  $\mathcal{P}$ .

If  $X$  is self-similar under  $\mathcal{P}$ , then, according to *Definition 1*, each equivalence class of  $p_i$  of  $\mathcal{P}$  is a contractive similarity of  $X$ . That is,

$$\forall_i \exists_j \{p_i = F_j(X)\}$$

Ferrari (2007) shows that all contractive similarities are invertible and their inverse functions are also affine transformations. Let's call  $W_j$  the inverse function of  $F_j$ . Thus, we say that,

$$\forall_j \{W_j(F_j(X)) = X\}$$

Equivalently, it's stated that,

$$\forall_i \exists_j \{W_j(p_i) = X\}$$

Let's call  $F_k$  the contractive similarity, such that equivalence class  $p_k$  equals  $F_k(X)$ . Next, transformation  $F_k$  is applied to both sides of the previous equation and we get that,

$$\forall_{i,k} \exists_j \{F_k(W_j(p_i)) = F_k(X)\}$$

Finally, let's call  $G$  the composition of  $F_k$  and  $W_j$ ,  $G = F_k \circ W_j$ , assuring that  $G$  is also an affine transformation, as it is a composition of two affine transformations, thus getting,

$$\forall_{i,k} \exists_G \{G(p_i) = F_k(X)\}$$

Equivalently, we say that,

$$\forall_{i,k} \exists_G \{G(p_i) = p_k\}$$

This way, we have shown in this manner that for a compact set to be self-similar under  $\mathcal{P}$  it is necessary that all equivalence classes of  $\mathcal{P}$  be affine transformations of one another, in pairs.

It is inferred from *Proposition 2* that H should only contain partitions in which all equivalence classes be affine in pairs. Notice now that when applying an affine transformation to any curve (ordered set of points) the amount of points on the curve (cardinality) will always be conserved. This means that  $\forall_S \forall_F \{ \#(S) = \#(F(S)) \}$ , where  $S$  is a curve,  $F$  is an affine transformation and  $\#(S)$ ,  $\#(F(S))$  are the corresponding cardinalities of  $S$  and  $F(S)$ . By doing this, we assert that *if two curves do not have the same cardinality, then one is not an affine transformation of the other and vice versa*. From this assertion we derive *Criterion 1*.

*Criterion 1: We include in H only those partitions in which all equivalence classes have the same cardinality. This cardinality we shall call  $n$ .*

Figure 3a shows a sixteen-point approximation of the Hilbert curve. Different examples of partitions of this curve are shown in Figures 3b, 3c, 3d and 3e. Of the four partitions shown, Figures 3b and 3e comply with *Criterion 1* with  $n=4$  and  $n=2$  respectively. Figures 3c and 3d do not comply with criterion 1 since their equivalence classes have different number of points.

Note that it is necessary that all equivalence classes  $p_i$  of a partition  $\mathcal{P}$  be contractive similarities of  $X$ , for  $X$  to be self-similar under  $\mathcal{P}$  (*Definition 1*). Since the contractive similarities do not alter the amount of elements of the original set, the cardinality of  $X$  under  $\mathcal{P}$  and that of each equivalence class must also be guaranteed to be equal. Let's denote by  $m$  the cardinality of  $X$  under  $\mathcal{P}$ , which is the number of equivalence classes of  $\mathcal{P}$ . From this we derive *Criterion 2*.

*Criterion 2: We include in H only those partitions whose equivalence classes have the same cardinality of  $X$  under  $\mathcal{P}$ .*

The partition in Figure 3e does not satisfy *Criterion 2* because it has eight equivalent classes, that is,  $m = 8$  and its equivalence classes have 2 points each, i.e.,  $n = 2$ .

The partition in Figure 3c does not satisfy *Criterion 2* because it has eight equivalent classes, that is,  $m=8$  and its equivalence classes have 2 points each, i.e.,  $n = 2$ .

According with the above assertions, one concludes that H only contains partitions whose equivalence classes have the same cardinality, and in addition, that this cardinality must be equal to the number of equivalence classes, that is,  $n = m$ . Note now that the amount of points of a curve  $X$ , divided by the number of equivalence classes equals the number of points of each equivalence class, that is,  $\#(X)/m = n$ . Replacing  $n = m$ , we get  $\#(X) = n^2$ ; from which *Criterion 3* is derived.

*Criterion 3: We include in H only those partitions whose equivalence classes are equal to the cardinality of each equivalence class that is equal in turn to the square root of the cardinality of curve  $X$ .*

This means that for a sixteen-point curve, only those partitions which have four equivalence classes are useful, where each equivalence class in turn has four points (Figure 3b).

So far, H includes all partitions that satisfy criteria 1, 2 and 3. This is considering all partitions of  $X$  with  $n$  equivalence classes and each equivalence class with  $n$  points, i.e., the number of partitions of  $X$  included in H equals all possible combinations of the number of points of  $X$  in  $n$  subsets (combinatorial number).

We will now resort to the arc-connectedness property to clean out set H. Figure 4 shows four partitions of the Hilbert curve. Each partition has four equivalence classes and each equivalence class contains four points. Figure 4d shows two equivalence classes that

are arc-connected and two equivalence classes that are not arc connected. On the other hand, Figure 4e shows one arc-connected equivalence class and three equivalence classes that are not arc-connected.

Note that affine transformations do not affect a curve's arc-connectedness. Thus, the arc-connected equivalence class of Figure 4e cannot be transformed by means of an affine transformation to any of the other equivalence classes of this partition. It is therefore excluded from H, according to Proposition 2. We state that in general, if a partition  $\mathcal{P}$  has a non-arc-connected equivalence class, then  $\mathcal{P}$  is not included in H. Criterion 4 is derived from this.

*Criterion 4: We include in H only those partitions whose equivalence classes are arc-connected sets.*

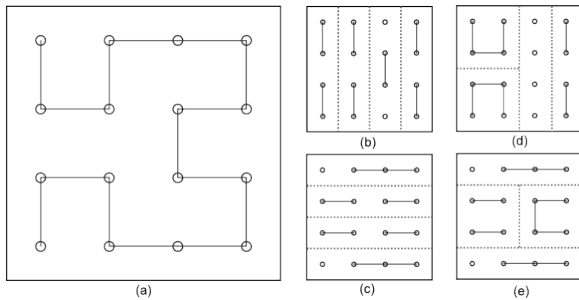


Figure 4. Non-arc-connected partitions of Hilbert curve

In Figure 5, there are four partitions of the Hilbert curve all of whose equivalence classes are arc-connected sets. The numbers in Figure 5a indicate the order in which are found the points located on the curve.

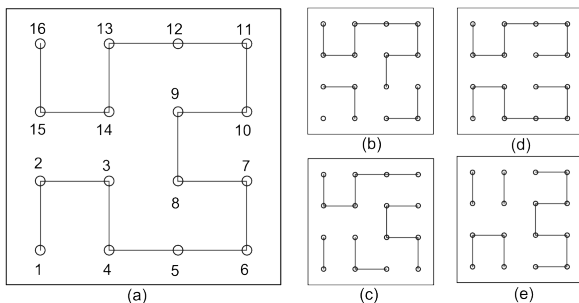


Figure 5. Connected partitions of Hilbert curve

The partition in Figure 5d contains two equivalence classes and each equivalence class contains eight points. The reader, by direct inspection, can verify that this is the only way to divide the curve of Figure 5a in two parts with the same number of points and that each part is arc-connected. Proposition 3 is derived from this.

*Proposition 3: If there is a partition of a curve X with m equivalence classes, such that each equivalence class is arc-connected and has n number of points, then that partition is unique.*

Sagan (Space-Filling Curves, 1994) defines an SFC as a continuous curve with a positive Jordan content. In this context, a continuous curve is a connected and locally connected compact curve (Sagan 1994: Space-Filling Curves, Chapter 1, Introduction).

Bing (1952) states that every continuous curve is a continuous transformation of an interval of a straight line. This leads us to conclude that SFCs are continuous transformations of a segment of a straight line and thus establish a relation of equivalence between them (Figure 6).

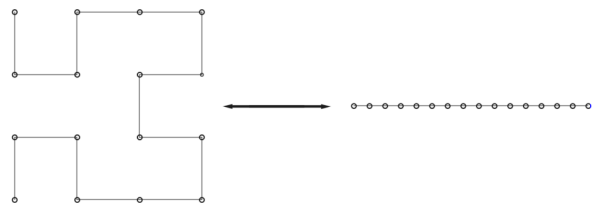


Figure 6. Equivalence of a SFC with a straight line interval.

Finally, we can use the theorem of unique segments (Bing 1952) to establish the existence of a unique equi-partition of the straight line interval to conclude that the partition is unique as proposition 3 states. From a discrete point of view, this means that there is only one way in which the straight line interval of Figure 6 can be divided into four parts with the same number of points.

Now, we integrate Criterion 3, Criterion 4 and Proposition 3. Criterion 3 indicates that in H we will

only include those partitions  $\mathcal{P}$  of  $X$  such that  $n=m=\sqrt{\#(X)}$ . Criterion 4 says that of these partitions, only are considered those whose equivalence classes are arc-connected. Lastly, *Proposition 3* states that if there is a partition of  $X$  satisfying both *Criterion 3* and *Criterion 4*, then that partition is unique. *CM* Partition of the Hilbert curve (Figure 3a) is shown in Figure 3b.

*Definition 2: Partition CM* of curve  $X$  is a partition of  $X$  that satisfies the following conditions:

it has  $n$  equivalence classes,

each equivalence class has  $n$  points,

and each equivalence class is an arc-connected set.

In addition, if *CM* exists, then it is unique.

The following is an algorithm to find partition *CM* of a given curve.

**Algorithm 1**

**Input:**  $X$ , curve represented by an ordered list of points.

**Output:** *CM*, structure storing ordered lists of points.

**procedure** CMpartition ( $X$ )

$$\{n \leftarrow \sqrt{\text{length}(X)}$$

$$CM \leftarrow \text{splitList}(X, n)$$

**return**  $CM$ }

Let's consider the example of Figures 3a and 3b. The curve in Figure 3a could be represented by the list of points in Table 1.

*Table 1: List of points of curve in Figure 3a.*

$x$	$y$
0,25	0,25
0,75	0,25
0,75	0,75
0,25	0,75
0,25	1,25
0,25	1,75
0,75	1,75
0,75	1,25
1,25	1,25
1,25	1,75
1,75	1,75
1,75	1,25
1,75	0,75
1,25	0,75
1,25	0,25
1,75	0,25

The *CM* partition of this curve, Figure 3b, is composed of 4 equivalence classes, i.e.,  $n=4$  in this case. The function splitList divides a list in  $n$  parts with equal number of elements and yields a list of lists of points. According to this,  $CM=\{L_1,L_2,L_3,L_4\}$ , where:

*Table 2: List of points of partition in Figure 3b.*

$L_1$		$L_3$	
$x$	$y$	$x$	$y$
0,25	0,25	1,25	1,25
0,75	0,25	1,25	1,75
0,75	0,75	1,75	1,75
0,25	0,75	1,75	1,25
$L_2$		$L_4$	
$x$	$y$	$x$	$y$
0,25	1,25	1,75	0,75
0,25	1,75	1,25	0,75
0,75	1,75	1,25	0,25
0,75	1,25	1,75	0,25

### 3.2 Self-Similarity Test of $X$ under $\mathcal{P}$

Once CM partition is identified, we verify if  $X$  is self-similar under this partition. Remember that  $X$  is self-similar if and only if it is self-similar under  $CM$ . To determine self-similarity of  $X$  and a partition  $\mathcal{P}$ , we will prove that each equivalence class of  $\mathcal{P}$  is an affine transformation of  $X$ . If all equivalence classes of  $\mathcal{P}$  are affine transformations of  $X$ , then  $X$  is self-similar under  $\mathcal{P}$ . We say that  $X$  is not self-similar under  $\mathcal{P}$  in case that any equivalence class of  $\mathcal{P}$  is not an affine transformation of  $X$ .

To determine if  $X$  is an affine transformation of an equivalence class  $p_i$  of  $\mathcal{P}$ , we need to review the concept of scale invariance. Note that there is not any affine transformation that transforms the Hilbert curve, Figure 3a, in any of the equivalence classes of Figure 3b. This does not mean though that Hilbert curve is not self-similar.  $X$  should not be considered in this case as being the original curve, but as partition  $\mathcal{P}$ . By doing this,  $X$  is understood as a set of equivalence classes and it is thus that it must be compared with each equivalence class  $p_i$ .

Figure 7b shows the  $CM$  partition of the Hilbert curve, composed of four equivalence classes of four points each. Figure 7a shows Hilbert curve considered as a set of equivalence classes of the partition shown in Figure 7b. To determine whether the Hilbert curve is self-similar or not, we must prove that each of the four equivalence classes in Figure 7b is an affine transformation of the highlighted curve in Figure 7a.

Given two curves  $X$  and  $Y$ ,  $X$  will be an affine transformation of  $Y$  if and only if there is a matrix  $D$  and a vector  $b$  such that  $\forall_i \{\vec{y}_i = D\vec{x}_i + b\}$  (Definition 1), where  $\vec{y}_i$  is the point  $i$  of curve  $Y$  and  $\vec{x}_i$  is the point  $i$  of curve  $X$ . To find  $D$  and  $b$  we can use a system of equations with six variables ( $D_{2 \times 2}, b_{2 \times 1}$ ) and  $k$  equations, where  $k$  is the number of points of the curves. If the system has a solution, then  $X$  is an affine transformation of  $Y$ . If it does not have a solution, then  $X$  is not an affine transformation of  $Y$ . Another way to do it is by choosing three non-collinear points and finding the affine transformation

transformation  $F$  which transforms these points of  $X$  in into the corresponding ones in  $Y$ . It is known that  $F$  exists and is unique for these three points. We find  $F(X)$  and state that  $X$  is an affine transformation of  $Y$  if and only if  $F(X)=Y$ .

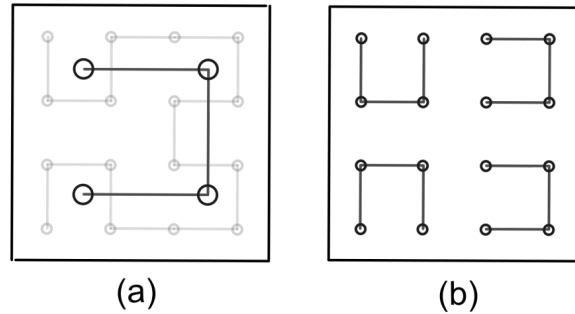


Figure 7. Self-Similarity of a curve under a partition

**Algorithm 2:** Evaluate self-similarity of a curve  $X$  under a partition  $\mathcal{P}$

**Input:**  $\mathcal{P}$ , partition of curve  $X$ , represented by a list of lists of points.

**Output:** *bool*, boolean indicating whether  $X$  is or is not self-similar under  $\mathcal{P}$ .

**procedure** selfsimilarityXunderP ( $\mathcal{P}$ )

```

{ $X_{scale} \leftarrow \text{ScaleP}(\mathcal{P})$ 
 $bool \leftarrow \text{AT}(X_{scale}, \mathcal{P})$ 
return  $bool$ }

```

Let's take the example in Figure 7a. Curve  $X$  is described in Table 1. Partition  $\mathcal{P}=\{L_1, L_2, L_3, L_4\}$  of Figure 7b is found in Table 2. The first step in the algorithm scales  $X$ ,  $\text{ScaleP}(\mathcal{P})$ , as shown in Figure 7a.  $X_{scale}$  is thus represented by:

Table 3.  $X_{scale}$  de la Figure 7a.

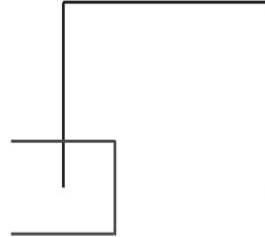
$x$	$y$
0,5	0,5
0,5	1,5
1,5	1,5
1,5	0,5



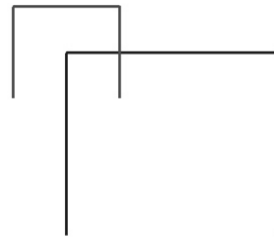
The second step in the algorithm,  $AT(X_{scale}, \mathcal{P})$ , evaluates whether  $X_{scale}$  is an affine transformation of all equivalence classes of  $\mathcal{P}$ . Figure 8 shows the affine transformation in matrix notation so that

$L_i = A_i X_{scale} + b_i$ . Since all equivalence classes are affine transformations of  $X$  it is then said that  $X$  is self-similar under  $\mathcal{P}$  and, as a result,  $X$  is a self-similar curve.

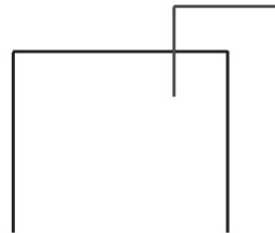
$L_1 = A_1 X_{scale} + b_1$		
$A_1$		$b_1$
0	0.5	0
0.5	0	0



$L_2 = A_2 X_{scale} + b_2$		
$A_2$		$b_2$
0.5	0	0
0	0.5	1



$L_3 = A_3 X_{scale} + b_3$		
$A_3$		$b_3$
0.5	0	1
0	0.5	1



$L_4 = A_4 X_{scale} + b_4$		
$A_4$		$b_4$
0	-0.5	2
-0.5	0	1

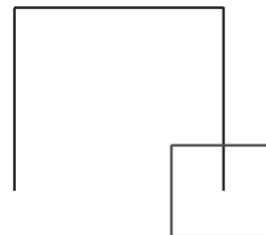


Figure 8: Affine transformations of  $X_{scale}$

#### 4. Applications

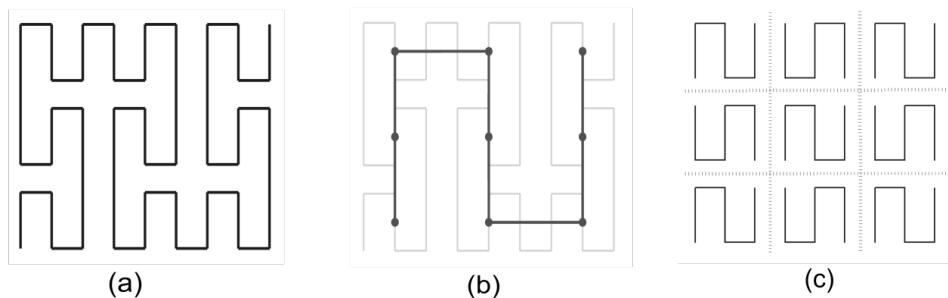
We will show in this section four applications of the self-similarity test. In order to do that, we will utilize Peano, Moore, Meander and Lebesgue curves, presented in (Sagan, 1994).

Figure 9a shows the Peano curve. Figure 9b shows  $X_{scale}$  of Peano curve using the  $CM$  partition. And  $CM$  partition is shown in Figure 9c, which was found using Algorithm 1. The same procedure is followed in Figures 10, 11 and 12.

To determine the self-similarity of each curve, we verify that  $X_{scale}$  is an affine transformation of each equivalence class of  $CM$  partition. To do this, we used the test of Algorithm 2 for each curve under the  $CM$  partition found with Algorithm 1. We found that Peano curve is self-similar, Moore curve is not self-similar, Meander curve is self-

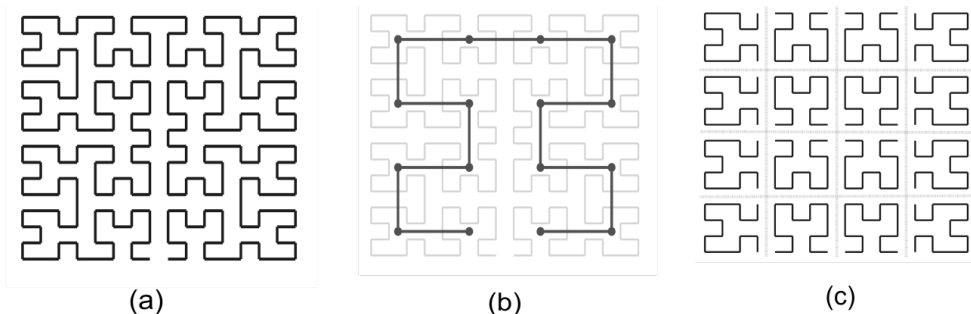
similar and Lebesgue curve is self-similar. This result is congruent with what we observed in Figures 9, 10, 11 and 12. Note that in Figures 9, 11 and 12, all equivalence classes of partition  $CM$  are affine transformations of  $X_{scale}$ , while in Figure 10,  $X_{scale}$  is not an affine transformation of any equivalence class of  $CM$  partition.

*Peano curve*



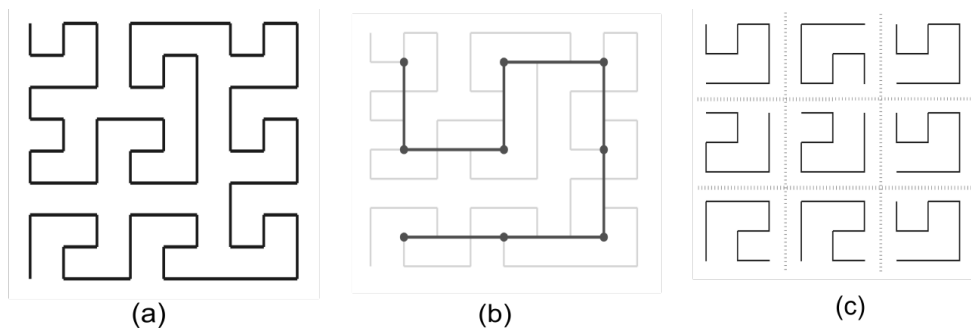
**Figure 9.** Self-Similarity Test for Peano curve.

*Moore curve*



**Figure 10.** Self-Similarity Test for Moore curve.

*Meander curve*



**Figure 11.** Self-Similarity Tests for Meander curve.

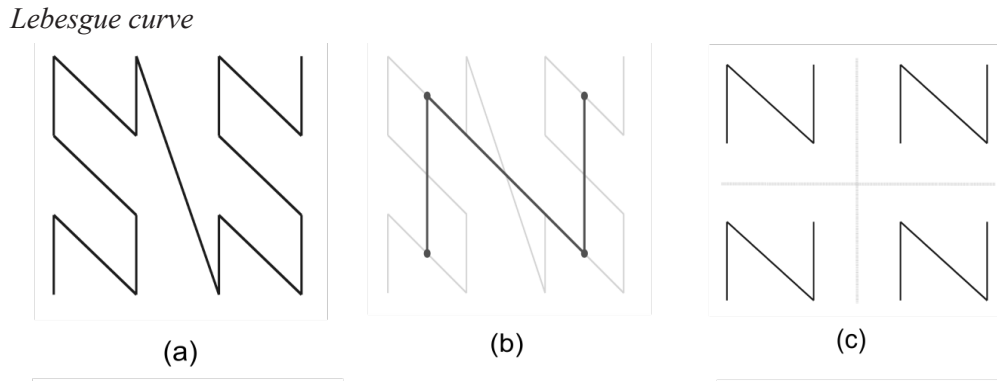


Figure 12. Self-Similarity Test for Lebesgue curve.

## 5. Conclusions

In this paper, we developed a test to determine self-similarity of an *SFC* and applied it to determine the self-similarity of four famous *SFCs*. In addition, we introduced several propositions and properties relating *SFCs*, self-similarity and affine transformations. These postulates can be used as a basis to formalize a theoretical structure of *SFCs* self-similarity. We established necessary and sufficient conditions for *SFC* self-similarity. This allows an extension of the work of McClure (2003). For further research, we suggest to use iterated systems of functions, both for generation of *SFCs* as for their application in different engineering fields. We also recommend to further study *SFC* self-similarity for less restrictive self-similarities, such as statistical self-similarity. This would permit the growth of applications in the fields of image processing and pattern recognition.

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