BREADTH EQUAL TO WIDTH IN A SIMPLE CLOSED CURVE

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INTRODUCTION

We study here the existence of chords with positive length which may be slided along a curve in R^2.

R. Fenn has studied this problem in piecewise analytic simple closed curves in R^2 [F] and has proved for such curves that there exists a chord with positive length which can be slided in a non trivial way along the whole curve, keeping its orientation, by proving first the existence of a chord with positive length which can be slided reversing orientation.

The length of such a chord, which may be slided reversing orientation, is unique and R. Fenn conjectures that it is the maximum of the lengths which may be slided keeping orientation.

D. G. Larman proves it for piecewise analytic curves [L], but the number of steps he needs in his lemmas may be infinite when the Jordan curve is a polygon, for he has to consider not only the points in the polygon in any parallel to the coordinate axis through the vertices of the polygon but the points in the polygon in any parallel to the coordinate axis. Even so his theorem is true assuming the lemmas are.

What we do here, is to prove Larman's lemmas for piecewise analytic curves using Alexander Duality, and see that the conjecture is also true for any continuous Jordan curve constructing piecewise linear approximations to the slidings.

Definitions

Let $h : S^1 \to R^2$ be a Jordan curve. A sliding of h is a simple closed curve $f : S^1 \to \Gamma \subset S^1 \times S^1$ non nulhomotopic such that $\| h(p_1 \circ f(\theta)) - h(p_2 \circ f(\theta)) \| = c > 0. \quad \forall\theta \in S^1$

Such curve doesn't meet the diagonal $\Delta \subset S^1 \times S^1$ and therefore is homotopic to a multiple of Δ. Because of the Jordan curve theorem, if $\Gamma \approx m\Delta$, $m \geq 0$, Γ selfintersects, so we can choose inside Γ a loop homotopic to Δ, and we will assume we have done it.

Let s be the symmetry in $S^1 \times S^1$ respect to the diagonal and $\upsilon : S^1 \times S^1 \to R^2$ the map given by $\upsilon(x, y) = ||h(x) - h(y)||$.

An inversion of h is an arc $\alpha : I \to S^1 \times S^1$ such that $\alpha(0) = s \circ \alpha(1)$ and $\upsilon(\alpha(\theta)) = b > 0 \; \forall\theta \in S^1$.

Thinking in an intuitive way we see that doing twice an inversion we have done a sliding. Coming to Mathematics, we see that the projection of α onto the Möbius band $S^1 \times S^1 - \Delta/s$ is a loop not nulhomotopic and therefore $\alpha * (s \circ \alpha)$ is a sliding.

We call width of a curve the maximum of the lengths of chords which may be slided along the curve in a non trivial way. The length of such chords is bounded by the diameter of the curve. but this diameter D may not be the maximum if $H_1(\psi^{-1}(D)) = 0$. We call breadth such maximum when the final orientation is the opposite to the original one, so the breadth is less or equal to the width.

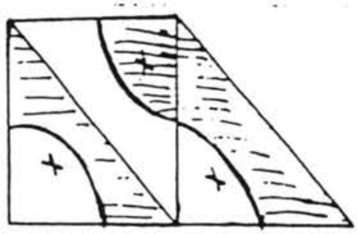
Results and Proofs

Theorem 1. Let α , α' be inversions with respective values $b > 0$,$b' > 0$, for a Jordan curve. then $b = b'$.

Proof

Let $\psi : S^1 \times S^1 \to R^2$ be given by $\psi(x, y) = \| h(x) - h(y) \|$. Then $\psi \circ \alpha(I) = b$ and $(\psi \circ \alpha')(I) = b'$. If $b \neq b'$, we consider a finite covering of $\alpha(I)$ by balls contained in $\psi^{-1}(b - \epsilon, b + \epsilon)$. being $\epsilon < \frac{b-b'}{2}$ and construct a simple analitic arc $\tilde{\alpha}$ contained in the union of such balls connecting $\alpha(0)$ and $\alpha(1)$. so that $\tilde{\alpha}(0, 1) \cap (s \circ \tilde{\alpha}(0, 1)) = \emptyset$. then $\psi(\tilde{\alpha}(I)) \subset (b - \epsilon, b + \epsilon)$. We do the same for α' and get $\tilde{\alpha}'$ analytic curve such that $\psi(\tilde{\alpha}'(I)) \subset (b' - \epsilon, b' + \epsilon)$.

The loop $\tilde{\alpha} * s \circ \tilde{\alpha}$ separates $S^1 \times S^1 - \Delta$ in two cylinders. Symmetrical points in $S^1 \times S^1 - \Delta$ are separated into the two cylinders, but $\tilde{\alpha}'$ connects at least two symetrical points and doesn't meet the diagonal, so $\tilde{\alpha}'(I) \cap (\tilde{\alpha}(I) \cup s \circ \alpha(I)) \neq \emptyset$. by the Jordan curve theorem, therefore $(b - \epsilon, b + \epsilon) \cap (b' - \epsilon, b' + \epsilon) \neq \emptyset$. contradiction.



We call this number b the breadth of the curve.

Theorem 2 Let $h : S^1 \to R^2$ be a Jordan curve of breadth $b > 0$. then

$$b = \sup_{\beta \in \Omega}\{\inf_{t \in I}\{(\psi \circ \beta)(t)\}\} = \inf_{\beta \in \Omega}\{\sup_{t \in I}\{(\psi \circ \beta)(t)\}\}$$

where Ω is the space of arcs in $S^1 \times S^1 - \Delta$ with symmetrical end points.

Proof

If there exists an arc $\alpha : I \to S^1 \times S^1 - \Delta$ such that $(\psi \circ \alpha)(I) = b$,

$$\sup_{\alpha\in\Omega}\{\inf_{t\in I}\{\psi \circ \alpha(t)\} \geq b \geq \inf_{\beta\in\Omega}\{\sup_{t\in I}\{\psi \circ \beta(t)\}.$$

On the other hand, let $\alpha' \in \Omega$, $\alpha'(I) = [b', b'']$, and $\tilde{\alpha}'$ a simple piecewise analytic aproximation as previously constructed such that $(\psi \circ \tilde{\alpha}')(I) \subset (b' - \epsilon, b'' + \epsilon)$. Considering that α joins symmetrical points respect to Δ in $S^1 \times S^1$ and that symmetrical points are separated by $\tilde{\alpha}' * s \circ \tilde{\alpha}'(I)$ in different cylinders, we have $\alpha(I) \cap (\tilde{\alpha}' * s \circ \tilde{\alpha}')(I) \neq \emptyset$, i. e. $b' - \epsilon \leq b \leq b'' + \epsilon \quad \forall \epsilon > 0$ and so $b' \leq b \leq b''$. therefore

$$\sup_{\beta\in\Omega}\{\inf_{t\in I}\{\psi \circ \beta(t)\} \leq b \leq \inf_{\beta\in\Omega}\{\sup_{t\in I}\{\psi \circ \beta(t)\}$$

and the theorem is proved.

We can verify that the last expression is positive in any simple closed curve.

Now we prove Fenn's conjecture in Larman's way, but with different proofs.

Proof of lemmas

Lemma 1:

Given two piecewise analytic closed curves: Γ, Γ^* in $S^1 \times S^1$, homotopic to the diagonal Δ with parametrizations $f : S^1 \to \Gamma$, $g : S^1 \to \Gamma^*$ there exist parametrizations $\alpha : S^1 \to S^1$, $\beta : S^1 \to S^1$ such that: $p_1 \circ f \circ \alpha = p_1 \circ g \circ \beta$.

Proof:

We consider $F : S^1 \times S^1 \to S^1 \times S^1$ given by $F(z) = (p_1 \circ f(z), p_1 \circ g(z))\ \forall z \in S^1$ and the map $\psi : S^1 \times S^1 \to \Re$ given by $\psi(x, y) = \| x - y \|$. We call $\Delta = \{(x, x) \setminus x \in S^1\} \subset S^1 \times S^1$ and $K = F^{-1}(\Delta) = (\psi \circ F)^{-1}(0)$. K is compact and also triangulable because $\psi \circ F$ is piecewise analytic[L_o]. So Čech cohomology and singular cohomology coincide for the pair (T, K). We have therefore the following commutative diagram:

$$H_1(T - K) \ \underrightarrow{H_1(i)} \ H_1(T)$$

(1) $$\wr\wr D_A \qquad\qquad \wr\wr D_P$$

$$H^1(T, K) \ \underrightarrow{H^1(j)} \ H^1(T)$$

D_A means Alexander Duality and D_P means Poincare duality.

On the other hand we have the exact cohomology sequence

(2) $$\longrightarrow H^1(T,K) \xrightarrow{H^1(j)} H^1(T) \xrightarrow{H^1(i)} H^1(K) \longrightarrow$$

which says that $\ker H^1(i) = \mathrm{im}\, H^1(j)$.

Given $\lambda \in H_1(T - K) \subset H_1(T)$, let λ^* be $D_P(\lambda) \in H^1(T)$, then we have $\lambda^* \in \ker H^1(i) \Longleftrightarrow \lambda^* \in \mathrm{im}\, H^1(j) \Longleftrightarrow \lambda \in H_1(T - K)$ according to (1) and (2), so

$$H^1(i)(\lambda^*) \neq 0 \Longleftrightarrow \lambda \in H_1(T - K)$$

We deduce that if any representative of the homology class of λ meets K there exist a curve m in K such that $m \cap \lambda \neq \phi$. Even more, if λ may be inserted as a generator in a basis of $H_1(T)$, considering integer coefficients we deduce that there exist a curve $m \subset K$ such that $m \cdot \lambda = 1$.

If λ is $\{e^{i\theta}, e^{-i\theta}\} \subset S^1 \times S^1$, $F(\lambda) = \{p_1 \circ f(e^{i\theta}),\ p_1 \circ g(e^{-i\theta})\} \subset S^1 \times S^1$ is a curve homotopic to λ because $\deg(p_1 \circ f) = \deg(p_1 \circ g) = 1$ as $\Gamma \sim \Delta$ and $\Gamma^* \sim \Delta$. So $\lambda \cdot \Delta = 1$ means $F(\lambda) \cdot \Delta = 1$ and $\lambda \cap K \neq \emptyset$. Therefore we get $m \subset K$ such that $m \cdot \lambda = 1$. Let m be given by a parametrization $(\alpha(e^{i\theta}), \beta(e^{i\theta})) \subset S^1 \times S^1$. Since $F(m) \subset \Delta$, $p_1 \circ f \circ \alpha(e^{i\theta}) = p_1 \circ g \circ \beta(e^{i\theta})$ c.q.d.

Lemma 2:

Given two piecewise analytic closed curves $f : S^1 \to \Gamma$, $g : S^1 \to \Gamma^*$ homotopic to Δ in $S^1 \times S^1$ there exist parametrizations $\alpha', \beta', \gamma : S^1 \to S^1$ such that

$$p_1 \circ f \circ \alpha' = p_1 \circ g \circ \beta'$$
$$p_2 \circ f \circ \alpha' = p_2 \circ g \circ \gamma$$

Proof:

Let α, β be the parametrizationsfound in lemma 1. We consider $f \circ \alpha$ and g and apply lemma 1 to p_2 so we obtain $\tilde{\alpha}$, $\tilde{\beta}$ such that

$p_2 \circ f \circ \alpha \circ \tilde{\alpha} = p_2 \circ g \circ \tilde{\beta}$

Now, we make $\alpha' = \alpha \circ \tilde{\alpha}, \beta' = \beta \circ \tilde{\alpha}, \gamma = \tilde{\beta}$ and we write

$p_1 \circ f \circ \alpha' = p_1 \circ f \circ \alpha \circ \tilde{\alpha} = p_1 \circ g \circ \beta \circ \tilde{\alpha} = p_1 \circ g \circ \beta'$

$p_2 \circ f \circ \alpha' = p_2 \circ f \circ \alpha \circ \tilde{\alpha} = p_2 \circ g \circ \tilde{\beta} = p_2 \circ g \circ \gamma$. *c.q.d.*

Lemma 3:

Given two piecewise analytic closed curves : Γ , $\Gamma^* \subset S^1 \times S^1$ homotopic to Δ, being Γ^* symmetric respect to the diagonal Δ given by parametrizations $f : S^1 \to \Gamma$, $g : S^1 \to \Gamma^*$, there exist parametrizations $\alpha'', \beta'', \gamma'' : S^1 \to S^1$ such that

(1) $$g \circ \beta''(\theta) = s(g \circ \beta''(\theta + \pi))$$

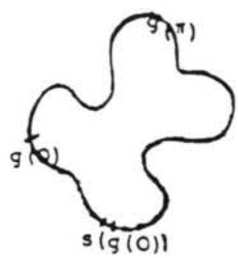
(2) $$p_1 \circ f \circ \alpha''(\theta) = p_1 \circ g \circ \beta''(\theta)$$

(3) $$p_2 \circ f \circ \alpha''(\theta) = p_2 \circ g \circ \gamma''(\theta)$$

We write θ instead of $e^{i\theta}$ for the sake of simplicity and denote by s the simmetry.

Proof:

Getting a symmetric parametrization of a symmetric simple closed curve is easy : you consider a point $x_0 \in \Gamma^*$and its symmetric $s(x_0) \in \Gamma^*$, you parametrize the arc joining both points by $l : I \to \Gamma^*$ and then you consider $l * s o l$ but it is not that easy getting a simmetric parametrization which keeps up with the first coordinate of another parametrization of a different curve Γ. So we modify the $g \circ \beta'$,previously obtained to make it symmetric.



First, we see, following Larman, that in any parametrization of a symmetric curve $g : S^1 \to \Gamma^*$ there exists θ_0 such that $g(\theta_0) = s(g(\theta_0 + \pi))$.For that we consider the points $g(0), g(\pi), s(g(0))$. If $s(g(0)) \neq g(\pi)$, choosing the orientation in the curve we can assume that $g(0) < sg(0) < g(\pi)$

As the symmetry respect to the diagonal keeps the orientation in the diagonal, and $\Gamma^* \sim \Delta$ it keeps also the orientation in Γ^* so $s(g(0)) < ssg(0) < sg(\pi)$ i. e. $g(0) < sg(\pi)$.

We have $g(\pi) > s \circ g(0)$, $g(\pi + \pi) < sg(\pi)$, then, according to mean value theorems. and considering the maps $s \circ g(\theta)$, $s \circ (\theta + \pi)$ we get the existence of $\theta_\circ \in (0, \pi)$ such that $sg(\theta_0) = g(\theta_0 + \pi)$.

In the following we assume g defined in the interval $[0, 2\pi]$

Now, let $\theta'_\circ$ be the sup $\{\theta \in (\theta_\circ, \theta_\circ + \pi) \mid g(\theta) = g(\theta_\circ)\}$ and

$$\bar{\theta}'_\circ = \inf \{\theta \in (\theta_\circ, \theta_\circ + \pi) \mid g(\theta) = g(\theta_\circ + \pi)\}$$
$$\theta''_\circ = \sup \{\theta \in (\theta_\circ + \pi, \theta_\circ + 2\pi) \mid g(\theta) = g(\theta_0 + \pi)\}$$
$$\bar{\theta}''_\circ = \inf \{\theta \in (\theta_\circ + \pi, \theta_\circ + 2\pi) \mid g(\theta) = g(\theta_\circ + 2\pi)\}$$

Making an adequate linear reparametrization we have

$$g|_{[\theta_0,\theta_0+\pi]} = g|_{[\theta_0,\theta'_0]} * g|_{[\theta'_0,\bar{\theta}'_0]} * g|_{[\bar{\theta}'_0,\theta_0+\pi]}$$

$$g|_{[\theta_0+\pi,\theta_0+2\pi]} = g|_{[\theta_0+\pi,\theta''_0]} * g|_{[\theta''_0,\bar{\theta}''_0]} * g|_{[\bar{\theta}''_0,\theta_0+2\pi]}$$

and we construct:

$$sg|_{[\theta_0+\pi,\theta''_0]} * g|_{[\theta_0,\theta'_0]} * g|_{[\theta'_0,\bar{\theta}'_0]} * g|_{[\bar{\theta}'_0,\theta_0+\pi]} * sg|_{[\bar{\theta}''_0,\theta_0+2\pi]}$$

$$g|_{[\theta_0+\pi,\theta''_0]} * sg|_{[\theta_0,\theta'_0]} * g|_{[\theta''_0,\bar{\theta}''_0]} * sg|_{[\bar{\theta}'_0,\theta_0+\pi]} * g|_{[\bar{\theta}''_0,\theta_0+2\pi]}$$

Both pairs of couples at the beginning and the end are symmetric with these parametrizations.

The central arcs don't touch Δ and begin and end in common perpendiculars to Δ. so making a $\frac{\pi}{2}$ turn. they may be considered as arcs in $I \times I$ beginning at $\{0\} \times I$ and ending at $\{1\} \times I$.

We transform $I \times I$ in $S^1 \times S^1$ by identifing opposed sides and we complete the arcs to simple closed curves $\bar{k}.\bar{l}$ by adding vertical segments in $\{0\} \times I \equiv \{1\} \times I$. These simple closed curves are homologons to a meridian.

We consider now:

$$F = (p_1 \circ \bar{k}, p_1 \circ \bar{l}) : S^1 \times S^1 \to S^1 \times S^1$$

$$K = F^{-1}(\Delta)$$

$$\lambda = \{e^{i\theta}.e^{-i\theta}\} \subset S^1 \times S^1$$

and follow the same reasonning as in lemma 1. Then we find $a.b.$ such that $p_1 \circ \bar{k} \circ a = p_1 \circ \bar{l} \circ b$. As $p_1 \circ \bar{k}(\theta) = 1 \Leftrightarrow p_1 \circ \bar{l}(\theta) = 1$. taking the restrictions of $\bar{k}$ and $\bar{l}$ to the intervals $S^1 - (p_1 \circ \bar{k})^{-1}(1)$ and $S^1 - (p_1 \circ \bar{l})^{-1}(1)$ we have $p_1 \circ k \circ a = p_1 \circ l \circ b$.

Also. as the arcs are piecewise differentiable and the tangent vectors are symmetric. we see that the set $H = \{t | k \circ a(t) = s \circ l \circ b(t)\} \supset \{0.1\}$ is open. taking into account the exponential map. and the commutative diagram:

$$T_p \approx T_s(p)$$

$$exp \downarrow \qquad \downarrow exp$$

$$\Gamma_1 \approx \Gamma_2$$

$$\Gamma_1 \approx \Gamma_2$$

H is closed because $H = (k \circ a, s \circ l \circ b)^{-1}(\Delta)$, so H is connected and not empty and therefore is the whole interval.

This result gives in $S^1 \times S^1$: $g|_{[\theta'_0,\dot{\theta}'_0]} \circ a = sg|_{[\theta''_0,\dot{\theta}'']} \circ b$ and glueing the couples of loops at the beginning and the end we have a symmetric parametrization.

We can assume that the new parametrization may be written as $g \circ \bar{\beta}$ where $\bar{\beta}$ makes S^1 go back to draw the loops introduced by a and then changes the speed in these arcs appropriately, such as we had to do in order to get $g|_{[\theta'_0,\dot{\theta}'_0]} \cdot g|_{[\theta''_0,\dot{\theta}''_0]}$.

Considering $g \circ \beta'$ instead of g, the desired parametrizations are $\beta'' = \beta' \circ \bar{\beta}$, $\alpha'' = \alpha' \circ \bar{\beta}$, $\gamma'' = \gamma \circ \bar{\beta}$.

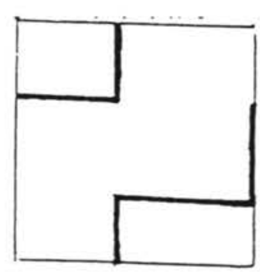
Now, we are going to study the variation of angle in the slided chord.

Let $h : S^1 \to J \subset \mathbb{R}^2$ be a simple closed curve, then a simple closed curve $f : S^1 \to \Gamma \subset S^1 \times S^1 - \Delta$ determines the sliding of the chord $[h(p_1 \circ f(\theta))h(p_2 \circ f(\theta))]$. The variation of the angle in this sliding is determined by

$$A(\theta) = \frac{h(p_1 \circ f(\theta)) - h(p_2 \circ f(\theta))}{||h(p_1 \circ f(\theta)) - h(p_2 \circ f(\theta))||}$$

lemma 4:
El grado de la función A es 1.

Proof:
We observe that Γ is homotopic to a curve like the one drawn in the picture in which there are four straight arcs, each of them with one constant coordinate. The corresponding slided chord moves only the origin or only the end in every step. The degree of H depends only on the homotopy type of Γ, so we can calculate its degree for this particular sliding.

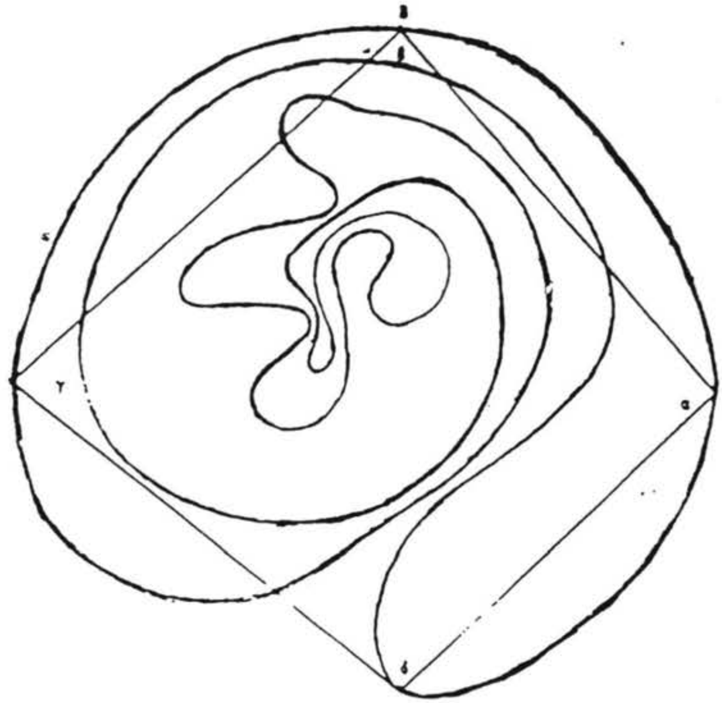


We can also choose the origins and ends in every step.

As the curve J is compact we can consider its absolute extremes respect to the coordinates x and y and call them A, B, C, D in the sense of orientation of the curve so that it completes a cycle in the clockwise direction.

We consider the chord $\overline{AD}$ and we slide the origin A up to C in the sense of the curve, getting $\overline{CD}$ in the first step. Then we slide D up to B in the same sense getting $\overline{CB}$ in the second step. After that we slide C up to A getting $\overline{AB}$ and finally we slide B up to D coming again to $\overline{AD}$.

As the curve is placed above the horizontal line through D, the angle in the first step is only δ as it is drawn in the picture, independently of the turns and rounds it has made. In the same way the angles in the following steps are γ, β, α, and the total angle is $\alpha + \beta + \gamma + \delta = 2\pi$. So deg $A = 1$.



Juan Peran has given a proof of lemma 5 based on lemma 1: let us call $N : C - \{0\} \to S^1$ the map

$$N(z) = \frac{z}{\| z \|}$$

This map verifies $\mathcal{N}(kz) = \mathcal{N}(z) \quad \forall k \in \Re^+$ and $\mathcal{N}(\mathcal{N}(z)) = \mathcal{N}(z)$.

Let us assume also that the origin is in interior of J.

Then, $A(\theta) = \mathcal{N}(h \circ p_1 \circ f(\theta) - h \circ p_2 \circ f(\theta))$.

Considering the curves in $S^1 \times S^1$: Γ given by $(\mathcal{N} \circ h \circ p_1 \circ f, \mathcal{N} \circ h \circ p_1 \circ f)$ and Γ^* given by $(-\mathcal{N} \circ h \circ p_2 \circ f, -\mathcal{N} \circ h \circ p_2 \circ f)$, by lemma 1, these exist $\alpha, \beta : S^1 \to S^1$ such that $\mathcal{N} \circ h \circ p_1 \circ f \circ \alpha = -\mathcal{N} \circ h \circ p_2 \circ f \circ \beta$ where $\deg\alpha = \deg\beta = 1$.

Then

$$A = \mathcal{N}(h \circ p_1 \circ f \circ \alpha - h \circ p_2 \circ f \circ \beta) = \mathcal{N}(\frac{h \circ p_1 \circ f \circ \alpha - h \circ p_2 \circ f \circ \beta}{\| h \circ p_1 \circ f \circ \alpha \| \| h \circ p_2 \circ f \circ \beta \|})$$

$$= \mathcal{N}(\frac{\mathcal{N}(h \circ p_1 \circ f \circ \alpha)}{\| h \circ p_2 \circ f \circ \beta \|} - \frac{\mathcal{N}(h \circ p_2 \circ f \circ \beta)}{\| h \circ p_1 \circ f \circ \alpha \|})$$

$$= \mathcal{N}[\mathcal{N}(h \circ p_1 \circ f \circ \alpha)(\frac{1}{\| h \circ p_2 \circ f \circ \beta \|} + \frac{1}{\| h \circ p_2 \circ f \circ \alpha \|}) = \mathcal{N}(h \circ p_1 \circ f \circ \alpha)$$

So deg H = 1.

A non trivial sliding (not going and coming back in the same way) preserving the orientation of the chord is given by a non-nulhomotopic curve $\Gamma \subset S^1 \times S^1 - \Delta$ by means of $h \times h : S^1 \times S^1 \to R^2 \times R^2$. This curve is, therefore, homotopic to a multiple of Δ. Considering $S^1 \times S^1 - \Delta$ as a cylinder in R^2, by the Jordan curve theorem we can choose a loop homotopic to Δ inside Γ which also gives a sliding.

We have pointed out that the breadth of a curve is less or equal than the width, so, we only have to prove that the width is less or equal than the breadth.

Theorem 3:

Let $\tilde{f} : S^1 \to \Gamma \subset S^1 \times S^1, g : S^1 \to \Gamma^* \subset S^1 \times S^1$ be slidings homotopic to Δ such that

$\| h(p_1(\tilde{f}(\theta))) - h(p_2(\tilde{f}(\theta))) \| = w > 0$ and $\| h(p_1(\tilde{g}(\theta))) - h(p_2(\tilde{g}(\theta))) \| = b > 0$ where $h : S^1 \to \Re^2$ is a Jordan curve and Γ^* is symmetric, then $w \leq b$.

Proof:

We interchange the tilde between the slidings and their aproximations, which are of the form described in Theorem 1, in order to simplify the notation in this proof.

Given $\epsilon, \delta \in R^+$, we can construct f, g piecewise analitic slidings which are ϵ and δ aproximations to $\tilde{f}$ and $\tilde{g}$. We can assume g symmetric, then by the previous lemmas we have α'', β'', γ'' so that

$$p_1 \circ f \circ \alpha'' = p_1 \circ g \circ \beta''$$
$$p_2 \circ f \circ \alpha'' = p_2 \circ g \circ \gamma''$$
$$g \circ \beta''(\theta) = s \circ g \circ \beta''(\theta + \pi)$$

as in lemma 4.

Denoting $f \circ \alpha'' = (p, q)$ we would have $g \circ \beta'' = (p, p^*)$, $g \circ \gamma'' = (q^*, q)$.

Taking the origin of the parametrization such that $g \circ \gamma''(0) = \sigma \circ g \circ \gamma''(\pi)$, we define $j : S^1 \to S^1 \times S^1$ as

$p_1 \circ j(\theta) = p_1 \circ g \circ \beta''(\theta) = p(\theta) \quad \forall \theta \in [0, 2\pi]$

$p_2 \circ j(\theta) = p_1 \circ g \circ \gamma''(\theta) = q^*(\theta) \quad \forall \theta \in [0, \pi]$

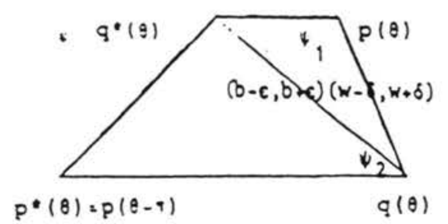
$p_2 \circ j(\theta) = p_1 \circ g \circ \gamma''(\theta - \pi) = q(\theta - \pi) \quad \forall \theta \in [\pi, 2\pi]$.

The curve j is well defined.

If j meets Δ, it must be $p(\theta) = q^*(\theta)$ or $p^*(\theta - \pi) = p(\theta) = q(\theta - \pi)$. In either cases we would have $(w - \epsilon, w + \epsilon) \cap (b + \delta, b - \delta) \neq \emptyset$. As we can do it for every $\epsilon > 0$ and for every $\delta > 0$, we have $w = b$. If j doesn't meet Δ, lemma 4 applied to j gives variation of the angle of j equal 2π. Calling $\eta(\theta) = var(j(\theta), j(\theta + \pi))$, we have $\eta(0) + \eta(\pi) = 2\pi$, so either both are π or one of them is bigger and other smaller than π. In any case, by the mean value theorem, there exists θ such that $\eta(\theta) = \pi$, what means that the chords $[p(\theta), q^*(\theta)]$ and $[p^*(\theta), q(\theta)]$ have opposite direction . We have the situation represented in one of the following pictures:

In the first picture the angles sum $\psi_1 + \psi_2$ is π. This picture shows that $\psi_1 \geq \pi/2$ we have $\| q^*(\theta) - q(\theta) \| > \| p(\theta) - q(\theta) \|$ i. e. $b + \delta \geq w - \epsilon$ for every ϵ and δ, therefore $b \geq w$. When $\psi_1 \leq \pi$ we consider the second picture which also shows that $\| p(\theta) - p^*(\theta) \| > \| p(\theta) - q(\theta) \|$ i. e. $w - \epsilon \leq b + \delta$ for every ϵ and δ, and $w \leq b$, i.e. the same conclusion.

(1)



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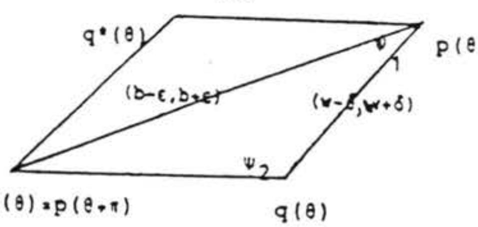

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