QUID 2017, pp. 2653-2663, Special Issue N°1- ISSN: 1692-343X, Medellín-Colombia

ALGEBRAIC EQUATIONS OF INTEGRAL AND ITS APPROXIMATION METHODS

(Recibido 05-06-2017. Aprobado el 07-09-2017)

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Abstract. Most mathematical models have been used in many problems of physics, biology, chemistry, engineering, etc. based on the integral equation. In this study, we examine the solution of integral algebraic equations using the differential transformation method, and we also examine the FDTM method for solution algebraic equations of the Volterra fuzzy integral, which this method evaluates the approximation solution by Taylor's finite series. In this study, we obtained the numerical algorithm, using the Maple software.

Citar, estilo APA: Khodaparast, E. (2017)**.** Algebraic equations of integral and its approximation methods. *Revista QUID (Special Issue),* 2653-2663.

1. INTRODUCTION

Equations of integral are one of the branches of mathematical science that has many applications in engineering and physics problems. The integral equations are appeared in many topics of physics, biology, chemistry and engineering. The integral equations were first introduced by Volterra in the early 1900s; Volterra was studying the phenomenon of population growth and the effect of inheritance that encountered such equations and selected this name for them. From that time on, scientists and researchers needed to solve these equations in many applied studies such as heat transmission, the phenomenon of dissemination, neutron diffusion, and so on.

2. LINEAR INTEGRAL EQUATIONS

An integral equation is an equation in which the unknown function u (x) is located under the integral symbol. An example of an integral equation in which u (x) is the unknown function that must be determined is in the following form.

$$(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} k\,(x,t)u(t)\,dt \quad (1)$$

$k(x,t)$Is the kernel of integral equation. α (x) and β (x) are integral limits. In Equation (1), the unknown function means u (x) has been appeared only below the integral symbol. And in other modes, the unknown function may also be present outside the integral symbol.

Linear integral equations for Fredholm

The standard form of Fredholm linear integral equations in which the lower limit and the upper limit of the integral are respectively the fixed numbers a and b, are as follows:

$$\emptyset\,(x)u(x) = f(x) + \lambda\int_a^b k(x,\mathrm{t})\,u\,(t)\,dt \quad , \quad a \leq x,t \leq b, \quad (2)$$

In which the kernel of the integral equation k (x, t) and the function f (x) are pre-identified, and λ is also a known parameter. Given that $\emptyset\,(x)$uses which of the following values, Fredholm linear integral equations are divided into two major categories:
When $\emptyset\,(x)$= 0, equation (2) is converted to the following equation.

$$f(x) + \lambda\int_a^b k(x,t)u(t)\,dt = 0 \quad , \quad (3)$$

This equation is called the Fredholm first order integral equation. When $\emptyset(x) = 1$, the equation (2) will be as follows

$$u(x) = f(x) + \lambda\int_a^b k(x,t)u(t)\,dt \quad (4)$$

This equation is called the Fredholm second order integral equation.

Volterra Linear Integral Equations

The standard form of the Volterra linear integral equations, means, the equations in which the upper and lower limit of integration, instead of to be a constant number, appears as a function of x, is as follows.

$$\emptyset\,(x)u(x) = f(x) + \lambda\int_a^x k(x,t)u(t)\,dt \quad (5)$$

In which the unknown function u (x) is under the symbol of integral in the linear form .It should be noted that (5) can be considered as a special case of Fredholm integral equations, so that the k (x, t) kernel to be assumed zero for t $x \in [a,b]$and $t < x$
The Volterra integral equations can be classified into two groups according to the value of $\emptyset\,(x)$.While $\emptyset\,(x)$= 0, equation (5) will be as follows.

$$f(x) + \lambda\int_a^x k(x,t)u(t)\,dt = 0 \quad (6)$$

This equation is called the Volterra integral of the first order.
When ∅ (x) = 1, then equation (5) will be as follows.

$$u(x) = f(x) + \lambda\int_a^x k(x,t)u(t)\,dt \quad (7)$$

This is called the Volterra second order integral equation.

3-INTEGRAL-DIFFERENTIAL EQUATIONS

In these equations, the unknown function u (x) appears on both sides. Of course, such equations also appear at the time of converting a differential equation to an integral equation. Some examples of integral-differential equations have been shown below.

$u''(x) = -x + \int_0^x (x-t)\, u(t)\, dt \quad , u(0) = 0, u'(0) = 1$ (8)

$u'(x) = -sin\,(x) - 1 - \int_0^x u(t)dt,, \; u\,(0) = 1$ (9)

$u'(x) = 1 - \frac{1}{3}x + \int_0^1 x\, t\, u(t)dt, \quad , \; u\,(0) = 1$ (10)

Equations (8) and (9) are called Volterra integral-differential equations and Equation (10) is called the Fredholm differential-integral equation. This classification has been done based on the integration limits.

4-Single integral equations

$f(x) = \lambda \int_{\alpha x}^{\beta x} k(x,t)u(t)dt,$ equation 11

Integral of first order

$u(x) = f(x) + \lambda \int_{\alpha x}^{\beta x} k(x,t)u(t)dt$The integral equation of the second order

Equation (12)

Integral of first order, Integral equation of second order (12) in which the lower limit, the upper limit, or both, are infinite integration are called single-integral equation. Moreover, if the kernel of the integral equations (11) and (12) to be at one point or at more points than the infinite integration domain, these equations are also called single-integral equations.

Some examples of single-integral equations have been shown below that the cause of being called them as single is infinitive of related integration domain.

$u(x) = 2\,x + 6\int_0^{\infty} \sin(x-t)u(t)dt,$ (13)

In these examples, the k (x, t) is kernel, while t → x, becomes infinite, therefore, the corresponding integral equations are called singular.

$x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\, dt\,,$ (*14*)

$x = \int_0^x \frac{1}{\sqrt{(x-t)^{\alpha}}} u(t)\, dt\,, \quad 0<\alpha<1$ (*15*)

$u(x) = 1 - 2\sqrt{x} - \int_0^x \frac{1}{\sqrt{x-t}} u(t)\, dt\,,$ (*16*)

It is important to note that the integral equations similar to (14) and (15), respectively, are called Abel integral equation and the generalized Abel integral equation. These single-integral equations were first introduced by a Norwegian mathematician in 1823, with the name of Abel. Equations of single integral, similar to Equation (16), are called as Volterra second order integral equation, weakly. Such equations appear in engineering applications and physics, such as heat transfer, crystal growth, and mechanic of fluids.

5. Homogeneous integral equations

If the Fredholm second order integral equation

$u(x) = f\,(x) + \lambda \int_a^b k(x,t)\, u\,(t)\, dt,$

And Volterra second-order integral equation

$u\,(x) = f\,(x) + \lambda \int_a^x k(x,t)\, u\,(t)\, dt,$

The condition f (x) = 0 to be established, then the resulting equation is called a homogeneous integral equation, otherwise the equation is called a non-homogeneous integral equation.

6. THE SOLUTION OF AN INTEGRAL EQUATION

The solution of an integral equation or differential integral equation is on the integrating distance of a function u (x) such that function in the given equation is valid. On other words, if the solution to be placed on the right side of the equation, then the two sides of the left and right of the equation to be equal, then u (x) is the solution of the equation.

7. VOLTERRA INTEGRAL EQUATIONS AND ITS SOLUTION METHODS

In this study, we focus on Volterra second order linear and non-homogeneous integral equations in the following form in which k (x, t) is the kernel of the equation and λ is a parameter.

$u\,(x) = f\,(x) + \lambda \int_0^x k(x,t)\, u\,(t)\, dt,$ (*18*)

The upper limits of integration in the Volterra integral equations are functions of x and, unlike the Fredholm integral equation, these limits are not constant. We assume that the kernel of considered integral equation is separable. The goal is to provide a variety of methods to determine the solution and solution u (x) of the integral equation (18).

The Adomian method of decomposition

In this method, the function u (x) is decomposed into the components that will be specified. The assumption u (x) is represented by the following power series.

$u\ (x) = \sum_{n=0}^{\infty} u_n\ (x)(19)$

We also assume that u (x) shows all sentences outside the integral symbol means:

$u_0\ (x) = f\ (x) \quad (20)$

By replacing the expression (19) in Equation (20) we will have:

$$\sum_{n=0}^{\infty} u_n\ (x) = f\ (x) + \lambda \int_0^x k(x,t)(\sum_{n=0}^{\infty} u_n\ (t))dt \quad , \quad (21)$$

Using the first few sentences we have in the above expansion

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f\ (x) + \lambda \int_0^x k(x,t)\, u_0(t)\, dt$$

$$+ \lambda \int_0^x k(x,t)\, u_1(t)\, dt$$

$$+ \lambda \int_0^x k(x,t)\, u_2(t)\, dt + \cdots \quad (22)$$

To determine the component$u_0(x), u_1(x)$, ..., we place from the unknown function u (x), which is completely determined by recursive equations and relationships.

$u_0(x) = f\ (x)$, (23)

$u_1(x) = \lambda \int_0^x k(x,t)\, u_0(t)\, dt$, (24)

$$u_2(x) = \lambda \int_0^x k(x,t)\, u_1(t)\, dt, \quad (25)$$

$$u_3(x) = \lambda \int_0^x k(x,t)\, u_2(t)dt, \quad (26)$$

And until the end...

The design that was discussed above for the determination of the components $u_0(x), u_1(x), u_2(x), \ldots$ from solution of the integral equation (18), u (x) can be written as follows:

$$u_0(x) = f\ (x) \quad , \quad (27)$$

$$u_{n+1}(x) = \lambda \int_0^x k(x,t)\, u_n(t)\, dt, \quad n \geq 0(28)$$

Given the equations (27) and (28), the components,$u_0(x), u_1(x), \ldots$, ... are immediately obtained by integration of functions, which are easily computable. After calculating these components, the solution and solution of the integral equation (18), u (x), is determined as a series in accordance with equation (19). Usually, by applying only the finite number of series sentences, we approximate it as $\sum_{n=0}^{k} u_n(x)$.

Example 1: Consider the following Volterra integral equation.

$u(x) = 1 + \int_0^x u(t)dt$, (29)

Solution)it is clear that $k\ (x\ ,t) = 1\ ,\ \lambda = 1$, $f(x) = 1$ by applying the analytic method to the solution series (19) and the recursive (27), (28) to determine the components$n \geq 0$and u_n we will have:

$u_0(x) = 1$

$u_1(x) = \int_0^x u_0(t)\ d\ t = \int_0^x d\ t = x$,(30)

$u_2(x) = \int_0^x u_1(t)\ \ t = \frac{1}{2!}x^2$, (31)

And until to the end, It should be noted that:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots, \quad (32)$$

We can simply get the solution in the form of a series that is identified as below.

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \ldots, \quad (33)$$

Therefore, the solution to the closed form according to the Taylor expansion for the function e^x will be as follows.

$u(x) = e^x(34)$

Example 2: We consider the following Volterra integral equation

$$u(x) = 1 + \frac{1}{2}\int_0^x x\, t^2 u\ (t)\ d\ t \quad (35)$$

حل)با توجه به روندی که قبلا بحث شد خواهیم داشت.

Solution) we will have following equation according to the process discussed previously.

$u_0(x) = 1$

(36)

$u_1(x) = \frac{1}{2} x \int_0^x t^2 \, d\,t,$

$u_1(x) = \frac{1}{6} x^4$,

(37)

$u_2(x) = \frac{1}{12} x \int_0^x t^6 \, d\,t = \frac{1}{84} x^8$,

(38)

$u_3(x) = \frac{1}{168} x \int_0^x t^{10} \, d\,t = \frac{1}{1848} x^{12}$

(39)

As a result, the solution and solution of the integral equation (35) will be in series as follows.

$u(x) = 1 + \frac{1}{6} x^4 + \frac{1}{48} x^8 + \frac{1}{1848} x^{12} + \ldots$

(40)

As can be seen, one cannot obtain a closed form for u (x). Although the decomposition method has shown that the technique is potent and reliable, but the decomposition method has improved, and the method is more efficient. The decomposition method has improved, the volume of computations will be greatly reduced, so that only the calculation of the two components$u_0(x)$,$u_1(x)$ is required, an improved technique is used for certain problems such as those in which the function f (x) in equation (18) contains at least two sentences.

8. DIFFERENTIAL TRANSFORMATION METHOD FOR SOLUTION VOLTERRA INTEGRAL EQUATIONS

Assume that u (x) is an unknown function, f (x) is given function and known function, k (x, t) is also the integral kernel that its value has been given. The Volterra linear integral equation of the first order is an integral equation in the following form [5]

$f(x) = \int_a^x k(x,t) u(t) dt$

(41)

The linear integral equation of the second order is also an integral equation in the following form:

$u(x) = f(x) + \int_a^x k(x,t) u(t) dt$

(42)

We use the DTM to solve the Volterra linear and nonlinear integral equations with a separable kernel.

$$k(x,t) = \sum_{j=0}^{n} m_j(x) N_j(t)$$

That for the Volterra first or second ordernonlinear integralequations, we can write:

$u(x) = f(x) + \sum_{j=0}^{n} k_j(x) \int_a^x g_j(t,u)(t) dt$

(43)

Differential transformation methods theorems

1. Theorem: Assume G (k), U (k) to be a differential transformation of the functions g (x), u (x), then we obtain.

If $f(x) = \int_a^x g(t) u(t) dt$, then

$F(k) = \sum_{l=0}^{k-1} G(l) \frac{U(k-l-1)}{k}$, $F(0) = 0$

(44)

If $f(x) = g(x) \int_a^x u(t) dt$, then

$F(k) = \sum_{l=0}^{k-1} G(l) \frac{U(k-l-1)}{k-l}$, $F(0) = 0$,

(45)

2

$$\begin{cases} 1 & if \quad k = m \\ 0 & if \quad k \neq m \end{cases}$$

2. Theorem: If$f(x) = x^m$, then

$F(k) = \delta(k - m) =$

(47)

Conversional Functions	*Main functions*

$F(k) = U(k) \pm V(k)$	$f(x) = u(x) \pm v(x)$
$F(k) = \alpha U(k)$	$f(x) = \alpha u(x)$
$F(k) = \sum_{l=0}^{k} V(l)U(k-l)$	$f(x) = u(x)\, v(x)$
$F(k) = (k+1)U(k+1)$	$f(x) = \frac{du(x)}{dx}$
$F(k) = (k+1)(k+2)(k+m)U(k+m)$	$f(x) = \frac{d^m u(x)}{dx^m}$
$F(k) = \frac{U(k-1)}{k} \quad ,k \geq 1, F(0) = 0$	$f(x) = \int_{x0}^{x} u(t)dt$
$F(x) = \delta(k-m)$	$f(x) = x^m$
$F(k\ ) = \frac{\lambda^k}{k!}$	$f(x) = exp(\lambda x)$
$F(k) = \frac{w^k}{k!} sin(\frac{\pi k}{2} + \alpha)$	$f(x) = sin\ (wx + \alpha)$
$F(k) = \frac{w^k}{k!} cos(\frac{\pi k}{2} + \alpha)$	$f(x) = cos\ (wx + \alpha)$

Table 1

Numerical solution of Volterra integral equations with differential transformation

Here are some examples of DTM applications for solution linear and nonlinear Volterra integral equations. Example

3: We consider the following Volterra integral equation [5].

$$u(x) = x + \int_0^x (t-x)u(t)dt, \qquad 0 < x < 1 \quad (47)$$

The following results are obtained using the theorems and Table 1:

$$u(x) = x + \int_0^x tu(t)dt - \int_0^x xu(t)dt,$$

$$U(k) = \delta(k-1) + \sum_{l=0}^{k-1} \quad \delta(l-1)\frac{u(k-l-1)}{k} - \sum_{l=0}^{k-1} \quad \delta(l-1)\frac{u(k-l-1)}{k}, k \geq 1 U(0) = 0 \ (48)$$

We have following equation by replacing a number for k:

$U\ (1) = 1, \qquad U\ (2) = 0,$

$U\ (3) = -\frac{1}{3!}, \qquad U\ (4) = 0,$

$U\ (5) = \frac{1}{3!}, \qquad U\ (6) = 0,$

$U\ (7) = -\frac{1}{7!}, \qquad U\ (8) = 0, \ldots$

Therefore, we obtain from $(f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k)$the solution of the integral equation (47):

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= sin\ (x),$$

Example 4: Consider the following non-linear Volterra integral equation: [5]

$u(x) + \int_0^x (u^2(t) + u(t))dt = \frac{3}{2} - \frac{1}{2} exp(-2x)$

(*49*)

According to the theorems and with the help of Table 1, we obtain:

$U(k) + \sum_{l=0}^{k-1} U(l) \frac{U(k-l-1)}{k} + \frac{U(k-1)}{k} = \frac{3}{2}\delta(k) - \frac{(-2)^k}{2\ k!}, \quad k \geq 1, \ u(0) = 0,$ (*50*)

Following values are obtained by replacing a value for k:

$U(1) = -1, U(2) = \frac{1}{2!},$

$U(3) = -\frac{1}{3!}, \qquad U(4) = \frac{1}{4!},$

$U(5) = -\frac{1}{5!}, \qquad U(6) = \frac{1}{6!}, \ldots$

We obtain the solution of equation from the equation ($f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k$) (56):

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \cdots$$

$= exp(-x)$

9-SOLUTION VOLTERRA FUZZY INTEGRAL EQUATIONS USING FUZZY DIFFERENTIAL TRANSFORMATION THEOREMS

Theorem: Assume that v (t), u (t) to be functions with a fuzzy value, so that their differential transformation with V (t), U (t) shows:

If f (t) = u (t) + v (t), then k∈K, F (k) = U (k) + V (k)

2) If f (t) = u (t) -v (t), then k∈K, F (k) = U (k) -V (k)

3) If f (t) = u (t) ⊖v (t), then k∈K, F (k) = U (k) ⊖V (k)

Theorem: If $U(k), G(k)$to be the differential transformation of the functions g, (u) where g is a function with positive value), if $f(x) = g(x)u(x)$, then

$F(k,r) = \sum_{l=0}^{k} G(l) \odot U(k-l-1,r), 0 \leq r \leq 1$ (8)

Proof: We have following equation using the definition of differential transformation:

$f(x,r) \approx (\sum_{k=0}^{n} G(k)(x-x_0)^k) \odot (\sum_{k=0}^{n} U(k,r)(x-x_0)^k)$

$= [G(0) + G(1)(x-x_0) + G(2)(x-x_0)^2 + \cdots + G(n)(x-x_0)^n] \odot$

$[U(0,r) + U(1,r)(x-x_0) + U(2,r)(x-x_0)^2 + \cdots U(n,r)(x-x_0)^n]$

$= [G(0)U(0,r)] + [G(0)U(1,r) + G(1)U(0,r)](x-x_0) +$

$[G(0)U(2,r) + G(1)U(1,r) + G(2)U(0,r)](x-x_0)^2 + \cdots$

$+ [G(0)U(n,r) + G(1)U(n-1,r) + \cdots + G(n-1)U(1,r) + G(n)U(0,r)]$

$(x-x_0)^n.$

As a result, we have:

$f(x,r) \approx \sum_{k=0}^{n} \sum_{l=0}^{k} G(l)\ U(k-l,r)\ (x-x_0)^k$

We obtain from the definition of differential transformation:

$$F(k,r) = \sum_{l=0}^{k} G(l)U(k-l,r).$$

9.3. Theorem: If $U(k), G(k)$to be the differential transformation of the functions g, (u) where g is a function with positive value), if$f(x) = g(x)u(x)$, then

$F(k,r) = \sum_{l=0}^{k} G(l) \odot U(k-l-1,r), 0 \leq r \leq 1$ (8)

Proof: We have following equation using the definition of differential transformation:

$f(x,r) \approx (\sum_{k=0}^{n} G(k)(x-x_0)^k) \odot (\sum_{k=0}^{n} U(k,r)(x-x_0)^k)$

$= [G(0) + G(1)(x-x_0) + G(2)(x-x_0)^2 + \cdots + G(n)(x-x_0)^n] \odot$

$[U(0,r) + U(1,r)(x-x_0) + U(2,r)(x-x_0)^2 + \cdots U(n,r)(x-x_0)^n]$

$= [G(0)U(0,r)] + [G(0)U(1,r) + G(1)U(0,r)](x-x_0) +$

$[G(0)\ U\ (2,r)+G\ (1\ )U(1,r) + G(2)U(0,r)]\ (x-x_0)^2+\cdots$

$+\quad [G(0)\ U\ (n,r)+G\ (1\ )U(n-1,r)+\cdots+G(n-1)U(1,r)+G(n)U(0,r)]$

$(x-x_0)^n$.

As a result, we have:

$f(x,r)\approx\sum_{k=0}^{n}\sum_{l=0}^{k}G(l)\ U\ (k-l,r)\ (x-x_0)^k$

We obtain from the definition of differential transformation:

$$F(k,r)=\sum_{l=0}^{k}G(l)U(k-l,r).$$

9.4 Theorem: Assume that $g\ \in\ \mathbb{E}$is a function with fuzzy value and $f(x,r)=\int_{x_0}^{x}g(t,r)dt$ then $k\geq 1$, $F(k)=\frac{G(k-1)}{k}$, where G (k), F (k) are the differential transformations g, f.

Proof: Using the definition of FDTM for $0\leq r\leq 1$,we have:

$f(x,r)=\int_{x_0}^{x}g(t,r)dt$ =$[\int_{x_0}^{x}\underline{g}(t,r)dt,\int_{x_0}^{x}\bar{g}(t)dt]$=

$[\int_{x_0}^{x}\sum_{k=0}^{\infty}\underline{G}\ (k,r)(t-x_0)^k dt,\int_{x_0}^{x}\sum_{k=0}^{\infty}\bar{G}\ (k,r)(t-x_0)^k dt]$=

$[[\sum_{k=0}^{\infty}\frac{\underline{G}(k,r)}{k+1}(t-x_0)^{k+1}]_{x_0}^{x},\quad [\sum_{k=0}^{\infty}\frac{\bar{G}(k,r)}{k+1}(t-x_0)^{k+1}]_{x_0}^{x}]=$

$[\sum_{k=0}^{\infty}\frac{\underline{G}(k,r)}{k+1}(x-x_0)^{k+1},\quad \sum_{k=0}^{\infty}\frac{\bar{G}(k,r)}{k+1}(x-x_0)^{k+1}]$.

We change the index k = 0 to k = 1, we deduce:

$f(x,r)$=$[\sum_{k=1}^{\infty}\frac{\underline{G}(k-1,r)}{k}(x-x_0)^k$, $\sum_{k=1}^{\infty}\frac{\bar{G}(k-1,r)}{k}(x-x_0)^k]$

$=\sum_{k=1}^{\infty}\frac{G(k-1,r)}{k}(x-x_0)^k$.

Finally, we obtain by using FDTM:

$\mathrm{F}(k,r)=\frac{G(k-1,r)}{k}$, $\qquad 0\leq r\leq 1\qquad ,k\geq 1$

9.5-**Theorem**: we suppose

$f(x)$= $\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}g(t)dtdx_1\dots dx_{n-1}$,

Then,$k\geq n$, $F(k)$=$\frac{(k-n)!}{k!}$ $G(k-n)$, where G (k), F (k) are the differential transformation of functions with fuzzy values of g, f,

Proof: we have using the FDTM definition: {3}

$\underline{f}(x,r)$ =$\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\underline{g}(t,r)dt\ dx_1\ dx_2\dots dx_{n-1}$

According to this: $\underline{g}\ (t,r)=\sum_{k=0}^{\infty}\underline{G}\ (k,r)\ (t-x_0)^k$

$=\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\sum_{k=0}^{\infty}\underline{G}\ (k,r)(t-x_0)^k(t,r)dt\ dx_1\dots dx_{n-1}$

$=\sum_{k=0}^{\infty}\quad\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\underline{G}(k,r)(t-x_0)^k(t,r)dt\ dx_1\dots dx_{n-1}$

Reminder:
$\{\int_0^x\int_0^{x_1}\dots\int_0^{x_{n-1}}f(x_n)\ dx_n\dots dx_1$=$\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}f(t)dt$

}:

$=\sum_{k=0}^{\infty}\quad\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\frac{\underline{G}(k,r)}{k+1}\quad(x-x_0)^{k+1}dx_1\dots dx_{n-1}$

=$\sum_{k=0}^{\infty}\quad\frac{\underline{G}(k,r)}{(k+1)(k+2)\dots(k+n)}(x-x_0)^{k+n}$.

$\bar{f}(x,r)$=$\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\overline{g}(t,r)dt\ dx_1\dots dx_{n-1}$

=$\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\sum_{k=0}^{\infty}\bar{G}\ (k,r)(t-x_0)^k dt\ dx_1\dots dx_{n-1}$

=$\sum_{k=0}^{\infty}\quad\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\overline{G}(k,r)(t-x_0)^k dt\ dx_1\ \ dx_2\dots dx_{n-1}$

$=\sum_{k=0}^{\infty}\quad\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1}\frac{\bar{G}(k,r)}{k+1}\quad(x-x_0)^{k+1}dx_1\dots dx_{n-1}$

…

=$\sum_{k=0}^{\infty}\quad\frac{\bar{G}(k,r)}{(k+1)(k+2)\dots(k+n)}(x-x_0)^{k+n}$.

Then, we have following equation by changing the index from k = 0 to k = n:

$\underline{f}(x,r)$ =$\sum_{k=n}^{\infty}\quad\frac{\underline{G}(k-n,r)}{(k+1-n)(k+2-n)\dots(k)}(x-x_0)^k=$

$\sum_{k=n}^{\infty}\frac{(k-n)!}{k!}\underline{G}(k-n,r)\ (x-x_0)^k,\qquad 0\le r\le 1$

$\bar{f}\,(x,r)=\sum_{k=n}^{\infty}\ \frac{\bar{G}(k-n,r)}{(k+1-n)(k+2-n)\dots(k)}(x-x_0)^k=$

$\sum_{k=n}^{\infty}\frac{(k-n)!}{k!}\bar{G}(k-n,r)\,(x-x_0)^k,\qquad 0\le r\le 1$

We obtain by using FDTM definition:

$\underline{F}(k,r)=\frac{(k-n)!}{k!}\underline{G}(k-n,r)\ (x-x_0)^k,\qquad 0\le r\le 1$

$\bar{F}(k,r)=\frac{(k-n)!}{k!}\bar{G}(k-n,r)\ (x-x_0)^k,\qquad 0\le r\le 1$

9.6. Theorem: Suppose G (k), U (k) is the differential transformation of the functions$g(x)$, $u(x)$, where g (x) is a function with positive value. If f$f(x)$= $\int_{x_0}^{x}g(t)u(t)dt$, then for f, $-i$ we derive {4}

$\underline{F}(k,r)=\sum_{l=0}^{k-1}\ G(l)\frac{\underline{U}(k-l-1,r)}{k},\qquad 0\le r\le 1$

(1)

$\bar{F}(k,r)=\sum_{l=0}^{k-1}\ G(l)\frac{\bar{U}(k-l-1,r)}{k},\qquad 0\le r\le 1$ (2)

For - ii, f, we derive:

$\underline{F}(k,r)=\sum_{l=0,even}^{k-1}\ G(l)\frac{\underline{U}(k-l-1,r)}{k}+\sum_{l=1,odd}^{k-1}\ G(l)\frac{\bar{U}(k-l-1,r)}{k},\quad 0\le r\le 1$ (3)

$\bar{F}(k,r)=\sum_{l=0,\text{even}}^{k-1}\ G(l)\frac{\bar{U}(k-l-1,r)}{k}+\sum_{l=1,odd}^{k-1}\ G(l)\frac{\underline{U}(k-l-1,r)}{k},\quad 0\le r\le 1$ (4)

The proof: we have using definition of FDTM forderivative -i, f, and for 0≤r≤1:

$F(0,r)=[\int_{x_0}^{x}g(t)u(t,r)dt]_{x=x_0}$=0,

$F(1,r)=\frac{d}{dt}[\int_{x_0}^{x}g(t)u(t,r)dt]_{x=x_0}=[\frac{d}{dt}\int_{x_0}^{x}g(t)\underline{u}(t,r)dt,\frac{d}{dt}\int_{x_0}^{x}g(t)\bar{u}(t,r)dt\ ]_{x=x_0}=[g(x)\underline{u}(x,r),g(x)\bar{u}(x,r)]_{x=x_0}=\ [G(0)U(0,r)].$

$F(2,r)=\frac{1}{2}\frac{d^2}{dx^2}[\int_{x_0}^{x}g(t)u(t,r)dt]_{x=x_0}=\frac{1}{2}[G(1)U(0,r)+G(0)U(1,r)\ ]_{x=x_0}$

For k we have:

We assume that $(t)u(t,r)=h(t,r)$, we obtain according to theorem (9-2):

$H(k,r)=\sum_{l=0}^{k}G(l)U(k-l,r)$,(5)

We conclude according to theorem (9.3), that$f(x,r)=\int_{x_0}^{x}h(t,r)dt$and

$F(k,r)=\frac{H(k-1,r)}{k},\qquad (6)$

We obtain by placing the equation (5) in (6):

$$F(k,r)=\sum_{l=0}^{k-1}\frac{G(l)U(k-l-1,r)}{k}.$$

$\underline{F}(k,r)=\frac{1}{k}\sum_{l=0}^{k-1}\ G(l)\underline{U}(k-l-1,r)$

$\bar{F}(k,r)=\frac{1}{k}\sum_{l=0}^{k-1}\ G(l)\bar{U}(k-l-1,r)$

For f- in the form of derivative$-ii$, it is also possible to easily obtain results.

$u(x)=f(x)+\int_{0}^{x}(x-t)u(t)dt$,*(1)*

Example 5: consider the following Volterra fuzzy integral equation

So that, $f(x,r)=[3+r,8-2r]$ and we have $k\ge 1$،$0\le r\le 1$.

Solution)

$u(x)=f(x)+x\int_{0}^{x}u(t)dt-\int_{0}^{x}tu(t)dt,$

We have according to Theorems 9-1 and 9-2:

$$\underline{U}(k,r)=(3+r)\delta(k)+\sum_{l=0}^{k-1}\delta(l-1)\frac{\underline{U}(k-l-1,r)}{k-l}$$

$$-\sum_{l=0}^{k-1}\delta(l-1)\frac{\underline{U}(k-l-1)}{k},$$

$$\overline{U}(k,r)=(8-2r)+\delta(k)+\sum_{l=0}^{k-1}\delta(l-1)\frac{\overline{U}(k-l-1,r)}{k-l}$$

$$-\sum_{l=0}^{k-1} \delta(l-1) \frac{\overline{U}(k-l-1)}{k},$$

As a result, we obtain by replacing value for k:

$\underline{U}(0,r) = 3 + r,$
$\underline{U}(1,r) = 0,$

$$\underline{U}(2,r) = \frac{3+r}{2}, \underline{U}(3+r) = 0,$$

$$\underline{U}(4,r) = \frac{3+r}{24}, \underline{U}(5,r) = 0,$$

$$\underline{U}(6,r) = \frac{3+r}{720}, \underline{U}(7,r) = 0,$$

...

And

$\overline{U}(0,r) = 8 - 2r, \overline{U}(1,r) = 0,$

$\overline{U}(2,r) = \frac{8-2r}{2}, \overline{U}(3,r) = 0,$

$\overline{U}(4,r) = \frac{8-2r}{24}, \overline{U}(5,r) = 0,$

$\overline{U}(6,r) = \frac{8-2r}{720}, \overline{U}(7,r) = 0,$

...

It is based on this that:

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots,$$

The – r cut solution of equation (1) is as follows:

$u(x,r) = [3 + r, 8 - 2r] \odot \cosh(x),$

This is the exact solution to the Volterra fuzzy integral equation. In addition, we have obtained the figure (Figure. 1), based on 0- cut and 1-cut (the obtained solution is trapezoidal)

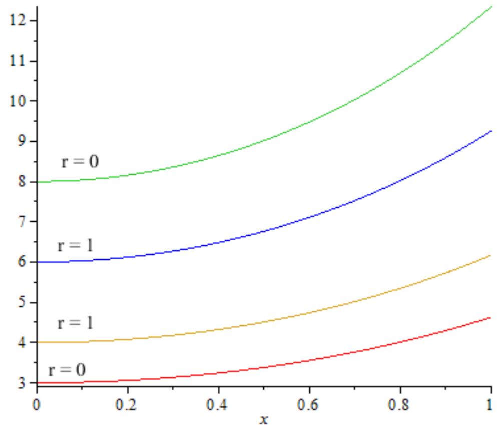

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Figure (1), Example 5 and x∈ [0,1]

Appendix

Maple program Example (5):

```
Restart;
f1:= x -> f1(x);
f1(x) :=( 3+r)*x^m;
f2:= x -> f2(x);
f2(x) :=( 8-2*r)*x^m;
M: =0;
N: =8;
For k from 0 to n do
If k=m then delta (k):=1 elif k<>m then delta (k):=0;
End if;
Cat ('________________________________________________');
Print ('k'=k);
Print ('delta (k)'=delta (k));
F1 (k) :=( 3+r)*delta (k);
F2 (k) :=( 8-2*r)*delta (k);
End do;
Cat ('________________________________________________');
For k from 0 to m+1 do
```

U1 (k):=F1 (k);

U2 (k):=F2 (k);

Cat ('____ ');

End do;

For k from 2 to n do

U1(k):=F1(k)+sum(delta(l-1)*(U1(k-l-1))/(k-l),l=0..k-1)-sum(delta(l-1)*((U1(k-l-1))/(k)),l=0..k-1);

U2(k):=F2(k)+sum(delta(l-1)*(U2(k-l-1))/(k-l),l=0..k-1)-sum(delta(l-1)*((U2(k-l-1))/(k)),l=0..k-1);

Cat ('____ ');

End do;

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