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Axially Symmetric Post-Newtonian Stellar Systems

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Abstract. We introduce a method to obtain self-consistent, axially symmetric disklike stellar models in the first post-Newtonian (1PN) approximation. By using in the field equations of the 1PN approximation a distribution function (DF) corresponding to a Newtonian model, two fundamental equations determining the 1PN corrections are obtained. The rotation curves of the corrected models differs from the classical ones and the corrections are clearly appreciable with values of the mass and radius of a typical galaxy. On the other hand, the relativistic mass correction can be ignored for all models.
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Sistemas estelares post-newtonianos axialmente simétricos

Resumen. Presentamos un método para obtener modelos estelares discoidales, axialmente simétricos, auto-consistentes en la primera aproximación post-Newtoniana (1PN). Usando en las ecuaciones de campo de la aproximación 1PN una función de distribución conocida (DF) que corresponde a un modelo Newtoniano, se obtienen dos ecuaciones fundamentales para determinar las correcciones 1PN. Las curvas de rotación de los modelos corregidos difieren de las clásicas y las correcciones son claramente apreciables con los valores de la masa y el radio de una galaxia típica. Por otro lado, la corrección relativista de la masa se puede ignorar para todos los modelos.
Palabras claves: Primer aproximación post-newtoniana, dinámicas estelar y galáctica.

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1. Introduction

The stars observed in the universe tend to cluster in huge self-gravitating systems, being the galaxies among the most noticeable and studied of them. Galaxies can be described by models whose mass is distributed symmetrically about an axis to a finite distance of it, i.e., the galaxy radius. Particularly useful models are those restricted to a plane or thin disk, since for many galaxies like the Milky Way its height is small compared to its radius. For galaxies in general, the average time between collisions or individual meetings (mean collision time) is greater than the system's life time. Many of the current models take Newton's law as their field equations [15, 16, 21, 22, 6, 10, 7, 8, 2, 9, 17]. However, many models have been developed recently under general relativity [12, 13, 14, 3, 4, 19, 20, 23], being one of the main motivations for including corrections made by general relativity the actual incompatibility between the rotation curves of theoretical models and the ones observed.

The post-Newtonian approximation introduces general relativity through a series over the speed (v/c). For Minkowski's background metric, the first post-Newtonian corrections (1PN) are included taking the terms

$$g_{00} \approx -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00}, \tag{1}$$

$$g_{i0} \approx \overset{3}{g}_{i0}, \tag{2}$$

$$g_{ij} \approx \delta_{ij} + \overset{2}{g}_{ij}, \tag{3}$$

where the upper index denotes the order of the power of (v/c) of the term. Using harmonic coordinates, the 1PN order potentials are defined as

$$\overset{2}{g}_{00} \equiv -2\phi/c^2, \tag{4}$$

$$\overset{3}{g}_{i0} \equiv \zeta_i/c^3, \tag{5}$$

$$\overset{4}{g}_{00} \equiv -2(\phi^2 + \psi)/c^4. \tag{6}$$

Henceforth, we consider only the stationary case, so the explicit time-dependent terms along with the potential vector $\boldsymbol{\zeta}$ disappear in all equations.

The stationary field equations in the 1PN approximation are

$$\nabla^2 \phi = \frac{4\pi G}{c^2} \overset{0}{T}{}^{00}, \tag{7}$$

$$\nabla^2 \psi = 4\pi G(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}), \tag{8}$$

where, in the classical limit, potential ψ vanishes and ϕ tends to the Newtonian potential

ϕ_N. Whereas that

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla\phi - \frac{1}{c^2}\left[\nabla\left(2\phi^2 + \psi\right) + 4\mathbf{v}(\mathbf{v}\cdot\nabla\phi) - v^2\nabla\phi\right] \tag{9}$$

is the stationary equation of motion.

A complete statistical description is achieved by knowing the distribution function (DF) of the system. The DF satisfies a continuity equation in the phase space called the Boltzmann equation. In the 1PN approximation, the collisionless Boltzmann equation (CBE) for a many identical particles system is given by [18]

$$v^i\frac{\partial F}{\partial x^i} - \frac{\partial\phi}{\partial x^i}\frac{\partial F}{\partial v^i} - \frac{1}{c^2}\left(\frac{\partial\phi}{\partial x^i}(4\phi + v^2) - \frac{\partial\phi}{\partial x^j}v^iv^j + \frac{\partial\psi}{\partial x^i}\right)\frac{\partial F}{\partial v^i} = 0. \tag{10}$$

According to Jeans theorem [1], the solution of colissionless Boltzmann equation is any function of the integrals of motion. Now, it is easy to verify that two isolated integrals a 1PN system with axial symmetry are given by

$$E = \frac{1}{2}v^2 + \Phi, \tag{11}$$

where

$$\Phi = \phi + \frac{2\phi^2 + \psi}{c^2}, \tag{12}$$

and

$$L_z = Rv_\varphi e^{-\phi/c^2} \approx Rv_\varphi(1 - \phi/c^2), \tag{13}$$

which can be interpreted as the 1PN generalizations of energy and the z component of angular momentum, respectively.

In addition, the DF must satisfy the condition of self-consistently generating the macroscopic mean values. In the post-Newtonian approximation, the following components of the energy-momentum tensor are need:

$$\overset{0}{T}{}^{00}(\mathbf{x}, t) = c^2\int F(\mathbf{x}, \mathbf{v}, t)d^3v, \tag{14}$$

$$\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} = 2\int (v^2(\mathbf{x}, t) + \phi(\mathbf{x}, t))F(\mathbf{x}, \mathbf{v}, t)d^3v. \tag{15}$$

Therefore, in the post-Newtonian approximation, self-consistent equilibrium models are defined by two scalar potentials, ϕ and ψ, together with a DF that satisfies 1PN CBE and relations (14) - (15) generating self-consistently the 1PN components of the energy-momentum tensor. In this paper we present a method to obtain post-Newtonian axially symmetric equilibrium models. The method uses thin disk models, allowing to solve the two differential equations with the Hunter's method [5], as shown in the next section. The solution of the field equations is obtained considering equations in vacuum, in which case, the energy-momentum tensor vanishes and the content of matter is expressed as a boundary condition on the fields in the disk, as shown below. Finally, in section 3 the first axially symmetric models in the 1PN approximation are presented.

2. Formulation of the method

For thin disks of finite radius the components of the energy-momentum tensor can be written as

$$\overset{0}{T}{}^{00} = c^2 \Sigma(R)\delta(z), \tag{16}$$

$$\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} = \sigma(R)\delta(z), \tag{17}$$

for $0 \leq R \leq a$, where δ is the Dirac delta function, and being zero for $R > a$. Therefore, the field equations reduce to two Laplace equations for the fields ϕ and ψ. It is demanded that the fields are even functions of z,

$$\phi(R, z) = \phi(R, -z), \qquad \psi(R, z) = \psi(R, -z), \tag{18}$$

and therefore, that the first derivatives with respect to z are odd functions of z.

Using Gauss' theorem with (16) and (17) gives,

$$\Sigma(R) = \frac{1}{2\pi G}\left(\frac{\partial \phi}{\partial z}\right)_{z=0^+}, \tag{19}$$

$$\sigma(R) = \frac{1}{2\pi G}\left(\frac{\partial \psi}{\partial z}\right)_{z=0^+}. \tag{20}$$

The problem is defined with the following boundary conditions: fields vanish at infinity, and at $z = 0$ its derivatives depend on the energy-momentum tensor components according to (19) and (20) for $0 \leq R \leq a$ and vanishing for $R > a$. Applying Hunter's method to each of the 1PN field equations, one can obtain exact analytical expressions for the potentials. The Hunter's method consists on obtaining solutions of the Laplace equation in oblate spheroidal coordinates. The oblate coordinates are related to the cylindrical by

$$R = a\sqrt{(1 + \xi^2)(1 - \eta^2)}, \tag{21}$$

$$z = a\xi\eta, \tag{22}$$

where $0 \leq \xi < \infty$ and $-1 \leq \eta \leq 1$. The disk is placed at $\xi = 0$, where, $\eta^2 = 1 - R^2/a^2$.

Following Hunter [5], the general solution of each Laplace equation satisfying previous boundary conditions can be written as

$$\phi(\xi,\eta) = -\sum_{n=0}^{\infty} A_{2n} q_{2n}(\xi) P_{2n}(\eta), \tag{23}$$

for the ϕ potential, and

$$\psi(\xi,\eta) = -\sum_{n=0}^{\infty} B_{2n} q_{2n}(\xi) P_{2n}(\eta), \tag{24}$$

for the ψ potential, where A_{2n} and B_{2n} are the constants required for each model, $P_{2n}(\eta)$ are the Legendre polynomials and $q_{2n}(\xi) = i^{2n+1} Q_{2n}(i\xi)$ are Legendre functions of second kind. Note that when using classical models, the expression for ϕ can be written taking constants of the form

$$A_{2n} = C_{2n} + D_{2n}/c^2, \tag{25}$$

where the C_{2n} constants define the Newtonian potential ϕ_N and the constants D_{2n} define the correction ϕ_{PN}. So that taking the limit $c \to \infty$, $\phi = \phi_N + \phi_{PN}$ is reduced to the Newtonian part only. The corresponding expressions for Σ and σ in oblate coordinates are

$$\Sigma = \frac{1}{2\pi a G \eta} \sum_{n=0}^{\infty} A_{2n}(2n+1) q_{2n+1}(0) P_{2n}(\eta), \tag{26}$$

$$\sigma = \frac{1}{2\pi a G \eta} \sum_{n=0}^{\infty} B_{2n}(2n+1) q_{2n+1}(0) P_{2n}(\eta). \tag{27}$$

Accordingly, Σ can also be written as the sum of a Newtonian part and a post-Newtonian correction: $\Sigma = \Sigma_N + \Sigma_{PN}$.

In order to obtain self-consistent models of axially symmetric thin disks in equilibrium, it is used a DF of the form $F = f(R, v_R, v_\varphi)\delta(z)\delta(v_z)$, which is zero for $R > a$. The DF reproduces Σ and σ through the equations

$$\Sigma(R) = \int\int f(R, v_R, v_\varphi) dv_R dv_\varphi, \tag{28}$$

$$\sigma(R) = 4\int\int E f(R, v_R, v_\varphi) dv_R dv_\varphi - 2\phi_N \Sigma_N, \tag{29}$$

where E is the integral of motion defined by (11). As in the Newtonian case, one could define a relative energy and a relative potential as

$$\varepsilon = -E + \Phi_0, \tag{30}$$

$$\Psi = -\Phi + \Phi_0, \tag{31}$$

where Φ_0 is a constant that is chosen so that ε and Ψ are always positive, i.e., such that $f > 0$ for $0 < \varepsilon \leq \Psi$.

The method to obtain self-consistent post-Newtonian models is to take (28) with (26) as the first fundamental equation and (29) with (27) as the second fundamental equation. In accordance with the order of magnitude of the approach, the second fundamental equation is solved using the Newtonian terms and then the first fundamental equation is solved using the terms up to 1PN order. Finally, note that for the second fundamental equation, the constants can be obtained directly by

$$B_{2i} = \frac{4i+1}{4i+2}\frac{2\pi a G}{q_{2i+1}(0)}\int_{-1}^{1} P_{2i}(\eta)\eta\sigma \mathrm{d}\eta, \tag{32}$$

where the integral depends on the particular model. On the other hand, in the first fundamental equation, the constants also appear on the right side of the equation (hence, it is necessary to obtain the constants differently for each model) and, in general, we can not use an explicit expression as done above. However, it is found that for all models treated it is finally possible to obtain expressions analog to (32) for the first fundamental equation, see (57) - (59) and (74) - (75). The DF in the 1PN approximation presents the same functional dependence on the integrals of motion that in the Newtonian models, that is, one uses the same DF but now with the 1PN energy and angular momentum. Of course, this is for the sake of correspondence with the Newtonian limit. Note that the correspondence principle must be satisfied by both the DF and its integrals.

It is possible to obtain alternative expressions for Σ and σ considering in particular the case where the DF depends on a linear combination of energy and angular momentum called Jacobi's integral, $J = \Omega L_z - E$. Jacobi's integral is interpreted classically as the energy measured from a frame of reference rotating with angular speed Ω [1]. In terms of the relative energy (30), we can write Jacobi's integral as

$$J = \varepsilon + \Omega L_z - \Psi_e(0), \tag{33}$$

where $\Psi_e(0)$ is the 1PN relative-effective potential evaluated in $\eta = 0$, defined by

$$\Psi_e = \Psi + \tfrac{1}{2}\Omega^2 R^2 \left(1 - 2\phi/c^2\right). \tag{34}$$

The Jacobi's integral takes values between zero and J_{max}, with

$$J_{max} = \Psi - \frac{1}{2}\Omega^2 a^2 \eta^2 \left(1 - \frac{2\phi}{c^2}\right) - \frac{\Omega^2 a^2 \phi}{c^2}. \tag{35}$$

Now, given that $2\pi dJ = dv_R dv_\varphi$, the relation (28) can be written as

$$\Sigma = 2\pi \int_0^{J_{max}} f(J) dJ. \tag{36}$$

Then, by using the expression

$$\int\int v_\varphi f dv_R dv_\varphi = 2\pi\langle v_\varphi\rangle \int_0^{J_{max}} f(J)dJ, \tag{37}$$

the second fundamental equation can be written as

$$\sigma = 2(2\Phi_0 - \Omega^2 a^2 - \phi_N + 2a\Omega\sqrt{1-\eta^2}\langle v_\varphi\rangle)\Sigma_N - 8\pi \int_0^{J_{max}} Jf(J)dJ, \tag{38}$$

with Σ_N given by (26) with $A_{2n} = C_{2n}$.

Finally, the circular speed necessary for the rotation curve can be obtained considering (9) for circular orbits. In that case, the circular speed is equal to v_ϕ and perpendicular to the gradient of the field ϕ. Therefore, the 1PN equation of motion (9) reduces to

$$\frac{v_\varphi^2}{R}\left[1 + \frac{R}{c^2}\frac{\partial \phi}{\partial R}\right]_{z=0} = \frac{\partial}{\partial R}\left[\phi + \frac{2\phi^2 + \psi}{c^2}\right]_{z=0}. \tag{39}$$

Then, in accordance with the 1PN order of approximation, the expression for the circular speed is

$$v_\varphi = \sqrt{R\frac{\partial \phi}{\partial R}\left(1 + \frac{4\phi}{c^2} - \frac{R}{c^2}\frac{\partial \phi}{\partial R}\right) + \frac{R}{c^2}\frac{\partial \psi}{\partial R}}\,\Bigg]_{z=0}. \tag{40}$$

Note that in the limit $c \to \infty$, the previous expression reduces to that of Newtonian theory $v_\varphi = \sqrt{R\partial\phi_N/\partial R}$.

3. Application to the Generalized Kalnajs Disks

We will now apply the previous formalism to the family of Generalized Kalnajs Disks (GKD) introduced by González and Reina in [2]. This family is characterized by mass densities of the form

$$\Sigma_N^{(m)} = \frac{3M}{2\pi a^2}\eta^{2m-1}, \tag{41}$$

where the index m of the model is any positive integer. The potential $\phi_N^{(m)}$ is given by (23) by taking the Newtonian limit in (25) and using

$$C_{2n}^{(m)} = \frac{MG\pi^{1/2}(4n+1)(2m+1)!}{a2^{2m+1}(2n+1)(m-n)!\Gamma(m+n+\frac{3}{2})q_{2n+1}(0)} \tag{42}$$

for $n \leq m$, and $C_{2n}^{(m)} = 0$ for $n > m$.

3.1. The 1PN model for the $m = 1$ GKD

The first disk of the family, the disk with $m = 1$, is the well known Kalnajs disk [10], with the mass distribution

$$\Sigma_N^{(1)} = \frac{3M\eta}{2\pi a^2}. \tag{43}$$

Then, as was shown by Kalnajs [11], the DF depends on the Jacobi's integral as

$$f(J) = \frac{3M}{4\pi^2 a^3}\left[2(\Omega_0^2 - \Omega^2)J\right]^{-1/2}, \tag{44}$$

where $\Omega_0^2 = 3\pi GM/4a^3$. Now, for the Kalnajs disk it holds that $\langle v_\varphi \rangle = \Omega R$, so that the disk spins like a rigid body.

Inserting expressions (43) and (44) into the second fundamental equation (38), and integrating the DF, one easily obtains the following system of equations:

$$B_0 - B_2 + B_4 = 0, \tag{45}$$

$$3B_2 - 10B_4 = 6MGa\Omega^2 - \frac{27\pi G^2 M^2}{4a^2}, \tag{46}$$

$$B_4 = \frac{9\pi G^2 M^2}{140a^2} - 30GM\Omega^2 a. \tag{47}$$

Likewise, for the first fundamental equation we have

$$D_0 - D_2 + D_4 = \frac{9\pi G^2 M^2 \Omega^2}{8a^2(\Omega_o^2 - \Omega^2)}, \tag{48}$$

$$\gamma D_2 + \vartheta D_4 = B_2 - \frac{15}{8}B_4, \tag{49}$$

$$\chi D_4 = \frac{35}{16}B_4 - \frac{3\pi G^2 M^2}{4a^2} - GM\Omega^2 a, \tag{50}$$

where

$$\gamma = \frac{24a^3(\Omega_o^2 - \Omega^2) - 9\pi MG}{9\pi MG}, \tag{51}$$

$$\vartheta = \frac{135\pi MG/8 - 80a^3(\Omega_o^2 - \Omega^2)}{9\pi MG}, \tag{52}$$

$$\chi = \frac{70[128a^3(\Omega_o^2 - \Omega^2) - 27\pi MG]}{864\pi MG}. \tag{53}$$

Therefore, we have a system of linear equations with an upper triangular matrix for the constants of each potential, ψ and ϕ_{PN}.

Solving for the constants explicitly, we obtain

$$B_0 = 2aGM(\pi-1)\Omega^2 - \frac{3G^2M^2\pi(15\pi-1)}{20a^2}, \tag{54}$$

$$B_2 = \tfrac{2}{7}aGM(7\pi-10)\Omega^2 - \frac{3G^2M^2\pi(21\pi-2)}{28a^2}, \tag{55}$$

$$B_4 = \frac{18GM}{35a}\left(\frac{GM\pi}{8a} - \frac{5a^2\Omega^2}{3}\right), \tag{56}$$

$$\begin{aligned} D_0 = &-\frac{27G^3\pi^2M^3}{8a^2\left(a^3\Omega^2-3GM\pi\right)} - \frac{47709G^3\pi^2M^3}{448a^2\left(128a^3\Omega^2-357GM\pi\right)} \\ &- \frac{3G^2\pi(1+80\pi)M^2}{320a^2} - \frac{9\left(28G^3M^3\pi^3-95G^3M^3\pi^2\right)}{28a^2\left(8a^3\Omega^2-21GM\pi\right)}, \end{aligned} \tag{57}$$

$$D_2 = -\frac{3G^2\pi(7\pi-13)M^2}{28a^2} - \frac{9\left(28G^3M^3\pi^3-95G^3M^3\pi^2\right)}{28a^2\left(8a^3\Omega^2-21GM\pi\right)}, \tag{58}$$

$$D_4 = \frac{27G^2M^2\pi\left(184\Omega^2a^3+39GM\pi\right)}{140a^2\left(128a^3\Omega^2-357GM\pi\right)}, \tag{59}$$

which defines the potential ψ, and the correction ϕ_{PN}, respectively. Hence we can get all the parameters of interest, in particular, we observed that the mass correction, Σ_{PN}, is negligible compared with the classical mass Σ_N. In contrast, the 1PN rotation curve obtained with (40), is visibly separated from the Newtonian one from a certain radius, the difference being maximum at the edge of the disk (see Fig. 1), where the difference between both reaches 10.3% approximately. Those have been obtained with the typical values of a galaxy such as the Milky Way.

3.2. The 1PN model for the $m = 2$ GKD

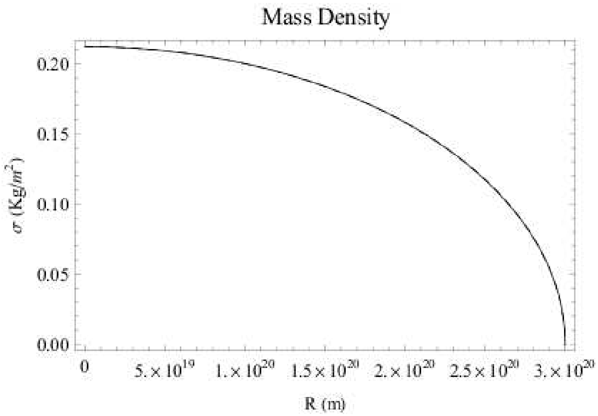
The procedure for the second disk is pretty much the same that for the first one. Again, the DF has a simple form when written as a function of Jacobi's integral,

$$f(J) = \frac{2M}{\sqrt{3}a^2}\left(\frac{10a^3}{G^3M^3\pi^{11}J}\right)^{1/4}, \tag{60}$$

for a mean circular speed $\langle v_\varphi\rangle = \Omega R = \sqrt{15\pi GM/32a^3}R$. This DF could be easily integrated to self-consistently obtain the mass density of the model, which also can be obtained from (41) with $m = 2$,

$$\Sigma_N^{(2)} = \frac{5M}{2\pi a^2}\left(1-\frac{R^2}{a^2}\right)^{3/2} = \frac{5M\eta^3}{2\pi a^2}. \tag{61}$$

the associated gravitational potential is given by the Newtonian limit ϕ_N of (23) with (42) and $m = 2$.



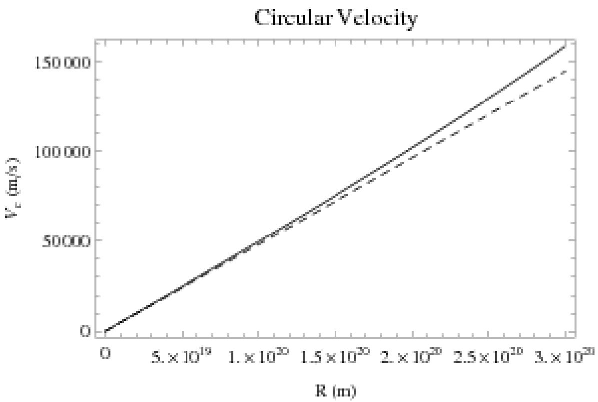


Figure 1. First we plot the Newtonian and 1PN mass density for the $m = 1$ GKD. The two curves seem to be superposed meaning that the mass correction is negligible. Then we plot the Newtonian (dashed line) and 1PN (full line) rotation curves for the same disk. The 1PN corrections are clearly notorious reaching a maximum at the border of the disc. Parameter values are: $M = 4 \times 10^{40} kg$, $a = 3 \times 10^{20} m$, $\Omega = 2 \times 10^{-13} Hz$.

Again, we replace the Newtonian terms into the second fundamental equation (38) to obtain

$$\sum_{n=0}^{4} B_{2n}(2n+1)q_{2n+1}(0)P_{2n}(\eta) = \sum_{n=0}^{4} \tilde{C}_{2i}\eta^{2i}, \tag{62}$$

where $\tilde{C}_0 = \tilde{C}_2 = 0$ and

$$\tilde{C}_4 = \frac{25\pi^2 10! G^2 M^2}{160a^2}, \tag{63}$$

$$\tilde{C}_6 = -\frac{75\pi^2 10! G^2 M^2}{160a^2} \tag{64}$$

$$\tilde{C}_8 = \frac{75\pi^2 10! G^2 M^2}{2240a^2}. \tag{65}$$

Multiplying (62) with a Legendre polynomial, integrating with respect to η, and using the orthogonality properties of the Legendre polynomials, we get

$$B_{2n} = \sum_{i=0}^{4} \frac{\sqrt{\pi}\tilde{C}_{2i} 2^{-2i-1}(4n+1)\Gamma(2n+1)}{q_{2n+1}(0)(2n+1)\Gamma(i-n+1)\Gamma(i+n+3/2)}. \tag{66}$$

We also could have used the expression (32).

After rearranging the terms, the first fundamental equation can be written as

$$\sum_{n=0}^{4}\{D_{2n}[\vartheta_{2n}P_{2n}(\eta) + q_{2n}(0)P_{2n}(0)] - B_{2n}q_{2n}(0)P_{2n}(\eta) - \hat{C}_{2n}\eta^{2n}\} = 0, \tag{67}$$

where,

$$\vartheta_{2n} = \frac{\pi(2j+1)}{32a^2 q_{2n+1}(0) - q_{2n}(0)}, \tag{68}$$

$$\hat{C}_0 = \frac{675\pi^2 G^2 M^2}{4096a^2} + \psi(0,0), \tag{69}$$

$$\hat{C}_2 = -\frac{1575\pi^2 G^2 M^2}{4096a^2}, \tag{70}$$

$$\hat{C}_4 = -\frac{1125\pi^2 G^2 M^2}{2048a^2}, \tag{71}$$

$$\hat{C}_6 = -\frac{2025\pi^2 G^2 M^2}{4096a^2}, \tag{72}$$

$$\hat{C}_8 = \frac{2025\pi^2 G^2 M^2}{8192a^2}. \tag{73}$$

Integrating and using orthogonality relations, the constants D_{2n} are given by

$$D_{2n} = \sum_{i=0}^{4} \frac{\sqrt{\pi}\hat{C}_{2i} 2^{-2i-1}(4n+1)\Gamma(2j+1)}{\vartheta_{2n}(2n+1)\Gamma(i-n+1)\Gamma(i+n+3/2)} + \frac{B_{2n}q_{2n}(0)}{\vartheta_{2n}}, \tag{74}$$

for $n > 0$, and

$$D_0 = \frac{1}{\vartheta_0 + \pi/2}\left[\sum_{i=0}^{4} \frac{\sqrt{\pi}\hat{C}_{2i} 2^{-2i-1}\Gamma(2j+1)}{\Gamma(i+1)\Gamma(i+3/2)} - \sum_{i=1}^{4} D_{2i}q_{2i}(0)P_{2i}(0)\right], \tag{75}$$

which defines the correction ϕ_{PN} to the Newtonian potential. The mass densities and rotation curves are plotted at Figure 2, note that the rotation curve is not only quantitative, but qualitatively different, presenting its maximum value more closely to the center of the disk and going upwards at the border. This is due to the fact that the 1PN curve includes more terms than the Newtonian one.

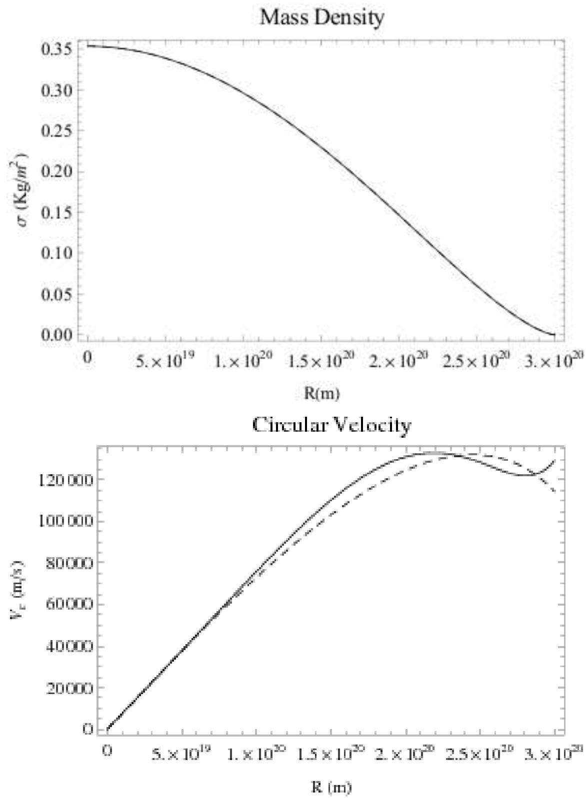


Figure 2. We plot the Newtonian and 1PN mass density, as well as the Newtonian (dashed line) and 1PN (full line) rotation curves for the $m = 2$ GKD, with parameter values: $M = 4 \times 10^{40} kg$, $a = 3 \times 10^{20} m$, $\Omega = 2 \times 10^{-13} Hz$. As we can see, the two graphics of mass density appear to be superposed, while that the 1PN rotation curve behaves completely different to the Newtonian one.

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