Some ring theoretical properties of skew Poincaré-Birkhoff-Witt extensions

Algunas propiedades de anillos de las extensiones PBW torcidas

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Abstract. In this paper we investigate a notion of Armendariz ring for skew Poincaré-Birkhoff-Witt extensions. We proceed with the study on the relationship between the ring theoretical properties of being Baer, quasi-Baer, p.p. and p.q.-Baer of a ring R and a skew PBW extension A over R.

Keywords: Armendariz, Baer, quasi-Baer, p.p and p.q.-Baer rings, skew Poincaré-Birkhoff-Witt extensions.

Resumen. En este artículo investigamos una noción de anillo de Armendariz para las extensiones torcidas de Poincaré-Birkhoff-Witt. Procedemos con el estudio de las relaciones entre las propiedades de Baer, quasi-Baer, p.p. y p.q.-Baer de un anillo R y una extensión PBW torcida sobre R.

Palabras claves: Armendariz, anillos de Baer, quasi-Baer, p.p y p.q.-Baer, extensiones torcidas de Poincaré-Birkhoff-Witt.

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1. Introduction

Kaplansky [14] defined a ring B as a *Baer* (resp. *quasi-Baer*, which was defined by Clark in [7]) ring if the right annihilator of every nonempty subset (resp. ideal) of B is generated by an idempotent (the objective of these rings is to abstract various properties of von Neumann algebras and complete $*$-regular rings; in [7], it was used the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra). Another generalization of Baer rings are the p.p.-rings. A ring B is called *right* (resp. *left*) *p.p* if the right (resp. *left*) annihilator of each element of B is generated by an idempotent (or

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equivalently, rings in which each principal right (resp. *left*) ideal is projective). Birkenmeir et. al., [5] defined a ring *right* (resp. *left*) *principally quasi-Baer* (or simply *right* (resp. *left*) *p.q-Baer*) ring if the right annihilator of each principal right (resp. left) ideal of B is generated by an idempotent. Note that in a reduced ring B, B is Baer (resp. p.p.-) if and only if B is quasi-Baer (resp. p.q.-Baer), see [3] for more details.

We can find several results about commutative and noncommutative Baer, quasi-Baer, p.p. and p.q.-Baer rings. Let us see some examples. If B is a reduced ring, then $B[x]$ is a Baer ring if and only if B is a Baer ring (in [2], Theorem B, it was shown an example to illustrate that the condition to be reduced is not superfluous). In the context of Ore extensions $B[x;\sigma,\delta]$ of injective type, i.e., when σ is injective, we found different works (cf. [3], [5], [4], [7], [10], [11], [12], and others). Some of these papers consider $\delta = 0$ (in this case, $B[x;\sigma]$ is called an Ore extension of endomorphism type) and σ an automorphism, or the case where σ is the identity. However, it is important to say that the Baerness and quasi-Baerness of a ring B and an Ore extension $B[x;\sigma,\delta]$ of B does not depend on each other. For instance, there exists a Baer ring B but the Ore extension $B[x;\sigma,\delta]$ is not right p.q.-Baer; similarly, there exist Ore extensions $B[x;\sigma,\delta]$ which are quasi-Baer, but B is not quasi-Baer (see Remark 4.5, [4], [9] and [12] for detailed examples).

A natural question for the notions of Baer, quasi-Baer, p.p.-rings, and p.q.-Baer, it is their behavior in the case of skew Poincaré-Birkhoff-Witt (PBW for short) extensions introduced by Gallego and Lezama [8] as a generalization of Ore extensions of injective type and PBW extensions (several homological and ring properties of these extensions have been investigated, see [8], [19], [18], [24] - [31], and others). In fact, it has been shown that skew PBW extensions contain remarkable examples of algebras such as the following: some Auslander-Gorenstein rings, examples of skew Calabi-Yau algebras, quantum polynomials, some kinds of quantum universal enveloping algebras, etc. (see [19], [26], and [30]). It it important to say that these extensions include several algebras which can not be expressed as Ore extensions (universal enveloping algebras of finite Lie algebras, diffusion algebras, and others, see [19] or [27] for a list of examples). With this in mind, we consider important to establish general results in a theory of Baerness and quasi-Baerness for several noncommutative rings. Precisely, in [27], a first treatment about these topics was established by the second author using a notion of *rigidness*, the Σ-*rigid* rings ([27], Definition 3.2), with the aim of establishing necessary and sufficient conditions to guarantee that all these properties are stable over skew PBW extensions. As a matter of fact, Σ-rigid rings have been also studied by the second author in [29], with the purpose of characterizing zip and reversible skew PBW extensions.

In this paper we investigate the (Σ,Δ)-Armendariz rings, which generalizes the Σ-rigid and Armendariz rings (a more general treatment can be found in [28]). As an application, we proceed with the study on the relationship between the ring theoretical properties of being Baer, quasi-Baer, p.p. and p.q.-Baer of a ring R and a skew PBW extension A over R. In this way, we generalize several results in the literature for Ore extensions of injective type and skew PBW extensions.

The paper is organized as follows. Section 2 contains the definition and some of the properties of the objects we are going to study, that is, the skew PBW extensions. In Section 3 we introduce the (Σ, Δ)-Armendariz rings (Definition 3.4) and a more general class of rings, the (Σ, Δ)-weak Armendariz rings (Definition 3.5). We show that every Σ-rigid ring is an (Σ, Δ)-Armendariz ring (Proposition 3.6), but the converse is false (Example 3.7). However, in Theorem 3.9, we prove the following equivalences: for a skew PBW extension A of a ring R, R is reduced and (Σ, Δ)-Armendariz $\Leftrightarrow$ R is Σ-rigid $\Leftrightarrow$ A is reduced. In this way, our Theorem 3.9 generalizes [20], Theorem A, and [6], Theorem 1 and Corollary 3. We also present some key results with the aim of proving that if R is a (Σ, Δ)-Armendariz ring, then A and R are Abelian rings (Proposition 3.12 and Theorem 3.13). Now, in Section 4, we investigate the properties of being Baer, quasi-Baer, p.p. and p.q.-Baer for skew PBW extensions over (Σ, Δ)-Armendariz rings. Since Σ-rigid rings are contained strictly in (Σ, Δ)-Armendariz rings, our treatment generalize different results in the literature. More precisely: (i) Theorem 4.1 generalizes [15], Theorem 10; [13], Theorem 21; [22], Theorem 13; and [27], Theorem 3.9. (ii) Theorem 4.2 generalizes [15], Theorem 9; [13], Theorem 22; [22], Theorem 14; and [27], Theorem 3.12. (iii) Theorem 4.3 generalizes [21], Propositions 3.2, 3.7, and Theorem 3.10. (iv) Theorem 4.4 generalizes [27], Theorem 3.13. The proofs presented in this paper follow the ideas presented in [13], [22], and [28].

2. Skew PBW extensions

Definition 2.1 ([8], Definition 1)**.** Let R and A be rings. We say that A *is a skew PBW extension of* R (also called a σ-PBW extension of R), which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ $(x^0 := 1)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Remark 2.2 ([8], Remark 2). (i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1 are unique. (ii) In Definition 2.1, $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.

Proposition 2.3 ([8], Proposition3). *Let A be a skew PBW extension of R. For every $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Definition 2.4 ([8], Definition 4, and [18], Definition 2.3). Let A be a skew PBW extension of a ring R.

(a) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by (iii'): for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; (iv'): for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$. (b) A is called *bijective* if σ_i is bijective, for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$. (c) A is called a skew PBW extension of *endomorphism type*, if $\delta_i = 0$, for every i. In addition, if σ_i is bijective, for each i, A is called a skew PBW extension of *automorphism type.*

Examples 2.5. If $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is an iterated Ore extension where

(i) σ_i is injective, for $1 \leq i \leq n$;

(ii) $\sigma_i(r)$, $\delta_i(r) \in R$, for every $r \in R$ and $1 \leq i \leq n$;

(iii) $\sigma_j(x_i) = cx_i + d$, for $i < j$, and $c, d \in R$, where c has a left inverse;

(iv) $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$, for $i < j$,

then $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \cong \sigma(R)\langle x_1, \dots, x_n\rangle$ ([19], p. 1212). In particular, note that skew PBW extensions of endomorphism type are more general than iterated Ore extensions $R[x_1; \sigma_1] \cdots [x_n; \sigma_n]$. On the other hand, skew PBW extensions are more general than Ore extensions of injective type (diffusion algebras, univesal enveloping algebras of finite Lie algebras, and others, are examples of skew PBW extensions which can not be expressed as iterated Ore extensions, see [19] for more details). Skew PBW extensions contains various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc. A detailed list of examples of skew PBW extensions is presented in [19], [26], and [30].

Definition 2.6 ([8], Definition 6). Let A be a skew PBW extension of R with injective endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.3.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^{\alpha} := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^{\alpha} \in \text{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\text{Mon}(A)$ (a total order on $\mathbb{N}^n$). For an element $x^{\alpha} \in \text{Mon}(A)$, $\exp(x^{\alpha}) := \alpha \in \mathbb{N}^n$. If $x^{\alpha} \succeq x^{\beta}$ but $x^{\alpha} \neq x^{\beta}$, we write $x^{\alpha} \succ x^{\beta}$. Every element $f \in A$ can be expressed uniquely as

$f = a_0+a_1X_1+\cdots+a_mX_m$, with $a_i \in R \setminus \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_mX_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$, and a_0 as the constant term of f. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [8], Section 3.

Proposition 2.7 ([8], Theorem 7). *Let A be a polynomial ring over R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(i) *for each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$, if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(ii) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$, if $p_{\alpha,\beta} \neq 0$.*

Remark 2.8. ([27], Remark 2.10) If $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, then

$$\begin{aligned}
a_iX_ib_jY_j = {} & a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
& + a_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
& + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
& + \cdots + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b)}x_n^{\alpha_{in}}x^{\beta_j}\\
& + a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b}x^{\beta_j}.
\end{aligned}$$

In this way, when we compute every summand of $a_iX_ib_jY_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

3. (Σ, Δ)-Armendariz rings and (Σ, Δ)-weak Armendariz rings

Following [16], an endomorphism σ of a ring B is called to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$, for $a \in B$. A ring B is said to be σ-rigid if there exists a rigid endomorphism σ of B. It is clear that any rigid endomorphism of a ring is a monomorphism, and σ-rigid rings are reduced ([12], p. 218). Properties of σ-rigid rings have been studied by several authors (c.f. [12]) and [16]). Now, from [23], a ring B is called an *Armendariz ring* if whenever two polynomials $f(x) = a_0 + a_1x_1 + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_tx^t \in B[x]$ with

$f(x)g(x) = 0$, then we have $a_ib_j = 0$, for every i, j. Motivated by the results established in several papers ([1], [2], [12] and [15]), in [13], it was defined B to be *σ-skew Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x_1 + \cdots + b_tx^t \in B[x;\sigma]$ satisfy $f(x)g(x) = 0$, then $a_i\sigma^i(b_j) = 0$, for every i, j. This definition and the results presented in [1], [2], [13] and [17] were generalized in [22] considering the (σ, δ)-skew Armendariz rings. More exactly, in [22], Definition 1, it was introduced the following notion: Let σ be an endomorphism and δ an σ-derivation of a ring B. B is called an *(σ, δ)-skew Armendariz* (or simply, (σ, δ)-Armendariz) ring, if for polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_tx^t$ in $B[x;\sigma,\delta]$, $f(x)g(x) = 0$ implies $a_ix^ib_jx^j = 0$, for every i, j. Note that every σ-skew Armendariz ring is (σ, δ)-skew Armendariz, since δ is the zero mapping, and every subring of an (σ, δ)-Armendariz ring is (σ, δ)-Armendariz. Also, every σ-rigid ring is (σ, δ)-Armendariz, but the converse does not hold ([13], Example 1).

In [22], Definition 2, it was also introduced the following definition: Let σ be an endomorphism and δ be an σ-derivation of B. B is called an *(σ, δ)-skew weak Armendariz* (or simply (σ, δ)-weak Armendariz) ring, if for linear polynomials $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$ in $B[x;\sigma,\delta]$, $f(x)g(x) = 0$ implies $a_ix^ib_jx^j = 0$, for $i, j \in \{0, 1\}$. It follows from the definitions that every (σ, δ)-Armendariz ring is (σ, δ)-weak Armendariz, and that every subring of an (σ, δ)-weak Armendariz ring is (σ, δ)-weak Armendariz. Nevertheless, an (σ, δ)-weak Armendariz ring is not necessarily (σ, δ)-Armendariz ([17], Example 3.2). It is important to say that in [22] it was presented an affirmative answer to a question formulated in [13], p. 115: it was proved that for a ring B with a monomorphism σ and σ-derivation δ, B is σ-rigid if and only if it is reduced and (σ, δ)-weak Armendariz.

As we saw above, for the Ore extensions $B[x;\sigma,\delta]$ the more general notions of Armendariz are the (σ, δ)-Armendariz and (σ, δ)-weak Armendariz, and since Ore extensions of injective type are particular examples of skew PBW extensions, we introduce these notions for this kind of extensions (Definitions 3.4 and 3.5, respectively). Before, we recall the notion of Σ-rigid ring and some key properties with the aim of showing that Σ-rigid rings are (Σ, Δ)-Armendariz rings.

Definition 3.1. Let B be a ring and Σ a family of endomorphisms of B. Σ is called a *rigid endomorphisms family* if $r\sigma^\alpha(r) = 0$ implies $r = 0$, for every $r \in B$ and $\alpha \in \mathbb{N}^n$. A ring B is called to be *Σ-rigid* if there exists a rigid endomorphisms family Σ of B.

Note that if Σ is a rigid endomorphisms family, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if B is a Σ-rigid ring and $r^2 = 0$ for $r \in B$, then $0 = r\sigma^\alpha(r^2)\sigma^\alpha(\sigma^\alpha(r)) = r\sigma^\alpha(r)\sigma^\alpha(r)\sigma^\alpha(\sigma^\alpha(r)) = r\sigma^\alpha(r)\sigma^\alpha(r\sigma^\alpha(r))$, i.e., $r\sigma^\alpha(r) = 0$ and so $r = 0$, that is, B is reduced (note that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, see [12], Example 9). With this in mind, we consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations in a skew PBW extension A of a ring R (see Proposition 2.3).

Proposition 3.2. ([27], Lemma 3.3) *Let B be an Σ-rigid ring and $a, b \in B$. Then:*

(i) *If $ab = 0$ then $a\sigma^{\alpha}(b) = \sigma^{\alpha}(a)b = 0$, for $\alpha \in \mathbb{N}^n$.*

(ii) *If $ab = 0$ then $a\delta^{\beta}(b) = \delta^{\beta}(a)b = 0$, for $\beta \in \mathbb{N}^n$.*

(iii) *If $ab = 0$ then $a\sigma^{\alpha}(\delta^{\beta}(b)) = a\delta^{\beta}(\sigma^{\alpha}(b)) = 0$, for every $\alpha, \beta \in \mathbb{N}^n$.*

(iv) *If $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$ for some $\theta \in \mathbb{N}^n$, then $ab = 0$.*

For the next proposition, suppose that the elements $c_{i,j}$ are invertible and they are at the center of R.

Proposition 3.3. ([27], Proposition 3.6) *Suppose that R is an Σ-rigid ring. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ be elements of a skew PBW extension A of R. Then $fg = 0$ if and only if $a_ib_j = 0$, for all $0 \le i \le m$, $0 \le j \le t$.*

Next, we define the key concepts of this paper.

Definition 3.4. Let A be a skew PBW extension of a ring R. We say that R is an (Σ, Δ)-*Armendariz ring*, if for polynomials $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ in A, the equality $fg = 0$ implies $a_iX_ib_jY_j = 0$, for every i, j.

Definition 3.5. Let A be a skew PBW extension of a ring R. We say that R is an (Σ, Δ)-*weak Armendariz ring*, if for linear polynomials $f = a_0 + a_1x_1 + \cdots + a_nx_n$ and $g = b_0 + b_1x_1 + \cdots + b_nx_n$ in A, the equality $fg = 0$ implies $a_ix_ib_jx_j = 0$, for every i, j.

Proposition 3.6. *Every Σ-rigid ring is (Σ, Δ)-skew Armendariz ring.*

Proof. The assertion follows from Propositions 3.2, 3.3, and Remark 2.8. $\square$

As a particular case of Proposition 3.6, we obtain [21], Lemma 2.5. The following example shows that there exists a non-Σ-rigid ring which is (Σ, Δ)-Armendariz.

Example 3.7. ([22], Example 9) Let σ be an endomorphism and δ be an σ-derivation of B. Let B be an σ-rigid ring and

$$B_3 := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in B \right\}.$$

Consider the endomorphism extended $\overline{\sigma}$ of B_3 defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$, and the extended derivation $\overline{\delta} : B_3 \to B_3$ given by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. Then B_3 is $(\overline{\sigma}, \overline{\delta})$-Armendariz and is not reduced, so it is not $\overline{\sigma}$-rigid.

Proposition 3.8. *If R is an (Σ, Δ)-weak Armendariz ring and $ab = 0$, then $\sigma^{\alpha}(a)\delta^{\alpha}(b) = \delta^{\alpha}(a)b = 0$, for every $\alpha \in \mathbb{N}^n$.*

Proof. Since it is sufficient to prove the case $\sigma_i(a)\delta_i(a) = \delta_i(a)b = 0$, the assertion follows from [22], Lemma 3. $\square$

In [20] and [6], both authors of these papers gave a positive answer to the following question formulated in [13], p. 115: Let σ be a monomorphism (or automorphism) of a (commutative) reduced ring B and B be a σ-skew Armendariz. Is B σ-rigid? The content of Theorem 3.9 is the generalization of this answer to skew PBW extensions. Again, we suppose that the elements $c_{i,j}$ in Definition 2.1 (iv) are invertible and commute with every element of R.

Theorem 3.9. *If A is a skew PBW extension of a ring R, then the following statements are equivalent:*

(i) *R is reduced and (Σ, Δ)-Armendariz;*

(ii) *R is Σ-rigid;*

(iii) *A is reduced.*

Proof. (ii) $\Leftrightarrow$ (iii) This is the content of [27], Proposition 3.5. (ii) We saw above that every Σ-rigid ring is reduced, so the assertion follows from Proposition 3.6. Let us prove (i) $\Rightarrow$ (ii) Suppose that R is reduced, (Σ, Δ)-Armendariz and is not Σ-rigid. Then there exists $\beta \in \mathbb{N}^n$ with $a\sigma^{\beta}(a) = 0$ and $a \neq 0$. Note that $\sigma^{\beta}(a)\sigma^{\beta}(\sigma^{\beta}(a)) = \sigma^{\beta}(a\sigma^{\beta}(a)) = 0$. Using that R is reduced, the equality $(\sigma^{\beta}(a)a)^2 = \sigma^{\beta}(a)a\sigma^{\beta}(a)a = 0$ implies $\sigma^{\beta}(a)a = 0$. Equivalently, since $a \neq 0$, σ^{β} is injective, and R is reduced, then $\sigma^{\beta}(a) \neq 0$ and $(\sigma^{\beta}(a))^2 \neq 0$. With this in mind, consider the elements $f = \sigma^{\beta}(a) + \sigma^{\beta}(a)x^{\beta}$, $g = a - \sigma^{\beta}(a)x^{\beta}$. Then

$$
\begin{aligned}
fg &= (\sigma^{\beta}(a) + \sigma^{\beta}(a)x^{\beta})(a - \sigma^{\beta}(a)x^{\beta}) \\
&= \sigma^{\beta}(a)a - (\sigma^{\beta}(a))^2x^{\beta} + \sigma^{\beta}(a)x^{\beta}a - \sigma^{\beta}(a)x^{\beta}\sigma^{\beta}(a)x^{\beta} \\
&= -(\sigma^{\beta}(a))^2x^{\beta} + \sigma^{\beta}(a)[\sigma^{\beta}(a)x^{\beta} + p_{\beta,a}] - \sigma^{\beta}(a)[\sigma^{\beta}(\sigma^{\beta}(a))x^{\beta} + q_{\beta,\sigma^{\beta}(a)}]x^{\beta} \\
&= \sigma^{\beta}(a)p_{\beta,a} - \sigma^{\beta}(a\sigma^{\beta}(a))x^{\beta}x^{\beta} - \sigma^{\beta}(a)q_{\beta,\sigma^{\beta}(a)}x^{\beta} \\
&= \sigma^{\beta}(a)p_{\beta,a} - \sigma^{\beta}(a)q_{\beta,\sigma^{\beta}(a)}x^{\beta},
\end{aligned}
$$

where $p_{\beta,a} = 0$ or $\deg(p_{\beta,a}) < |\beta|$, if $p_{\beta,a} \neq 0$, and $q_{\beta,\sigma^{\beta}(a)} = 0$ or $\deg(q_{\beta,\sigma^{\beta}(a)}) < |\beta|$, if $q_{\beta,\sigma^{\beta}(a)} \neq 0$. Since $a\sigma^{\beta}(a) = \sigma^{\beta}(a)a = 0$, Remark 2.8 and Proposition 3.8 guarantee that $\sigma^{\beta}(a)p_{\beta,a} = \sigma^{\beta}(a)q_{\beta,\sigma^{\beta}(a)}x^{\beta} = 0$, so $fg = 0$. By assumption, R is (Σ, Δ)-Armendariz, that is, $-(\sigma^{\beta}(a))^2 = 0$, but $-(\sigma^{\beta}(a))^2 \neq 0$, i.e., we have obtained a contradiction. Hence, R is Σ-rigid. $\square$

Corollary 3.10. ([22], Theorem 6) *A ring B with a monomorphism σ, is an σ-rigid ring if and only if it is a reduced (σ, δ)-weak Armendariz ring.*

Next, we present some key results about (Σ,Δ)-Armendariz rings which are very important in Section 4.

Proposition 3.11. *If R is an (Σ,Δ)-weak Armendariz ring, then $\sigma_i(e)=e$ and $\delta_i(e)=0$, for every idempotent element e of R.*

Proof. Consider an idempotent element e of R. Then $\delta_i(e)=\sigma_i(e)\delta_i(e)+\delta_i(e)e$. Let $f,g\in A$ given by $f=\delta_i(e)+0x_1+\cdots+0x_{i-1}+\sigma_i(e)x_i+0x_{i+1}+\cdots+0x_n$, and $g=e-1+(e-1)x_1+\cdots+(e-1)x_n$, respectively. Recall that $\delta_i(1)=0$, for every i. Let us show that $fg=0$:

$$\begin{aligned}fg&=\delta_i(e)(e-1)+\sum_{j=1}^{n}\delta_i(e)(e-1)x_j+\sigma_i(e)x_i(e-1)+\sum_{j=1}^{n}\sigma_i(e)x_i(e-1)x_j\\&=\delta_i(e)(e-1)+\sum_{j=1}^{n}\delta_i(e)(e-1)x_j+\sigma_i(e)[\sigma_i(e-1)x_i+\delta_i(e-1)]\\&+\sum_{j=1}^{n}\sigma_i(e)[\sigma_i(e-1)x_i+\delta_i(e-1)]x_j.\end{aligned}$$

Equivalently,

$$\begin{aligned}fg&=\delta_i(e)(e-1)+\sum_{j=1}^{n}\delta_i(e)(e-1)x_j+\sigma_i(e)[(\sigma_i(e)-\sigma_i(1))x_i+\delta_i(e)]\\&+\sum_{j=1}^{n}\sigma_i(e)[(\sigma_i(e)-\sigma_i(1))x_i+\delta_i(e)]x_j\\&\quad\delta_i(e)(e-1)+\sum_{j=1}^{n}\delta_i(e)(e-1)x_j+\sigma_i(e)[\sigma_i(e)x_i-x_i+\delta_i(e)]\\&+\sum_{j=1}^{n}\sigma_i(e)[\sigma_i(e)x_i-x_i+\delta_i(e)]x_j\\&=\delta_i(e)e-\delta_i(e)+\sum_{j=1}^{n}(\delta_i(e)e-\delta_i(e))x_j+\sigma_i(e)x_i-\sigma_i(e)x_i+\sigma_i(e)\delta_i(e)\\&+\sum_{j=1}^{n}(\sigma_i(e)x_i-\sigma_i(e)x_i+\sigma_i(e)\delta_i(e))x_j\\&=\delta_i(e)e-\delta_i(e)+\sum_{j=1}^{n}\delta_i(e)ex_j-\sum_{j=1}^{n}\delta_i(e)x_j+\sigma_i(e)\delta_i(e)+\sum_{j=1}^{n}\sigma_i(e)\delta_i(e)x_j\\&=\sigma_i(e)\delta_i(e)+\delta_i(e)e-\delta_i(e)+\left(\sum_{j=1}^{n}(\sigma_i(e)\delta_i(e)+\delta_i(e)e-\delta_i(e))\right)x_j\\&=0.\end{aligned}$$

Since R is (Σ, Δ)-weak Armendariz, we obtain $\delta_i(e)(e-1) = 0$, i.e., $\delta_i(e)e = \delta_i(e)$, and hence $\sigma_i(e)\delta_i(e) = 0$.

Now, consider the elements s and t of A given by $s = \delta_i(e) - (1-\sigma_i(e))x_i$ and $t = e + \sum_{j=1}^{n} ex_j$, respectively. Then $st = 0$. Let us see:

$$\begin{aligned}
st &= \delta_i(e)e + \delta_i(e)e\sum_{j=1}^{n} x_j - (1-\sigma_i(e))x_i e - (1-\sigma_i(e))x_i e\sum_{j=1}^{n} x_j \\
&= \delta_i(e)e + \delta_i(e)e\sum_{j=1}^{n} x_j - x_i e + \sigma_i(e)x_i e - x_i e\sum_{j=1}^{n} x_j + \sigma_i(e)x_i e\sum_{j=1}^{n} x_j \\
&= \delta_i(e)e + \delta_i(e)e\sum_{j=1}^{n} x_j - (\sigma_i(e)x_i + \delta_i(e)) + \sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \\
&- (\sigma_i(e)x_i + \delta_i(e))\sum_{j=1}^{n} x_j + \sigma_i(e)(\sigma_i(e)x_i + \delta_i(e))\sum_{j=1}^{n} x_j,
\end{aligned}$$

or what is the same,

$$\begin{aligned}
st = {}& \delta_i(e)e + \delta_i(e)e\sum_{j=1}^{n} x_j - \sigma_i(e)x_i - \delta_i(e) + \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) \\
& - \sigma_i(e)x_i\sum_{j=1}^{n} x_j - \delta_i(e)\sum_{j=1}^{n} x_j + \sigma_i(e)x_i\sum_{j=1}^{n} x_j + \sigma_i(e)\delta_i(e)\sum_{j=1}^{n} x_j.
\end{aligned}$$

Since $\delta_i(e) = \delta_i(e)e$ and $\sigma_i(e)\delta_i(e) = 0$, then $st = 0$. By (Σ, Δ)-weak Armendariz condition we know that $\delta_i(e)e = 0$, which shows that $\delta_i(e) = 0$.

Consider the elements $u, v \in A$ given by $u = 1 - e + (1-e)\sigma_i(e)x_i$ and $v = e + (e-1)\sigma_i(e)x_i$. Then

$$\begin{aligned}
uv = {}& e + (e-1)\sigma_i(e)x_i - e^2 - e(e-1)\sigma_i(e)x_i + (1-e)\sigma_i(e)x_i e \\
& + (1-e)\sigma_i(e)x_i(e-1)\sigma_i(e)x_i \\
& = e\sigma_i(e)x_i - \sigma_i(e)x_i - e\sigma_i(e)x_i + e\sigma_i(e)x_i + (1-e)\sigma_i(e)(\sigma_i(e)x_i + \delta_i(e)) \\
& + (1-e)\sigma_i(e)(\sigma_i(e)x_i - x_i + \delta_i(e))\sigma_i(e)x_i \\
& = -\sigma_i(e)x_i + e\sigma_i(e)x_i + \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - e\sigma_i(e)x_i - e\sigma_i(e)\delta_i(e) \\
& + [\sigma_i(e)x_i - \sigma_i(e)x_i + \sigma_i(e)\delta_i(e) - e\sigma_i(e)x_i + e\sigma_i(e)x_i - e\sigma_i(e)\delta_i(e)]\sigma_i(e)x_i \\
& = 0.
\end{aligned}$$

Hence, by the (Σ, Δ)-weak Armendariz condition, $(1-e)(e-1)\sigma_i(e) = 0$, i.e., $e\sigma_i(e) = \sigma_i(e)$.

Now, let $w = e + e(1-\sigma_i(e))x_i,\ z = 1 - e - e(1-\sigma_i(e))x_i$ be elements of A. Then

$$
\begin{aligned}
wz &= e - e^2 - e^2(1-\sigma_i(e))x_i + e(1-\sigma_i(e))x_i - e(1-\sigma_i(e))x_i e \\
&\quad - e(1-\sigma_i(e))x_i e(1-\sigma_i(e))x_i \\
&= -e(1-\sigma_i(e))x_i e[1+(1-\sigma_i(e))x_i] \\
&= -e(1-\sigma_i(e))(\sigma_i(e)x_i+\delta_i(e))[1+(1-\sigma_i(e))x_i] \\
&= (-e\sigma_i(e)x_i + e\sigma_i(e)x_i)[1+(1-\sigma_i(e))x_i] \\
&= 0,
\end{aligned}
$$

since that $\delta_i(e) = 0$ and $\sigma_i(e)\sigma_i(e) = \sigma_i(e)$. Then, (Σ, Δ)-weak Armendariz condition implies $e(-e(1-\sigma_i(e))) = 0$, which shows that $e\sigma_i(e) = e$, and so $\sigma_i(e) = e$. □

Proposition 3.12. *Let A be a skew PBW extension of an (Σ, Δ)-Armendariz ring R. If $e = e_0 + e_1X_1 + \cdots + e_mX_m$ is an idempotent element of A, then $e = e_0$.*

Proof. The equalities $e(1-e) = 0$ and $(1-e)e = 0$ can be written as $(e_0 + e_1X_1 + \cdots + e_mX_m)((1-e_0) - e_1X_1 - \cdots - e_mX_m) = 0$ and $((1-e_0) - e_1X_1 - \cdots - e_mX_m)(e_0 + e_1X_1 + \cdots + e_mX_m) = 0$, respectively. By the (Σ, Δ)-Armendariz condition, $e_0(1-e_0) = 0$, $e_0e_i = 0$, and $(1-e_0)e_i = 0$, for every i, so $e_i = 0$, for $1 \le i \le n$, and hence $e = e_0 = e_0^2$. □

From Theorem 3.9 we know that a ring R is Σ-rigid if and only if A is reduced. The next result, Theorem 3.13, extend Theorem 3.9 and also [13], Proposition 20, and [22], Theorem 12, to the context of (Σ, Δ)-Armendariz rings.

Theorem 3.13. *If A is a skew PBW extension of an (Σ, Δ)-Armendariz ring R, then A is an Abelian ring.*

Proof. We know that for every idempotent element e of R, $\sigma_i(e) = e$ and $\delta_i(e) = 0$, for each $i = 1, \ldots, n$ (Proposition 3.11). Now, from Proposition 3.12 it follows that every idempotent element of A is an idempotent element of R, which means that it is sufficient to prove the assertion for R, that is, we want to see that R is Abelian. With this in mind, let C be the set of idempotent elements of R. Note that $efR \cap (1-f)(1-e)C = 0$, for every $e, f \in C$. More exactly, if this is not the case, then $ef(-t) = (1-f)(1-e)s \in efR \cap (1-f)(1-e)C = 0$ for some $t \in R$ and $s \in C$. Let $g = e + \sum_{i=1}^{n}(1-f)x_i$ and $h = (1-e)s + \sum_{i=1}^{n} ftx_i$. Then

$$
\begin{aligned}
gh &= e(1-e)s + \sum_{i=1}^{n} eftx_i + \sum_{i=1}^{n}(1-f)x_i(1-e)s \\
&\quad + \Big(\sum_{i=1}^{n}(1-f)x_i\Big)\Big(\sum_{i=1}^{n} ftx_i\Big) \\
&= \sum_{i=1}^{n} eftx_i + \sum_{i=1}^{n}(1-f)(1-e)sx_i \\
&\quad + \Big(\sum_{i=1}^{n}(1-f)[f\sigma_i(t)x_i + f\delta_i(t)]\Big)\Big(\sum_{i=1}^{n} x_i\Big) \\
&= \sum_{i=1}^{n} eftx_i + \sum_{i=1}^{n}(1-f)(1-e)sx_i \\
&\quad + \Big(\sum_{i=1}^{n}(1-f)f\sigma_i(t)x_i + (1-f)f\delta_i(t)\Big)\Big(\sum_{i=1}^{n} x_i\Big) \\
&= 0.
\end{aligned}
$$

Since R is (Σ,Δ)-Armendariz, then $eft=0$, whence $efR\cap(1-f)(1-e)A=0$. Now, note that if $fe=0$ for two elements f,e of C, then $ef=0$. This follows from the following facts: $-ef=(1-f)(1-e)f\in efR\cap(1-f)(1-e)C=0$. If $k=e+er(1-e)$ and $l=(1-e)+(1-e)re$, with $r\in R$, then $k^2=(e+er(1-e))(e+er(1-e))=e^2+e^2r(1-e)+er(1-e)e+er(1-e)er(1-e)=e+er(1-e)=k$; $l^2=(1-e+(1-e)re)(1-e+(1-e)re)=1-e+(1-e)re-e+e^2-e(1-e)re+(1-e)re-(1-e)re^2+(1-e)re(1-e)re=1-e+(1-e)re=l$; $(1-e)k=(1-e)(e+er(1-e))=e+er(1-e)-e^2-e^2r(1-e)=0$; $el=e(1-e+(1-e)re)=e-e^2+e(1-e)re=0$. Since k,l and $1-e$ are idempotent elements, then $k(1-e)=le=0$, i.e., $(e+er(1-e))(1-e)=(1-e+(1-e)re)e=0$, or equivalently, $e-e^2+er(1-e)=e-e^2+(1-e)re^2=0$. Hence $er=ere=re$, that is, R is Abelian. $\square$

4. Baer, quasi-Baer, p.p. and p.q.-rings over skew PBW extensions

As an application of the treatment developed in Section 3, next we study the relationship between the properties of being Baer, quasi-Baer, p.p. and p.q.-Baer of a ring R and a skew PBW extension A over R. We generalize several results in the literature for Ore extensions ([13], [15], [21] and [22]) and for skew PBW extensions ([27]). Recall that for a subset C of a ring B, $r_B(C)$ denotes the right annihilator of C in B, that is, $r_B(C)=\{r\in B \mid cr=0, \text{ for all } r\in C\}$.

First, our Theorem 4.1 extends [15], Theorem 10; [13], Theorem 21; [22], Theorem 13; and [27], Theorem 3.9, to the skew PBW extensions over (Σ,Δ)-

skew Armendariz rings.

Theorem 4.1. *If A is a skew PBW extension of an (Σ, Δ)-Armendariz ring R, then R is a Baer ring if and only if A is a Baer ring.*

Proof. Suppose that A is Baer, and let B be a nonempty subset of R. Then $r_A(B) = eA$ for some idempotent $e \in R$ (Proposition 3.12). Hence, $r_R(B) = r_A(B) \cap R = eA \cap R = eR$, i.e., R is Baer. Conversely, if R is Baer, from Theorem 3.13 we know that R is Abelian, and since every Abelian Baer ring is reduced, the assertion follows from [27], Theorem 3.9. $\square$

Now, our Theorem 4.2 extends [15], Theorem 9; [13], Theorem 22; [22], Theorem 14; and [27], Theorem 3.12.

Theorem 4.2. *If A is a bijective skew PBW extension of an (Σ, Δ)-Armendariz ring R, then R is a p.p.-ring if and only if A is a p.p.-ring.*

Proof. Suppose that A is a p.p.-ring. If r is an element of R, there exists an idempotent element e of R such that $r_A(\{a\}) = eA$ (Proposition 3.12), and hence $r_R(\{a\}) = eR$, that is, R is p.p.

Conversely, suppose that R is a p.p.-ring. Consider a nonzero element $f = a_0 + a_1X_1 + \cdots + a_mX_m$ of A. Then, there exists an idempotent $e_k \in R$ with $r_R(\{a_k\}) = e_kR$, for every $0 \le k \le m$. Let $e := e_0e_1 \cdots e_m$. Note that $e^2 = e$ and $eR = \bigcap_{k=0}^{m} r_R(\{a_k\})$, since R is Abelian (Proposition 3.12 and Theorem 3.13). Now, by Remark 2.8 and Proposition 3.11, we have $fe = 0$, that is, $eA \subseteq r_A(f)$. Now, if $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in r_A(f)$, then $fg = 0$, i.e., $a_iX_ib_jY_j = 0$, for every i, j. Since $a_iX_ib_jY_j = a_i[\sigma^{\alpha_i}(b_j)X_i + p_{\alpha_i,b_j}]Y_j = a_i\sigma^{\alpha_i}(b_j)X_iY_j + a_ip_{\alpha_i,b_j}Y_j = a_i\sigma^{\alpha_i}(b_j)[c_{\alpha_i,\beta_j}x^{\alpha_i+\beta_j} + p_{\alpha_i,\beta_j}] + a_ip_{\alpha_i,b_j}Y_j = 0$, whence $a_i\sigma^{\alpha_i}(b_j)c_{\alpha_i,\beta_j} = 0$. Using the bijectivity of R, $a_i\sigma^{\alpha_i}(b_j) = 0$ (where p_{α_i,b_j} and p_{α_i,β_j} are as in Proposition 2.7), whence $\sigma^{\alpha_i}(b_j) \in r_R(\{a_i\}) = e_iR$, for every $0 \le i \le m$, $0 \le j \le t$. Hence, $b_j \in eR = \bigcap_{i=0}^{m} r_R(\{a_i\})$, for all $0 \le j \le t$, which implies $g \in eA$. We conclude that $eA = r_A(\{f\})$, i.e., A is a p.p.-ring. $\square$

Next theorem generalizes [21], Propositions 3.2, 3.7, and Theorem 3.10.

Theorem 4.3. *If A is a bijective skew PBW extension of an (Σ, Δ)-Armendariz ring R, then R is a quasi-Baer ring if and only if A is a quasi-Baer ring.*

Proof. Suppose that A is quasi-Baer. Let I be an ideal of R. Since A is quasi-Baer, there exists an idempotent $e \in A$ with $r_A(IA) = eA$, where $e = e_0 + e_1X_1 + \cdots + e_mX_m$. By Proposition 3.12, we obtain $e_0 \in r_R(I)$, which shows that $e_0R \subseteq r_R(I)$. Now, if $a \in r_R(I)$, then $a \in r_A(IA) \cap R = e_0A \cap R$, i.e., $a = e_0g$, for some $g \in A$ given by $g = b_0 + b_1Y_1 + \cdots + b_tY_t$. Necessarily, $a = e_0b_0$, that is, $a \in e_0R$, whence $r_R(I) \subseteq e_0R$. This fact concludes the proof.

Conversely, suppose that R is quasi-Baer. Let I be a nonzero ideal of A. Let us see that $r_A(I) = eA$ for some idempotent element e of R. Consider the set $I' = \{0\} \cup \{\text{lc}(f) \mid f \in I\}$. Note that I' is a nonzero left ideal of R. Now, if we take elements $a \in I'$ and $r \in R$, we know that there exists an element $f \in I$ given by the expression $f = a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1} + aX_m$. If $\alpha_m = (\alpha_{m1}, \ldots, \alpha_{mn})$, let $-\alpha_m^{\rm op} := (-\alpha_{mn}, \ldots, -\alpha_{m1})$. Since A is bijective, we consider the expression

$$\begin{aligned} f\sigma^{-\alpha_m^{\rm op}}(r) &= a_0\sigma^{-\alpha_m^{\rm op}}(r) + a_1X_1\sigma^{-\alpha_m^{\rm op}}(r) + \cdots + a_{m-1}X_{m-1}\sigma^{-\alpha_m^{\rm op}}(r) \\ &\quad + aX_m\sigma^{-\alpha_m^{\rm op}}(r) \\ &= a_0\sigma^{-\alpha_m^{\rm op}}(r) + a_1[\sigma^{\alpha_1}(\sigma^{-\alpha_m^{\rm op}}(r))X_1 + p_{\alpha_1,\sigma^{-\alpha_m^{\rm op}}(r)}] \\ &\quad + \cdots + a_{m-1}[\sigma^{\alpha_{m-1}}(\sigma^{-\alpha_m^{\rm op}}(r))X_{m-1} + p_{\alpha_{m-1},\sigma^{-\alpha_m^{\rm op}}(r)}] \\ &\quad + a[\sigma^{\alpha_m}(\sigma^{-\alpha_m^{\rm op}}(r))X_m + p_{\alpha_m,\sigma^{-\alpha_m^{\rm op}}(r)}], \end{aligned}$$

that is,

$$\begin{aligned} f\sigma^{-\alpha_m^{\rm op}}(r) &= a_0\sigma^{-\alpha_m^{\rm op}}(r) + a_1\sigma^{\alpha_1}(\sigma^{-\alpha_m^{\rm op}}(r))X_1 + a_1p_{\alpha_1,\sigma^{-\alpha_m^{\rm op}}(r)} \\ &\quad + \cdots + a_{m-1}\sigma^{\alpha_{m-1}}(\sigma^{-\alpha_m^{\rm op}}(r))X_{m-1} + a_{m-1}p_{\alpha_{m-1},\sigma^{-\alpha_m^{\rm op}}(r)} \\ &\quad + arX_m + ap_{\alpha_m,\sigma^{-\alpha_m^{\rm op}}(r)}, \end{aligned}$$

where $p_{\alpha_j,\sigma^{-\alpha_m^{\rm op}}(r)} = 0$, or $\deg(p_{\alpha_j,\sigma^{-\alpha_m^{\rm op}}(r)}) < |\alpha_j|$ if $p_{\alpha_j,\sigma^{-\alpha_m^{\rm op}}(r)} \neq 0$, for $1 \leq j \leq m$ (Proposition 2.7 (i)). Since $\operatorname{lc}(f\sigma^{-\alpha_m^{\rm op}}(r)) = ar$, I' is a two-sided ideal of R. By assumption, there exists an idempotent $e \in R$ with $r_R(I') = eR$. With this in mind, let us show that $eA \subseteq r_A(I)$. Let $f \in I$ given by the expression above. By Propositions 2.3 and 3.11, we have

$$\begin{aligned} fe &= (a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1} + aX_m)e \\ &= a_0e + a_1X_1e + \cdots + a_{m-1}X_{m-1}e + aX_me \\ &= a_0e + a_1eX_1 + \cdots + a_{m-1}eX_{m-1} + aeX_m. \end{aligned}$$

Since $r_R(I') = eR$, we obtain $ae = 0$. Note that $fe = a_0e + a_1eX_1 + \cdots + a_{m-1}eX_{m-1}$ is an element of I with $\operatorname{lc}(fe) = a_{m-1}e$, i.e., $a_{m-1}e \in I'$, and using the equality $r_R(I') = eR$, we have $a_{m-1}e = 0$, which means that $fe = a_0e + a_1X_1e + \cdots + a_{m-2}eX_{m-2}$. Again, since fe is an element of I where $\operatorname{lc}(fe) = a_{m-2}e \in I'$, then $a_{m-2}e = 0$. Continuing in this way, we can see that $fe = 0$, so $eA \subseteq r_A(I)$.

Next, let us show that $r_A(I) \subseteq eA$. Let $f \in I$ given by the expression above, and $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in r_A(I)$, whence $fg = 0$. Let us see that $g = eg$. Set $h := g - eg$. Then $fh = 0$, $h = b_0 - eb_0 + (b_1 - eb_1)Y_1 + \cdots + (b_t - eb_t)Y_t$, and

$$\begin{aligned}
fh &= (a_0 + a_1X_1 + \cdots + aX_m)(b_0 - eb_0 + (b_1 - eb_1)Y_1 + \cdots + (b_t - eb_t)Y_t)\\
&= a_0(b_0 - eb_0) + a_0(b_1 - eb_1)Y_1 + \cdots + a_0(b_t - eb_t)Y_t + a_1X_1b_0\\
&+a_1X_1(b_1 - eb_1)Y_1\\
&+ \cdots + a_1X_1(b_t - eb_t)Y_t + \cdots + aX_m(b_0 - eb_0)\\
&+ aX_m(b_1 - eb_1)Y_1 + \cdots + aX_m(b_t - eb_t)Y_t\\
&= a_0(b_0 - eb_0) + a_0(b_1 - eb_1)Y_1 + \cdots + a_0(b_t - eb_t)Y_t + a_1[\sigma^{\alpha_1}(b_0)X_1 + p_{\alpha_1,b_0}]\\
&+ a_1[\sigma^{\alpha_1}(b_1 - eb_1)X_1 + p_{\alpha_1,b_1-eb_1}]Y_1 + \cdots + a_1[\sigma^{\alpha_1}(b_t - eb_t)X_1 + p_{\alpha_1,b_t-eb_t}]Y_t\\
&+ \cdots + a[\sigma^{\alpha_m}(b_0 - eb_0)X_m + p_{\alpha_m,b_0-eb_0}] + a[\sigma^{\alpha_m}(b_1 - eb_1)X_m + p_{\alpha_m,b_1-eb_1}]Y_1\\
&+ a[\sigma^{\alpha_m}(b_t - eb_t)X_m + p_{\alpha_m,b_t-eb_t}]Y_t,
\end{aligned}$$

that is,

$$\begin{aligned}
fh &= a_0(b_0 - eb_0) + a_0(b_1 - eb_1)Y_1 + \cdots + a_0(b_t - eb_t)Y_t + a_1\sigma^{\alpha_1}(b_0)X_1 + a_1p_{\alpha_1,b_0}\\
&+ a_1\sigma^{\alpha_1}(b_1 - eb_1)[c_{\alpha_1,\beta_1}x^{\alpha_1+\beta_1} + p_{\alpha_1,\beta_1}] + a_1p_{\alpha_1,b_1-eb_1}Y_1\\
&+ \cdots + a_1\sigma^{\alpha_1}(b_t - eb_t)[c_{\alpha_1,\beta_t}x^{\alpha_1+\beta_t} + p_{\alpha_1,\beta_t}] + a_1p_{\alpha_1,b_t-eb_t}Y_t\\
&+ \cdots + a\sigma^{\alpha_m}(b_0 - eb_0)X_m + ap_{\alpha_m,b_0-eb_0}\\
&+ a\sigma^{\alpha_m}(b_1 - eb_1)[c_{\alpha_m,\beta_1}x^{\alpha_m+\beta_1} + p_{\alpha_m,\beta_1}]\\
&+ ap_{\alpha_m,b_1-eb_1}Y_1 + a\sigma^{\alpha_m}(b_t - eb_t)[c_{\alpha_m,\beta_t}x^{\alpha_m+\beta_t} + p_{\alpha_m+\beta_t}] + ap_{\alpha_m,b_t-eb_t}Y_t,
\end{aligned}$$

whence $\operatorname{lc}(fe) = a\sigma^{\alpha_m}(b_t - eb_t)c_{\alpha_m,\beta_t} = 0$, i.e., $a\sigma^{\alpha_m}(b_t - eb_t) = 0$, and so $\sigma^{\alpha_m}(b_t - eb_t) \in r_R(I') = eR$. Hence, $\sigma^{\alpha_m}(b_t - eb_t) = e\sigma^{\alpha_m}(b_t - eb_t)$, that is, $b_t - eb_t = e(b_t - eb_t)$, and so $b_t - eb_t = 0$. Using a similar reasoning we can see that $b_j - eb_j = 0$, for $0 \leq j \leq t$, which shows that $h = 0$, that is, $g = eg$. This proves that $r_A(I) \subseteq eA$, so A is a quasi-Baer ring. □

Finally, our Theorem 4.4 generalizes [27], Theorem 3.13.

Theorem 4.4. *If A is a bijective skew PBW extension of an (Σ, Δ)-Armendariz ring R, then R is a p.q.-Baer ring if and only if A is a p.q.-Baer ring.*

Proof. First, suppose that A is a p.q-Baer ring. We want to see that R is p.q-Baer. Let sR be a principal right ideal in R. Since A is p.q-Baer, there exists an idempotent $e \in A$ such that $r_A(sA) = eA$, where $e = e_0 + e_1X_{\alpha_1} + \cdots + e_mX_{\alpha_m}$, but since e is an idempotent element, by Proposition 3.12 we have $e = e_0$, so $e_0 \in r_R(sR)$, that is, $eR \subseteq r_R(sR)$. Now, let us see that $r_R(sR) \subseteq e_0R$. Let $b \in r_R(sR)$. Since $b \in A$, we have $b = e_0g$ with $g = c_0 + c_1X_{\beta_1} + \cdots + c_tX_{\beta_t}$, but necessarily $g \in R$ so $b = ec_0$, i.e., $b \in e_0R$, which shows that $r_R(sR) \subseteq e_0R$.

Conversely, suppose that R is p.q-Baer, and let us see that A is p.q-Baer also. Let gA be a principal right ideal in A, with $g = a_0 + a_1X_{\alpha_1} + \cdots + a_mX_{\alpha_m}$, and $\exp(X_{\alpha_m}) = \alpha_m = (\alpha_{m1}, \alpha_{m2} ..., \alpha_{mn})$. We want to show that $r_A(gA) = eA$ with e an idempotent element in A. Consider $J = \{0\} \cup \{\operatorname{lc}(h) \mid h \in gA\}$. We claim that $J = a_mR$. If $a_nb \in a_nR$, since A is a bijective skew PBW extension of R, we can consider the element $\sigma^{-\alpha^{op}}(b)$ where $-\alpha^{op} = (-\alpha_{mn}, ..., -\alpha_{m1})$.

Then $\mathrm{lc}(g\sigma^{-\alpha^{op}}(b)) = a_n b$, so we obtain that $a_n b \in J$. Now it is easy to see that $J \subseteq a_m R$ because the leading monomial of a product is in fact the product of the leading polynomials. Hence, we obtain that J is a principal ideal of R generated by a_m, so by hypothesis we have $r_R(J) = eR$, for some idempotent e of R, it is enough to see that $r_A(gA) = eA$. Let us prove $eA \subseteq r_A(gA)$, that is, $(gp)e = 0$, for every $gp \in gA$, but we see that $pe = ep$, for every polynomial $p \in A$ (Proposition 3.11), so we have to see that $ge = 0$. With this in mind, consider $ge = (a_0 + a_1X_{\alpha_1} + \cdots + a_mX_{\alpha_m})e = a_0e + a_1X_{\alpha_1}e + \cdots + a_mX_{\alpha_m}e = a_0e + a_1eX_{\alpha_1} + \cdots + a_{m-1}eX_{\alpha_{m-1}} + a_meX_{\alpha_m}$ but $a_me = 0$ so $ge = a_0e + a_1eX_{\alpha_1} + \cdots + a_{m-1}eX_{\alpha_{m-1}}$, and then $a_{m-1}e$ is the leading coefficient of ge so $a_{m-1}e \in J$, so we obtain $ge = a_0e + a_1eX_{\alpha_1} + \cdots + a_{m-1}eX_{\alpha_{m-1}} = a_0e + a_1eX_{\alpha_1} + \cdots + a_{m-1}eeX_{\alpha_{m-1}} = a_0e + a_1eX_{\alpha_1} + \cdots + a_{m-2}eX_{\alpha_{m-2}}$. Again, since $a_{m-2}e \in J$, then $a_{m-2} = 0$. Continuing in this way we conclude that $ge = 0$, so $eA = r_A(gA)$. Finally, let us show that $r_A(gA) \subseteq eA$. Let $h \in r_A(gA)$, $h = b_0 + b_1Y_1 + ... + b_tY_t$, where $gf = 0$. Taking $z := h - eh$, we obtain $gz = 0$, and using the same argument used in Theorem 4.3 we conclude that $h = eh$. This proves that $r_A(gA) \subseteq eA$, so A is a principally quasi Baer ring. □

Remark 4.5. (i) ([9], Example 2.8). Let $B = \Bbbk[t]$ be the polynomial ring over a field $\Bbbk$ and σ be the endomorphism given by $\sigma(f(t)) = f(0)$. Then B is quasi-Baer but the ring $B[x;\sigma]$ is not a quasi-Baer ring. This example shows that the assumption on R (injective endomorphisms) is not a superfluous condition in Theorem 4.3. Another examples which show the importance of rigidness of R can be found in [12], Examples 9 and 10 (1).

(ii) ([4], Example 1.6). There is a ring B and a derivation δ of B such that $B[x;\delta]$ is a Baer ring but B is not quasi-Baer. Let $B = \mathbb{Z}_2[t]/\langle t^2\rangle$, with the derivation δ such that $\delta(\bar{t}) = 1$ where $\bar{t} = t + \langle t^2\rangle$ in B, and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Consider the Ore extension $B[x;\delta]$. If we set $e_{11} = \bar{t}x$, $e_{12} = \bar{t}$, $e_{21} = \bar{t}x^2 + x$, and $e_{22} = 1 + \bar{t}x$ in $B[x;\delta]$, then they form a system of matrix units in $B[x;\delta]$. Now the centralizer of these matrix units in $B[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $B[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So the ring $B[x;\delta]$ is a Baer ring, but B is not quasi-Baer.

(iii) Since prime rings are quasi-Baer, if A is a bijective skew PBW extension of a prime ring R, then A is prime ([18], Corollary 4.2, or [25], Proposition 3.3) and hence quasi-Baer.

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