gw-prime submodules

Submódulos gw-primos

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Abstract. In this work, gw-prime submodules of a module over a commutative ring with identity are defined. This class of submodules is a generalization of weakly prime submodules. After examining general properties of gw-prime submodules, their relation with valuation modules are investigated.

Keywords: gw-modules, gw-submodules, weakly prime submodules, valuation modules, valuation rings.

Resumen. En este artículo, definimos los submódulos gw-primos de un módulo sobre un anillo conmutativo con identidad. Esta clase de submódulos es una generalización de los submódulos débilmente primos. Después de examinar las propiedades generales de los submódulos gw-primos, buscamos su relación con los módulos de valuación.

Palabras claves: gw-módulos, gw-submódulos, submódulos débilmente primos, módulos de valuación, anillos de valuación.

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1. Introduction

Throughout this paper all rings are assumed to be commutative with identity. Let M be an R-module. A submodule P of M is said to be prime if whenever $am \in P$ for $a \in R$ and $m \in M$, either $m \in P$ or $aM \subseteq P$. In recent years, several generalizations of prime submodules are obtained, see [1, 6, 7, 11, 13]. Among those generalizations, the concept of weakly prime submodules distinguishes itself with its useful properties. Weakly prime submodules are first introduced by Behboodi and Koohy in [6] and since then studied extensively by many authors as [2, 3, 5]. A submodule N of an R-module M is called a weakly prime submodule if for every subset K of M and $x, y \in R$ the inclusion

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$xyK \subseteq N$ implies $xK \subseteq N$ or $yK \subseteq N$. For a generalization of weakly prime submodules the reader is referred to [12], in which weakly prime submodules are mentioned as classical prime submodules.

In this paper, we define gw-prime submodules as a generalization of weakly prime submodules. An R-module M is said to be a gw-prime module if whenever $abK = 0$ for $a, b \in R$ and K a submodule of M, either $a^2K = 0$ or $b^2K = 0$ holds. A submodule N of M is a gw-prime submodule if M/N is a gw-prime R-module. This definition can be interpreted as such: A submodule N is a gw-prime submodule if and only if for each $a, b \in R$ and each submodule K of M, the inclusion $abK \subseteq N$ implies $a^2K \subseteq N$ or $b^2K \subseteq N$.

In Section 2, the concepts of gw-prime module and submodule are introduced and their general properties are investigated. In Corollary 2.3, we give a further equivalent condition for being a gw-prime submodule. Namely, a submodule N of M is a gw-prime submodule if and only if for every $m \in M$ and $a, b \in R$ the inclusion $abm \in N$ implies $a^2m \in N$ or $b^2m \in N$. Moreover, the behaviour of gw-prime submodules under localization and taking direct products are studied.

In Section 3, the relation between gw-prime submodules and valuation modules is examined. An R-module M is said to be torsion free if for each $0 \neq m \in M$ and $r \in R$, the equation $rm = 0$ implies r to be a zero-divisor. Let R be an integral domain with quotient field K and M a torsion-free R-module. For $y = \frac{r}{s} \in K$ and $x \in M$, write $yx \in M$ if there exists $m \in M$ such that $rx = sm$. The module M is called a valuation module in [10], if for each $y \in K$, either $yM \subseteq M$ or $y^{-1}M \subseteq M$ holds. In Theorem 3.2, a sufficient condition is given for a torsion-free module over an integral domain to be a valuation module. Namely, if every principal submodule of a torsion-free module over an integral domain is gw-prime, then the module is a valuation module. The question when a converse is possible is asked and answered in affirmative in the cases of torsion-free uniserial modules and multiplication modules. Following [10], an R-module M is called uniserial if its submodules are totally ordered by inclusion. In Proposition 3.4, it is shown that every submodule of a torsion-free uniserial module is gw-prime. An R-module M is called a multiplication module if each submodule N of M is of the form IM for some ideal I of R, see [8]. In Corollary 3.6, valuation multiplication modules are characterized in terms of gw-prime submodules.

In Section 4, gw-prime submodules of the R-module R are investigated and a characterization of valuation rings is acquired.

2. gw-prime submodules

In this section we define gw-prime modules and submodules, and examine some of their general properties.

Definition 2.1. An R-module M is called a gw-prime module if whenever $abK = 0$ for $a, b \in R$ and K a submodule of M, either $a^2K = 0$ or $b^2K = 0$

holds. A submodule N of M is called a gw-prime submodule if M/N is a gw-prime R-module.

Proposition 2.2. *Let M be an R-module. M is a gw-prime module if and only if for each $m \in M$ and $a, b \in R$ the equation $abm = 0$ implies $a^2m = 0$ or $b^2m = 0$.*

Proof. The necessity part is clear. For the sufficiency part, let K be a submodule of M and $a, b \in R$ satisfying $abK = 0$. Then for each $k \in K$, we have $abk = 0$. By assumption, for each $k \in K$ we have either $a^2k = 0$ or $b^2k = 0$. Set

$$A = \bigcup_{\substack{k \in K \\ a^2k=0}} \{k\} \qquad \text{and} \qquad B = \bigcup_{\substack{k \in K \\ b^2k=0}} \{k\}.$$

Observe that A and B are submodules of M and $K = A \cup B$. Then either $A \subseteq B$ or $B \subseteq A$ must hold. Without loss of generality, assume $A \subseteq B$. Then $K = B$, and so, for each $k \in K$ the equality $b^2k = 0$ holds. Thus we obtain $b^2K = 0$. $\square$

Observe that a submodule N of M is a gw-prime submodule if and only if for each $a, b \in R$ and each submodule K of M, the inclusion $abK \subseteq N$ implies $a^2K \subseteq N$ or $b^2K \subseteq N$. As a corollary to the preceding proposition we equipped with one more equivalent condition for being a gw-prime submodule.

Corollary 2.3. *Let M be an R module and N a submodule of M. Then N is a gw-prime submodule if and only if for each $m \in M$ and each $a, b \in R$, $abm \in N$ implies $a^2m \in N$ or $b^2m \in N$.*

Throughout the text, the preceding statement and the formal definition of gw-prime submodule will be used interchangeably.

Following [6], a submodule N of an R-module M is called a weakly prime submodule if for every subset K of M and $x, y \in R$ the inclusion $xyK \subseteq N$ implies $xK \subseteq N$ or $yK \subseteq N$. We note that every weakly prime submodule is a gw-prime submodule.

Now, we give some general properties of gw-prime submodules.

Proposition 2.4. *Let N be a submodule of an R-module M.*

i. *If N is a gw-prime submodule, then $\operatorname{rad}((N : M))$ is a prime ideal.*

ii. *If $\phi : K \to M$ is an R-module homomorphism and N is a gw-prime submodule, then $\phi^{-1}(N)$ is a gw-prime submodule of K.*

iii. *If $\phi : M \to K$ is an R-module epimorphism and N is a gw-prime submodule of M containing $Kerf$, then $\phi(N)$ is a gw-prime submodule of K.*

iv. *If N is a gw-prime submodule and K is a submodule of M contained in N, then N/K is a gw-prime submodule of M/K.*

v. If K' is a gw-prime submodule of M/N then $K' = K/N$ for some gw-prime submodule K of M.

Proof. i. Assume that N is a gw-prime submodule. Let $x, y \in R$ satsifying $xy \in rad((N : M))$. Then we have $x^n y^n M \subseteq N$ for some $n \in \mathbb{N}$. Since N is gw-prime, either $x^{2n} M \subseteq N$ or $y^{2n} M \subseteq N$. That means $x \in rad((N : M))$ or $y \in rad((N : M))$. Hence $rad((N : M))$ is a prime ideal.

ii. Let $x, y \in R$ and $k \in K$ such that $xyk \in \phi^{-1}(N)$. Then $\phi(xyk) \in N$. Since ϕ is an R-module homomorphism, we have $xy\phi(k) \in N$. Since N is gw-prime we obtain $x^2\phi(k) \in N$ or $y^2\phi(k) \in N$. This implies $x^2 k \in \phi^{-1}(N)$ or $y^2 k \in \phi^{-1}(N)$. Hence $\phi^{-1}(N)$ is a gw-prime submodule.

iii. Let $x, y \in R$ and k be an element of K such that $xyk \in \phi(N)$. Then, there exists $n \in N$ satsifying $\phi(n) = xyk$. Since ϕ is an epimorphism, there exists $m \in M$ such that $\phi(m) = k$. Then, we have $\phi(xym) = \phi(n)$. This implies $n - xym \in \ker f \subseteq N$. Thus, we obtain $xym \in N$. Since N is a gw-prime submodule, either $x^2 m \in N$ or $y^2 m \in N$. Therefore, we get $x^2 k = x^2\phi(m) = \phi(x^2 m) \in \phi(N)$ or $y^2 k \in \phi(N)$.

iv. and v. follow from part (ii) and (iii) with the natural epimorphism $\phi : M \to M/N$. □

Proposition 2.5. *Let M be an R-module and N a submodule of M. Let S be a multiplicatively closed subset of R. If N is a gw-prime submodule, then $S^{-1}N$ is a gw-prime submodule of $S^{-1}M$.*

Proof. Suppose that N is a gw-prime submodule and S a multiplicatively closed subset of R. Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ and $\frac{m}{u} \in S^{-1}M$ satisfying $\frac{a}{s}\frac{b}{t}\frac{m}{u} \in S^{-1}N$. Then there is an $n \in N$ and $y, z \in S$ such that $yzabm - ystun = 0$. This implies $yzabm \in N$. Since N is a gw-prime submodule either $y^2 z^2 a^2 m \in N$ or $b^2 m \in N$. In the former case we obtain

$$\left(\frac{a}{s}\right)^2 \frac{m}{u} = \frac{y^2 z^2 a^2 m}{y^2 z^2 s^2 u} \in S^{-1}N,$$

and in the latter case we get

$$\left(\frac{b}{t}\right)^2 \frac{m}{u} = \frac{b^2 m}{t^2 u} \in S^{-1}N.$$

Thus, we conclude that $S^{-1}N$ is a gw-prime submodule of $S^{-1}M$. □

Recall that a submodule N of an R-module M is said to be primary if whenever $rm \in N$ either $m \in N$ or $r^n \in (N : M)$. In this case $P = (N : M)$ is a prime ideal of R and N is called P-primary. The previous proposition has a converse for primary submodules.

Corollary 2.6. *Let M be an R-module and N a P-primary submodule of M for a prime ideal P of R. Then N is a gw-prime submodule if and only if NM_P is a gw-prime submodule of M_P.*

Proof. Necessity part follows from Proposition 2.5. For the sufficiency, assume that NM_P is a gw-prime submodule of M_P. Since N is P-primary we have $NM_P \cap M = N$. Then, applying Proposition 2.4(ii) with the homomorphism $\phi : M \to M_P$, we get N gw-prime. $\square$

In the following results we investigate gw-prime property for direct products of modules.

Proposition 2.7. *Let M_1 and M_2 be R-modules and N_1 and N_2 proper submodules of M_1 and M_2, respectively.*

i. $N = N_1 \times M_2$ is a gw-prime submodule of $M = M_1 \times M_2$ if and only if N_1 is a gw-prime submodule of M_1.

ii. If $N = N_1 \times N_2$ is a gw-prime submodule of $M = M_1 \times M_2$ then N_1 is a gw-prime submodule of M_1 and N_2 is a gw-prime submodule of M_2.

Proof. (i) and (ii): Apply Proposition 2.4(ii), (iii) with the natural projection homomorphisms $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$.

$\square$

Proposition 2.8. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where M_1 is an R_1-module and M_2 is an R_2-module. Let N_1 and N_2 be proper submodules of M_1 and M_2, respectively.*

i. $N = N_1 \times M_2$ is a gw-prime submodule of M if and only if N_1 is a gw-prime submodule of M_1.

ii. If $N = N_1 \times N_2$ is a gw-prime submodule of M then N_1 is a gw-prime submodule of M_1 and N_2 is a gw-prime submodule of M_2.

Proof. (i) Assume that N is a gw-prime submodule of M. Let $a, b \in R_1$ and $m \in M_1$ such that $abm \in N_1$. Then $(a,1)(b,1)(m,0) = (abm,0) \in N$ and hence $(a^2m,0) \in N$ or $(b^2m,0) \in N$. This implies $a^2m \in N_1$ or $b^2m \in N_1$. Therefore N_1 is a gw-prime submodule of M_1. Conversely assume that N_1 is a gw-prime submodule of M_1. Let $(a_1,a_2),(b_1,b_2) \in R$ and $(n,m) \in M$ satisfying $(a_1,a_2)(b_1,b_2)(n,m) \in N$. Then $a_1b_1n \in N_1$. Since N_1 is gw-prime, $a_1^2n \in N_1$ or $b_1^2n \in N_1$. This implies $(a_1,a_2)^2(n,m) \in N$ or $(b_1,b_2)^2(n,m) \in N$. Thus N is a gw-prime submodule of M.

(ii) Assume that N is a gw-prime submodule of M. Let $a, b \in R_1$ and $m \in M_1$ such that $abm \in N_1$. Then $(abm,0) = (a,1)(b,1)(m,0) \in N$. Since N is gw-prime $(a^2m,0) = (a,1)^2(m,0) \in N$ or $(b^2m,0) = (b,1)^2(m,0) \in N$. Therefore $a^2m \in N_1$ or $b^2m \in N_1$. The assertion for N_2 follows similarly. $\square$

3. gw-Prime Submodules and Valuation Modules

In this section, we investigate the relation between gw-prime submodules and valuation modules.

Recall that a submodule N of am module M is said to be principal if $N = Rm$ for some $m \in M$. The following proposition shows that to decide whether all submodules of a given module are gw-prime or not, it is enough to examine only the principal submodules.

Proposition 3.1. *Let M be an R-module. Then every principal submodule of M is gw-prime if and only if every submodule of M is gw-prime.*

Proof. Assume that every principal submodule of M is gw-prime. Let N be a submodule of M and $xym \in N$ for $x, y \in R$ and $m \in M$. Then $xym = n$ for some $n \in N$, and hence $xym \in Rn$. Since Rn is gw-prime, either $x^2m \in Rn \subseteq N$ or $y^2m \in Rn \subseteq N$. Thus, N is a gw-prime submodule. The other part is clear. □

In [10], valuation modules over integral domains are introduced. Let R be an integral domain with quotient field K and M a torsion-free R-module. For $y = \frac{r}{s} \in K$ and $x \in M$, write $yx \in M$ if there exists $m \in M$ such that $rx = sm$. Then M is a valuation R-module if for each $y \in K$, one of the inclusions $yM \subseteq M$ and $y^{-1}M \subseteq M$ holds.

Theorem 3.2. *Let R be an integral domain and M a torsion-free R-module. If every principal submodule of M is gw-prime, then M is a valuation module.*

Proof. Let K be the quotient field of R and $y = \frac{a}{b} \in K$. For $m \in M$, assume that $ym \notin M$. Then for all $n \in M$, we have $am \neq bn$. The set $L = Rabm$ is a principal, hence gw-prime submodule. Then $Ra^2m \subseteq L$ or $Rb^2m \subseteq L$. If $Ra^2m \subseteq L$, then there exists $c \in R$ such that $a^2m = cabm$. Since M is torsion-free and R is an integral domain, we get $am = cbm$, a contradiction. Therefore, the inclusion $Rb^2m \subseteq L$ must hold. Then, we have $b = ra$ for some $r \in R$. Thus, $y^{-1} = \frac{b}{a} = r$ and hence $y^{-1}M = rM \subseteq M$. □

The converse of Theorem 3.2 is not true in general as can be seen in the following example:

Example 3.3. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{(p)}$, where p is a prime number and $Z_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p \text{ does not divide } b\}$. For $y = \frac{r}{q} \in \mathbb{Q}$ with $(r, q) = 1$ and $m = \frac{a}{b} \in M$, if $ym \notin M$ then p divides q. Since $(r, q) = 1$ we have p not divide r. Hence $y^{-1}M \subseteq M$. Therefore, the $\mathbb{Z}$-module M is a valuation module. Observe that $\left(\frac{6}{1}\right)$ is a submodule of M, however, it is not gw-prime since $2.3\left(\frac{1}{1}\right) \subseteq \left(\frac{6}{1}\right)$ and neither $4\left(\frac{1}{1}\right) \subseteq \left(\frac{6}{1}\right)$ nor $9\left(\frac{1}{1}\right) \subseteq \left(\frac{6}{1}\right)$. Hence $\left(\frac{6}{1}\right)$ is not a gw-prime submodule.

Now, we are to investigate when we could obtain a converse for Theorem 3.2. The following propositions shows that if the set of submodules of a module is totally ordered by inclusion, then every submodule is gw-prime. Following [9], an R-module M is called uniserial if its submodules are totally ordered by inclusion. Note that a torsion-free uniserial module is a valuation module, see [10].

Proposition 3.4. *If M is a torsion-free uniserial R-module then every submodule of M is gw-prime.*

Proof. Let L and N be submodules of M and $x, y \in R$ such that $xyL \subseteq N$. Assume $x^2L \nsubseteq N$. Since M is uniserial we have $N \subseteq x^2L$. Then $xyL \subseteq x^2L$. Since M is torsion-free we have $yL \subseteq xL$. Thus $y^2L \subseteq xyL \subseteq N$. □

An R-module M is called a multiplication module if each submodule N of M is of the form IM for some ideal I of R, see [8]. It is proved in [10, 2.7] that the set of submodules of a valuation multiplication module are totally ordered by inclusion. Thus as a corollary to the preceding proposition, we observe in the following results that a converse for Theorem 3.2 is possible in the case of multiplication modules.

Corollary 3.5. *If M is a valuation multiplication R-module, then every submodule of M is gw-prime.*

As a result, we have the following corollary characterizing valuation multiplication modules over an integral domain.

Corollary 3.6. *Let M be a torsion-free multiplication module over an integral domain R. The following are equivalent:*

i. M is a valuation module.

ii. Every submodule of M is gw-prime.

iii. Every principal submodule of M is gw-prime.

The following proposition shows that a principal submodule which is a valuation module in its own right must be a *gw*-prime submodule.

Proposition 3.7. *Let R be an integral domain and M a torsion-free R-module. Let $m \in M$. If Rm is a valuation R-module, then it is a gw-prime submodule of M.*

Proof. Let $m \in M$ and assume that Rm is a valuation R-module. For $a, b \in R$ and $k \in M$, suppose that $abk \in Rm$. Then $abk = sm$ for some $s \in R$. Set $y = \frac{a}{b} \in K$ where K is the quotient field of R. Then $ym \in Rm$ or $y^{-1}Rm \subseteq Rm$. The latter inclusion implies that $y^{-1}m \in Rm$. Therefore, we obtain $a^2k = yabk = ysm \in Rm$ or $b^2k = y^{-1}abk = y^{-1}sm \in Rm$. Hence Rm is a *gw*-prime submodule of M. □

The following corollary is a consequence of Theorem 3.2 and Proposition 3.7.

Corollary 3.8. *Let R be an integral domain and M a torsion-free R-module. If every principal submodule of M is a valuation R-module, then M is a valuation module.*

4. Application of gw-Prime Structure to Rings

In this section we define gw-prime ideal in a ring R which corresponds to gw-prime submodule in the R-module R.

Definition 4.1. An ideal I of R is called a gw-prime ideal if for any $a, b, c \in R$, whenever $abc \in I$ either $a^2c \in I$ or $b^2c \in I$.

We note that ring theoretic analogues of the above results are valid for gw-prime ideals. For example, gw-prime ideals have prime radicals, and gw-prime property is preserved under ring homomorphisms, preimages of ring homomorphisms, localizations and direct sums. In addition, interesting results arise in the case of valuation domains. As a corollary to Proposition 3.1, Theorem 3.2 and Proposition 3.4, we have the following characterization of valuation rings.

Theorem 4.2. *Let R be an integral domain. The following are equivalent:*

i. R is a valuation ring.

ii. Every principal ideal of R is gw-prime.

iii. Every ideal of R is gw-prime.

In [4], a similar characterization of valuation rings is obtained by using 2-prime ideals. An ideal I of a ring R is said to be a 2-prime ideal if for $a, b \in R$ whenever $ab \in I$ either $a^2 \in I$ or $b^2 \in I$ holds. Observe that every gw-prime ideal is a 2-prime ideal. However the converse is not true in general as the following example shows:

Example 4.3. In [4, 4.8], it is shown that $I = (x^4, x^2y, xy^2, y^4)$ is a 2-prime ideal of the ring $R = k[x, y]$. However, it is not a gw-prime ideal since

$$xy(x + y) = x^2y + xy^2 \in I$$

but neither $x^2(x + y) = x^3 + x^2y$ nor $y^2(x + y) = xy^2 + y^3$ is in I.

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