A multiscale formulation for FEM and IgA

Formulación multiescala para el método de los elementos finitos con análisis isogeométrico

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Abstract. A numerical method is formulated based on Finite Elements, Isogeometric Analysis and a Multiscale technique. Isogeometric Analysis, which uses *B-Splines* and *NURBS* as basis functions, is applied to evaluate its performance. The analyzed PDE is Poisson's Equation. The method starts with a coarse mesh which is refined to obtain each scale, considering every current scale mesh's element as a subdomain to the following scale. Local problems of each subdomain are solved independently, and the system is executed iteratively. Isogeometric analysis shows to have a better performance regarding approximation error and convergence in the iterative method that was derived here, which favorably influences computational cost.

Keywords: multiscale, isogeometric analysis, finite elements, Poisson, *B-splines*, *NURBS*, numerical analysis, *FLOP*.

Resumen. En este artículo se formula un método numérico basado en elementos finitos, análisis isogeométrico y técnica multiescala. El análisis isogeométrico utilizando *B-splines* y *NURBS* como funciones base se aplica para evaluar su funcionamiento. La ecuación diferencial parcial analizada es la de Poisson. El método inicia con una malla burda la cual es refinada para obtener cada escala, considerando cada elemento de la malla como un subdominio de la escala posterior. Los problemas locales de cada subdominio se resuelven independientemente y el sistema de forma iterativa. El análisis isogeométrico muestra un buen comportamiento en lo que respecta a errores de aproximación y convergencia del método iterativo que fue derivado, lo cual favorece el costo computacional.

Palabras claves: multiescala, análisis isogeométrico, elementos finitos, Poisson, B-splines, NURBS, análisis numérico.

Mathematics Subject Classification: 65-XX.

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1. Introduction

As a part of the development of the finite element method (FEM), some years ago it was noticeable the convenience of getting a closer integration between this technique and computer aided design (CAD) [5]. As a result of such need Isogeometric Anlysis (IgA) appeared, which uses the mathematical principles of FEM and employs CAD functions to represent geometric entities, like *NURBS* [10]. This recent method has very attractive properties regarding continuity and refinement [5]. Regarding computational resources, several approaches have been proposed, such as the multiscale finite elements [6]. In this work, the formulation to be obtained is related to the theory behind the variational multiscale method, which splits the solution and its vectorial space in a coarse component and a fine component, and yields a solution from an interaction of both of them [7].

The Variational Multiscale Method (VMS), named by Hughes [7], succeeds in finding a theoretical reason to add a stabilizing term to the variational formulation of boundary value problems typically treated with numerical methods other than finite elements. This term has the purpose of stabilize the solution or capturing some features of the solution that cannot be represented with the shape functions used in the classic Galerkin scheme [7].

The cited author splits the exact solution of a boundary value problem into two parts: the resolvable scales and unresolvable scales. The former are the ones that may be captured by a finite dimensional space, as those employed by residual-based numerical methods (like FEM); the latter are those which, with the typical formulation of finite elements, cannot be captured and finally are not observed in the numerical solution. This notion of 'scales' is the one to be considered when working with VMS.

According to the character of the differential operator of the boundary value problem, that is, whether it consists of a diffusive or a convective operator, the unresolvable scales contribute not only the effect of the fine features to the numerical solution, but it may imply the stability of the computational technique used on the resolvable scales. As a proof of such circumstance the classical Galerkin scheme is considered and the fact that this method offers spurious or non-stable solutions in fluid mechanics problems and, in general, in any problem of transport equations, where a convective operator is always involved, which, according to analyses made by several authors, due to not having the property of being self-adjoint, the operator nature leads the Galerkin method not to capturre an important part of the solution [5].

Hughes later showed that the derivation of VMS is the theoretical basis necessary for the explanation of how some stabilized methods worked, like the SUPG (*Streamline Upwind Petrov-Galerkin*), which were previously developed to enhance the solution of fluid problems, but without actually having a full understanding on how they were able to stabilize the numerical output [3, 7].

The scale separation proposed and derived on those works, has been taken as the basis to propose scale separation in other contexts, such as the discontinuous

Galerkin method, or other related techniques, where there are resolvable and unresolvable scales for the so named numerical fluxes and traces. According to the reviewed publications, it is possible to formulate a method which first treats every scale independently, somehow taking into account the other scales' effect, and later sum them up to get a global approximation [4, 9].

In this paper, the starting point to state a new numerical method which combines finite elements and multiscale schemes is the definition of a specific boundary value problem of basic applied mechanics, just like some other numerical techniques are first analyzed [11]. After that, the used notation and important definitions are presented, and finally a two-dimensional problem is resolved as an example; the obtained results are analyzed and the technique performance is assessed.

2. Boundary value problem

The chosen boundary value problem to be analyzed is Poisson's Equation for a scalar field ϕ, which is stated in its strong form as follows:

Let Ω be an open domain in $\mathbb{R}^d$ (being d equals to 1, 2 or 3), whose boundary is denoted as Γ, see Figure 1). Find $\phi = \phi(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{aligned} \mathcal{L}\phi &= f \text{ in } \Omega \subset \mathbb{R}^d \\ \phi &= g \text{ on } \Gamma_g \subset \Gamma = \partial\Omega \\ -\partial_n\phi = -\nabla\phi \cdot \mathbf{n} &= h \text{ on } \Gamma_h \subset \Gamma = \partial\Omega. \end{aligned} \tag{1}$$

where differential operator $\mathcal{L} = -\kappa\Delta$ is dependent on the diffusivity field $\kappa = \kappa(\mathbf{x})$ and the laplaciann operator Δ, the portion of the boundary with a Dirichlet type boundary condition is Γ_g and the one with a Neumann type boundary condition is Γ_h, so that $\Gamma = \Gamma_g \cup \Gamma_h$, and $\mathbf{n}$ is the unit vector normal to Γ (outwards).

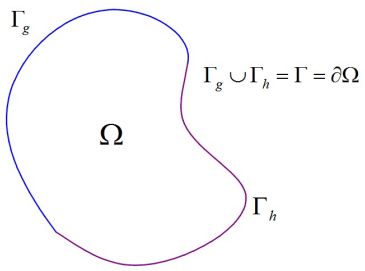

Figure 1: Domain of the boundary value problem

The corresponding weak form is stated as:

Find $\phi = \phi(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ such that

$$0 = (\nabla w, \kappa \nabla \phi)_\Omega - (w, f)_\Omega + (w, \hbar)_{\Gamma_h} - (w, \kappa \partial_n \phi)_{\Gamma_g} \tag{2}$$

where w is an arbitrary scalar function (test function), which weights the residual of the strong equation over the whole domain; bilinear form $(\cdot, \cdot)_\square$ represents the integral of the product of two functions over certain subset of $\mathbb{R}^d$ specified as $\square$. For instance, for two functions $a(\mathbf{x})$ and $b(\mathbf{x})$, the bilinear form is defined

$$(a, b)_\square = \int_\square ab d\square. \tag{3}$$

Usually the integration space $\square$ used in this kind of formulations, is the support of an element or a subdomain Ω (the length, the area or the volume, according to the number of space dimensions), or its boundary Γ (endpoints, or the delimiting line or surface).

2.1. Scale Separation

The exact solution ϕ belongs to a infinite dimensional vector space $\mathcal{V}$, that contains all the defined functions in the problem domain Ω. The space elements are the so-called trial functions [8]. The solution may be divided into a finite number of parts, which are going to be named scales, whose overall sum is equal to the full exact solution, which is the global solution. The definition above can be expressed thus:

$$\phi = \phi_0 + \phi_1 + \ldots + \phi_S, \tag{4}$$

wherein there are $S + 1$ scales. Similarly, each of these parts corresponds to a different subspace that spans just one part of the full space, that is, its vector basis is containing just some of the elements contained in the global solution space. Accordingly the space which includes the global solution is the result of the sum of all scale subspaces into which the global one has been divided.

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1 + \ldots + \mathcal{V}_S, \tag{5}$$

such that $\phi_s \in \mathcal{V}_s$ for $s = 0, 1, \ldots, S$.

In order to set the first term of (2) into a symmetric bilinear form (i.e. an inner product, whose two arguments are w and ϕ), it is necessary to define w as a function belonging to the same space as ϕ, that is, the solution space $\mathcal{V}$ [1, 12]. The definition of a in inner product is important so as to be able to get a symmetric system of equations after discretization and variational methods have been applied in certain fashion to be shown below.

Once this is done, it is possible to separate scales in test functions w too like it was done with ϕ.

$$w = w_0 + w_1 + \ldots + w_S. \tag{6}$$

2.2. Scale separation in the discrete problem

The approximate solution proposed in this paper, as well as in the analytical case, is assumed to have several scales, and thus there are finite dimensional spaces to which every solution scale belongs.

The concept of scale, as it was mentioned above, is specifically related to the mesh size with which the boundary value problem domain has been discretized. Scale 0 consists of the coarsest mesh, and as the scale number increases, so does the refinement level. The finest scale is the S-th one, yielding a total number of finite dimensional scales $S + 1$.

An important aspect of the herein described solution, is the decomposition of the current domain into subdomains at the next scale. This procedure means that every element at scale s is considered to be a subdomain for scale $s + 1$. Thus every subdomain may have its own inner mesh, independent of rest of the subdomains, but the solution on this portion of the global domain has to be coupled to the others' by means of certain numerical approach. The depiction of this method is shown in Figure 2, where additionally there is displayed the sum of all of the solution scales ($s = 0, 1, 2$) to find the global solution, and it is compared to the exact solution.

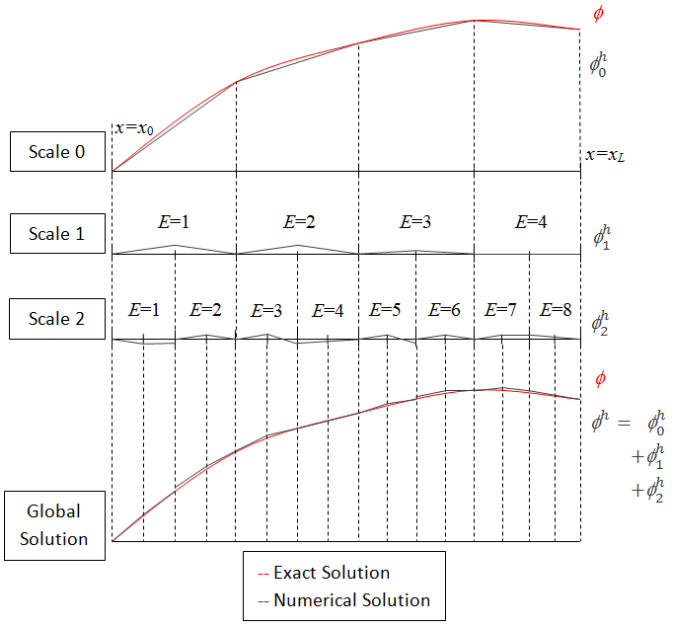

Figure 2: Global and scale solutions of a multiscale FEM one-dimensional problem

From the perspective of vector spaces, the discontinuity between any adjacent

subdomains at the fine scales lead to a numerical solution space $\mathcal{V}_s^h$ $(s > 0)$ which is not contained in the space of the exact solution $\mathcal{V}$. The numerical solution space of each scale is composed by multiple spaces whose support matches one subdomain. This means that such a space $\mathcal{V}_s^h$ is given by:

$$\mathcal{V}_s^h = V_s^{h,1} \oplus V_s^{h,2} \oplus \ldots \oplus V_s^{h,n^{el,s-1}}, \tag{7}$$

where n_{s-1}^{el} is the number of subdomains at scale s.

According to section 2.1 the space whose support comprise some subdomain, may comprise a finite dimensional subset of the first Sobolev space $(H^1(\Omega))$. Symbolically, the space associated to every subdomain E at scale s corresponds to:

$$\mathcal{V}_s^{h,E} \subset H^1(\Omega_{s-1}^E). \tag{8}$$

As a result of combining (7) and (8) it yields as follows:

$$\mathcal{V}_s^h \subset \bigoplus_{E=1}^{n_{s-1}^{el}} H^1(\Omega_{s-1}^E) \neq H^1(\Omega). \tag{9}$$

3. Definitions and notation

3.1. Indices

From this point on a notation convention is being held with the purpose of prevent ambiguities when handling the indices on the variables and symbols that denote some parameter of the problem. In all cases the subindex is representing the number of scale at which the concerning variable is defined, while the superindex corresponds to the rest of parameters that requires index specification (i.e. nodes, control points, elements, subdomains, etc.).

3.2. Main Notation

The definition of variables and the basic notation for the present formulation are next presented.

- Polynomial degree for the parametric coordinates ξ and η: p and q, respectively
- Number of elements at scale s: n_s^{el}
- A^{th} basis function at scale s: $N_s^A(\mathbf{x})$
- Nodal values of control variables: d_s^A
- Domain of element e at scale s: Ω_s^e (see Fig. (3.2 and Fig. (3.2)

- Set of element domains at scale s: $\tilde{\Omega}_s = \bigcup_{e=1}^{n_s^{el}} \Omega_s^e$
- Set of boundaries at scale s: $\Gamma_s = \bigcup_{e=1}^{n_s^{el}} \partial\Omega_s^e$
- Set of interfacial boundaries at scale s:
  - For $s = 0$: $\Gamma_0' = \Gamma_0 \backslash \Gamma$ (see Fig. (3.2))
  - For $s > 0$: $\Gamma_s' = \Gamma_j \backslash \Gamma \backslash \left(\bigcup_{r=0}^{s-1} \Gamma_r' \right)$ (see Fig. (3.2))

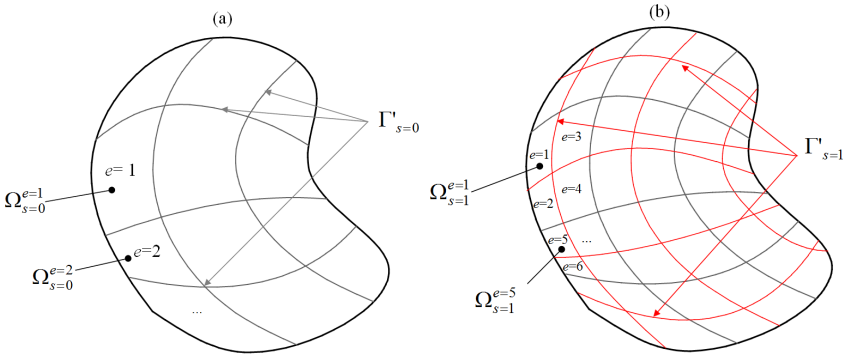

Figure 3: Scheme of mesh and its related notation: (a) coarse scale, (b) fine scales

- Set of interfacial boundaries in subdomain E at scale s: $\Gamma_s'^E = \left(\bigcup_{e=1}^{n_s^{el,E}} \partial\Omega_s^e \right) \cap \Gamma_s'$
- Local Neumann boundary of subdomain E at scale s: $\Gamma_{h,s}^E = \partial\Omega_s^E \cap \Gamma_h$
- Local Dirichlet boundary of subdomain E at scale s: $\Gamma_{g,s}^E = \partial\Omega_s^E \cap \Gamma_g$
- Local set of interfaces of a subdomain E at scale s: $\Gamma_s^{E,int} = (\Gamma_s' \backslash \Gamma) \cap \partial\Omega_s^E = \partial\Omega_s^E \backslash \left(\Gamma_{h,s}^E \cup \Gamma_{g,s}^E \right)$
- Global approximate solution:

$$\phi^h(\mathbf{x}) = \sum_{r=0}^{S} \phi_r^h(\mathbf{x}) \tag{10}$$

- Approximate solution of the coarse scale:

$$\phi_0^h(\mathbf{x}) = \sum_{A=1}^{n_0^{np}} N_0^A(\mathbf{x}) d_0^A \tag{11}$$

- Approximate solution of scale s ($s > 0$):

$$\phi_s^h(\mathbf{x}) = \sum_{E=1}^{n_{s-1}^{el}} \left(\sum_{A=1}^{n_s^{np,E}} N_0^A(\mathbf{x}) d_s^A \right)^E \tag{12}$$

  Here, E represents the support of each element in the immediately previous scale $(s-1)$, that is, according to the scale separation, each of the n_{s-1}^{el} existing subdomains at scale s. Every E has an associated number of elements ($n_s^{sd,E}$) and a number of nodes or control points $n_s^{np,E}$ (for $s = 0$ as there is one subdomain only, which covers the whole domain, the notation reduces to n_0^{np}), over which there are defined the corresponding basis functions ($N_s^A(\mathbf{x})$) and the nodal values or the control variables (d_s^A).

Assuming the existence of a scalar function u supported both on the analyzed subdomain and on the contiguous one, it may subject of two operators that are valid only on their interfaces:

- $\{u\}$ is the average of function u on a specified interface, that is,

$$\{u\} := \frac{1}{2}(u^+ + u^-). \tag{13}$$

- $[\![\partial_n u]\!]$ represents the jump of function derivative $\partial_n u$ across a specified interface, that is,

$$[\![\partial_n u]\!] := (\nabla u)^+ \cdot \mathbf{n}^+ + (\nabla u)^- \cdot \mathbf{n}^-. \tag{14}$$

- $[\![u]\!]$ represents the jump of function u across a specified interface, that is,

$$[\![u]\!] := u^+ - u^-. \tag{15}$$

3.3. Trace

The approximate trace of the function ($\hat{\phi}$) must be defined on the set of boundaries of each scale s ($0 \leq s \leq S$), and its formula can be derived as follows:

$$\hat{\phi}_s \ := \ \{\phi_{r\geq s}^h\} - \frac{h_s}{2\alpha}[\![\partial_n \phi_{r\geq s}^h]\!] \text{ en } \Gamma'_s, \tag{16}$$

where h_s is the mesh parameter at scale s. The expression $u_{r\geq s}$ and those with similar subindices state for a sum of the functions over the scales attaining the condition given there with respect to the current scale s; for instance:

$$u_{r\geq s} \ := \ \sum_{r\geq s} u_r \quad 0 \leq r, s \leq S. \tag{17}$$

4. Variational formulation

4.1. Global problem

Global problem refers to the statement of the variational problem of the full domain taking the coarsest basis as the trial functions and test functions. Before finding the definite formulation including the weak Dirichlet imposition terms (which come up from applying a technique similar to the one in [2]), it will be first shown, starting from the strong formulation and using an approach alike to Galerkin's, the origin of the different terms in the final expression after the scale separation.

$$0 = (w_o^h, \mathcal{L}\phi^h - f)_{\Omega_0} \tag{18}$$

$$0 = \left(w_o^h, \mathcal{L}\left(\sum_{r=0}^{S} \phi_r^h \right) - f \right)_{\Omega_0} \tag{19}$$

$$0 = \left(w_o^h, \sum_{r=0}^{S} \mathcal{L}\phi_r^h - f \right)_{\Omega_0} \tag{20}$$

$$\begin{aligned} 0 = {} & (w_o^h, \mathcal{L}\phi_0^h)_{\Omega_0} - (w_o^h, f)_{\Omega_0} \\ & + (w_o^h, \mathcal{L}\phi_{r>0}^h)_{\Omega_0} \end{aligned} \tag{21}$$

$$\begin{aligned} 0 = {} & (w_o^h, \mathcal{L}\phi_0^h)_{\Omega_0} - (w_o^h, f)_{\Omega_0} \\ & + (\mathcal{L}^* w_o^h, \phi_{r>0}^h)_{\Omega_0} \end{aligned} \tag{22}$$

$$\begin{aligned} 0 = {} & (\nabla w_o^h, \kappa \nabla \phi_0^h)_{\Omega_0} + (w_o^h, \hbar)_{\Gamma_h} - (w_o^h, f)_{\Omega_0} \\ & + (\mathcal{L}^* w_o^h, \phi_{r>0}^h)_{\Omega_0}. \end{aligned} \tag{23}$$

The first three terms are the ones that bring up from the typical scheme by Galerkin [8]. The last term represents the integral over the domain of the product of $\phi_{r>0}^h$ by adjoint operator $\mathcal{L}^*$ applied on the test function, as it was necessary to avoid the integration of $\mathcal{L}\phi_{r>0}^h$, due to the discontinuity of the fine scales across the internal interfaces. In the following equation, the mentioned term is being subjetct of the substitution of $\mathcal{L}^*$ by $\mathcal{L}$ thanks to the self-adjointness of the operator in Poisson's equation [7].

$$
\begin{aligned}
\Rightarrow 0 = {} & (\nabla w_0^h, \kappa \nabla \phi_0^h)_{\Omega_0} + (\mathcal{L} w_o^h, \phi_{r>0}^h)_{\Omega_0} - (w_0^h, f)_{\Omega_0} + (w_o^h, \hbar)_{\Gamma_h} \\
& - \left(\kappa \partial_n w_0^h, \phi^h - \mathcal{g}\right)_{\Gamma_g} \\
& - \left(\kappa w_0^h, \partial_n \phi^h - \frac{2\alpha}{h_0}(\phi^h - \mathcal{g})\right)_{\Gamma_g} \\
& - \left(\kappa [\![\partial_n w_0^h]\!], \phi_0^h - \left(\{\phi_{r\geq 0}^h\} - \frac{h_0}{2\alpha} [\![\partial_n \phi_{r\geq 0}^h]\!]\right)\right)_{\Gamma_0'} \qquad (24) \\
0 = {} & (\nabla w_0^h, \kappa \nabla \phi_0^h)_{\Omega_0} + (\mathcal{L} w_o^h, \phi_{r>0}^h)_{\Omega_0} - (w_0^h, f)_{\Omega_0} + (w_o^h, \hbar)_{\Gamma_h} \\
& - \left(\kappa \partial_n w_0^h, \phi_0^h - \mathcal{g}\right)_{\Gamma_g} \\
& - \left(\kappa w_0^h, \partial_n \phi^h - \frac{2\alpha}{h_0}(\phi^h - \mathcal{g})\right)_{\Gamma_g} + \left(\kappa [\![\partial_n w_0^h]\!], \{\phi_{r>0}^h\} - \frac{h_0}{2\alpha} [\![\partial_n \phi_{r\geq 0}^h]\!]\right)_{\Gamma_0'}. \qquad (25)
\end{aligned}
$$

4.2. Local problem

Local problem refers to the variational problem that is to be solved on every subdomain at every fine scale. In order to derive such problem, the chosen test functions are the ones belonging to the current scale space, that is, w_s^h for the scale s. Afterwards, the scale separation is used with the purpose of having bilinear forms that handle with w_s^h and ϕ_s^h.

$$
\begin{aligned}
\Rightarrow 0 = {} & (\nabla w_s^h, \kappa \nabla \phi_s^h)_{\Omega_{s-1}^E} + (w_s^h, \mathcal{L} \phi_{r<s}^h - f)_{\Omega_{s-1}^E} \\
& + (\mathcal{L} w_s^h, \phi_{r>s}^h)_{\Omega_{s-1}^E} + (w_s^h, \hbar + \kappa \partial_n \phi_{r<s}^h)_{\Gamma_{h,s-1}^E} \\
& - \left(\kappa \partial_n w_s^h, \phi_s^h - (\mathcal{g} - \phi_{r<s}^h)\right)_{\Gamma_{g,s-1}^E} \\
& - \left(\kappa w_s^h, \partial_n \phi_{r\geq s}^h - \frac{2\alpha}{h_s}(\phi_{r\geq s}^h - (\mathcal{g} - \phi_{r<s}^h))\right)_{\Gamma_{g,s-1}^E} \\
& - \left(\kappa \partial_n w_s^h, \frac{1}{2} [\![\phi_s^h \mathbf{n}]\!] \cdot \mathbf{n} - \left(\{\phi_{r>s}^h\} - \frac{h_{s-1}}{2\alpha} [\![\partial_n \phi_{r\geq s-1}^h]\!]\right)\right)_{\Gamma_{s-1}^{E,int}} \\
& - \left(\kappa w_s^h, \partial_n \phi_{r\geq s}^h - \frac{\alpha}{2h_s} [\![\phi_{r\geq s}^h \mathbf{n}]\!] \cdot \mathbf{n} - \frac{h_{s-1}}{2h_s} [\![\partial_n \phi_{r\geq s-1}^h]\!]\right)_{\Gamma_{s-1}^{E,int}} \\
& + \left(\kappa [\![\partial_n w_s^h]\!], \{\phi_{r>s}^h\} - \frac{h_s}{2\alpha} [\![\partial_n \phi_{r\geq s}^h]\!]\right)_{\Gamma_s'^E}. \qquad (26)
\end{aligned}
$$

4.3. System Solution

In the current case, an easily applicable type of cycle to seek a numerical solution by means of an iterative strategy can be alike to the multigrid V-cycle, but inverting the direction of the scale succesion; this means that a cycle will

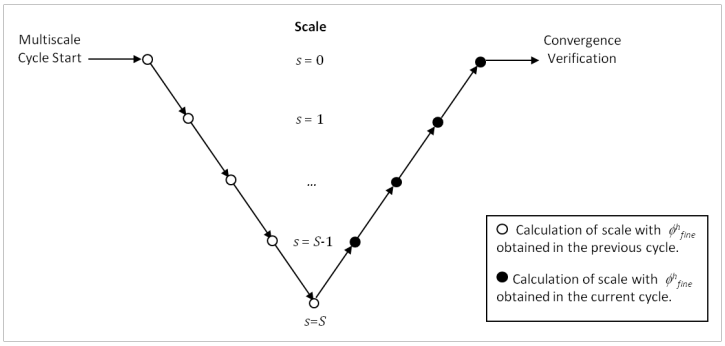

Figure 4: Schematic solving procedure of an iteration of the proposed multiscale method, based on a V-cycle

begin at scale 0 and will advance towards the finest scale S and must go back to the coarsest one, solving scale by scale all of the subdomains' local problems at each step of the cycle. This technique is schematically presented in Figure 4. Additionally, on this figure it is possible to appreciate the management of the memory data for the current calculations: The sum of the function scales that are considered fine with respect to the present scale is denoted (ϕ^h_{fine}) and this data is taken from the computer memory in the way displayed in Figure 4.

The criterion of convergence will simply consider the evaluation of the euclidean norm of the difference of the nodal (or control) variable vectors at every iteration with respect to the previous one. The expression to calculate such a residual (ρ) is given by:

$$\rho = \sum_{s=0}^{S} \left(\sum_{E=1}^{n_{s-1}^{el}} \left| d_s^{E,(it)} - d_s^{E,(it-1)} \right| \right). \tag{27}$$

The goal of this iterative solver is that ρ becomes less than a predefined admissible residal ρ_{adm}. If this condition is not accomplished within a maximal number of iterations it_{max}, then the algorithm will stop running and will proceed to notify that there was not convergence and print out the last numerical vectors calculated as the solution.

5. Numerical results

This problem consists of an annular shape domain that sweeps 90°. The choice of the boundary conditions and the load function will be properly made having in mind the desired exact solution, which is meant to be dependent on the radial coordinate only. The domain geometry and its boundary conditions of both Dirichlet type and Neumann type are described in Figure 5.

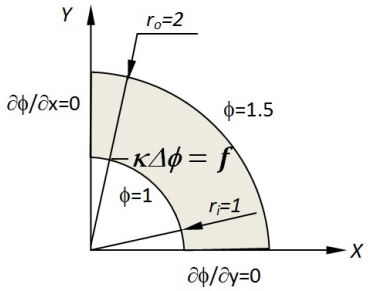

Figure 5: Example's two-dimensional boundary value problem

Load field f is stated as a function of radial direction, r. Thus such function is defined as $f(r) = r^2 = x^2 + y^2$. A constant diffusivity value is again assumed, and equals to one. The exact solution to this problem is presented in eq. (28).

$$\begin{aligned} \phi(r) &= \frac{-1}{16} r^4 + C_1 \ln \frac{r}{r_i} + C_2 \\ C_2 &= a + \frac{1}{16} r_i^4 \\ C_1 &= \frac{b - C_2 + \frac{1}{16} r_o^4}{\ln \frac{r_o}{r_i}}. \end{aligned} \tag{28}$$

Constants a and b are the imposed quantities by the Dirichlet boundary conditions for $r = r_o$ and $r = r_i$ respectively, see Figure 5.

In this example the potential of geometry representation by *NURBS* is clearly aprreciated, as it is required only one quadratic element to reproduce the curved boundary of this domain that in this case consist of circumpherential sections. The parameters to draw the patch corresponding to this problem are the ones registered in table 1. The control points are presented on table 2. For a wide understanding of the concepts used in IgA, the reader is encouraged to review a quite relevant reference concerning this novel method [5].

Parameter	**Coordinate ξ**	**Coordinate η**
Polynomial degree	$p = 2$	$q = 2$
Knot vector	$\Xi = \{0, 0, 0, 1, 1, 1\}$	$H = \{0, 0, 0, 1, 1, 1\}$
Number of control points	$n = 3$	$m = 3$

Table 1: Coarse mesh parameters with *NURBS* for a single two-dimensional element

In Figure 6 the two-scale and three-scale refined meshes are displayed. This refinement consists of turning every element into a subdomain with a 3x2 element mesh.

Control point No.	x	y	w
1	1	0	1
2	1.5	0	1
3	2	0	1
4	1	1	$\sqrt{2}/2$
5	1.5	1.5	$\sqrt{2}/2$
6	2	2	$\sqrt{2}/2$
7	0	1	1
8	0	1.5	1
9	0	2	1

Table 2: Control points of the coarse mesh with *NURBS* for the example problem

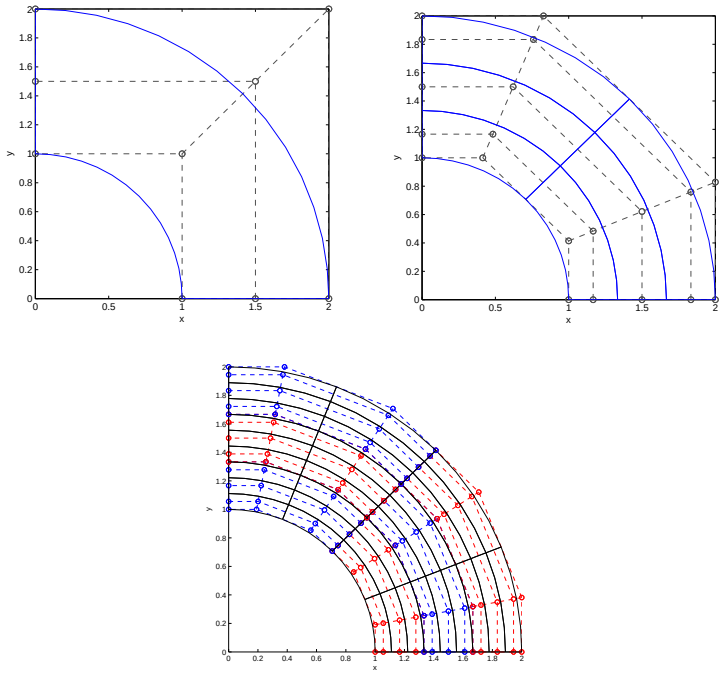

Figure 6: Problem's domain represented in one scale (top), two scales (middle) and three scales (bottom)

The algorithm ran 13 iterations, using a penalty constant of $\alpha = 20$, and an admisible residual of $\rho_{adm} = 10^{-8}$.

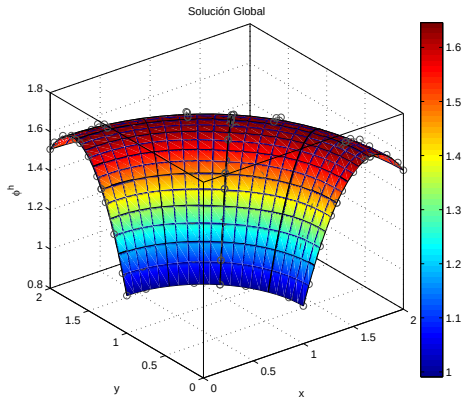

Figure 7: Global solution of the example problem with *NURBS* basis

6. Conclusions

According to the results, the approximation error in IA is generally smaller than that in FEM. There were few cases where the opposite occurred and the difference was very small.

The large number of factors that influence the approximation error makes it difficult to analyze whether the new method can generally behave better than the traditional one.

It was concluded that the potential of the method is mainly in computational cost, with the condition that the systems to be compared require a high level of refinement.

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