Preference-based Evolutionary Multi-objective Optimization for Solving Fuzzy Portfolio Selection Problems

ANA B. RUIZ

abruiz@uma.es
Department of Applied Economics (Mathematics), Universidad de Málaga, Spain

RUBÉN SABORIDO

ruben.saborido-infantes@polymtl.ca
Polytechnique Montréal Researchers in Software Engineering, École Polytechnique de Montréal, Canada

JOSÉ D. BERMÚDEZ

bermudez@uv.es
Department of Statistics and Operational Research, Universitat de València, Spain

MARIANO LUQUE

mluque@uma.es
Department of Applied Economics (Mathematics), Universidad de Málaga, Spain

ENRIQUETA VERCHER

enriqueta.vercher@uv.es
Department of Statistics and Operational Research, Universitat de València, Spain

RESUMEN: En este trabajo, abordamos el problema de selección de carteras de inversión desde una perspectiva multiobjetivo y consideramos algoritmos evolutivos de optimización multiobjetivo basados en preferencias para generar carteras eficientes teniendo en cuenta preferencias del inversor. Por un lado, proponemos un modelo de optimización para la selección de carteras con tres objetivos, en el que se han considerado números fuzzy de tipo LR para modelizar la incertidumbre de los futuros beneficios. Las funciones objetivo a optimizar son el beneficio esperado (a maximizar) y dos medidas del riesgo de la inversión (ambas a minimizar): la semi-desviación media absoluta por debajo del beneficio esperado y el valor de riesgo (medida de la peor pérdida esperada en un horizonte dado). Además de la restricción presupuestaria, se ha introducido una restricción que limita la cardinalidad de las carteras y cotas superiores e inferiores para la inversión en cada activo. El problema de optimización multiobjetivo resultante, denominado por sus siglas en inglés como modelo MASdVaR (mean-absolute semi-deviation value-at-risk model), es no lineal y no convexo y, por ello, se ha aplicado el algoritmo evolutivo basado en preferencias WASF-GA para generar carteras de inversión eficientes. En WASF-GA, se consideran valores de aspiración que el decisor desea alcanzar en cada objetivo para expresar las preferencias. Para el modelo MASdVaR, los valores de aspiración considerados corresponden a un perfil de inversor conservador. Los resultados obtenidos para un caso basado en el mercado de inversión español demuestran que las carteras eficientes obtenidas que alcanzan los mayores beneficios, mejoran los valores de referencia considerados para las dos medidas del riesgo. Por otro lado, aquellas carteras con beneficios más moderados presentan valores de riesgo menores, pero sin dejar de satisfacer las aspiraciones del inversor, por lo que representan estrategias de inversión menos arriesgadas.

Palabras Clave: Selección de carteras de inversión, Algoritmos evolutivos de optimización multiobjetivo basados en preferencias, Solución Pareto óptima, Distribución de credibilidad, Variables fuzzy tipo LR.

ABSTRACT: In this work, we address the multi-criteria paradigm of the portfolio selection problem and consider a preference-based evolutionary multi-objective optimization algorithm to find Pareto optimal portfolio solutions based on the investor preferences. Firstly, we propose a three-objective optimization model for portfolio selection, in which the uncertainty of the portfolio returns is modelled by means of an LR-power fuzzy variable. In the model, three criteria are considered, which are the credibility expected value of the returns (to be maximized) and two measures of the risk (both to be minimized): the below-mean absolute semi-deviation and the fuzzy value-at-risk. Besides the budget constraint, a cardinality constraint and lower and upper bound constraints for the assets are also considered. The resulting model, called a credibility mean-absolute semi-deviation value-at-risk (MASdVaR) model, is a non-linear and non-convex multi-objective optimization problem which is solved by means of the preference-based evolutionary algorithm WASF-GA. In WASF-GA, the preferences are expressed by means of aspiration values that the decision maker would like to achieve for the objectives. In the MASdVaR model, the investor aspiration values considered for the objectives are calculated assuming a conservative profile. The results obtained for a case study based on the Spanish stock market show that the portfolios generated with the highest expected returns improve the aspiration values considered for the two risk measures. Besides, portfolios with intermediate values for the expected return achieve lower values for the risk measures, but they still improve their aspiration values and thus represent less risky investment options for the investor.

Keywords: Portfolio selection; Preference-based evolutionary multi-objective optimization; Pareto optimal solutions; Credibility distributions; LR-fuzzy variables.

1. Introduction

Solving problems arising in real-world applications, such as e.g. economy, engineering or industry, involves optimizing several objective functions simultaneously. Usually the objectives are in conflict and there is no solution which optimizes all of them at the same time. These problems are called multi-objective optimization problems and, instead of searching for one optimal solution of the problem, the interest is focused on a set of Pareto optimal solutions, at which none of the objectives can be improved without a sacrifice in, at least, one of the others. The set of all the Pareto optimal solutions (in the objective space) is known as the Pareto optimal front.

The so-called portfolio selection problem is an example of a multi-objective optimization problem, in which an investor desires to know how her/his capital should be allocated between the different assets available in the market in order to guarantee the maximum expected return of the investment with the minimum risk. The multi-objective nature of these problems is unquestionable and, since the classical mean-variance optimization portfolio selection problem of Markowitz was proposed in [21], portfolio theory has evolved considerably. Nowadays more complex and sophisticated models are used which include more than two criteria to find Pareto optimal portfolios [3, 26, 31, 36], thus making the problem more difficult to solve. Also, besides the variance of the returns, alternative statistical measures of the risk of the investment have been employed, such as e.g. the semi-variance, the absolute semi-deviation or the value-at-risk (see [13, 30] and references therein). However, not all the relevant information for portfolio selection can be obtained just by optimizing the returns and the risk simultaneously. Usually, information about trading and about the requirements of the investor is considered in the model by means of the introduction of new constraints, such as cardinality constraints (to limit the number of assets participating in the feasible portfolios), lower and upper bound constraints (to set lower and upper limits for the amount invested in each asset) and so on.

However, in portfolio selection, deciding the approach to quantify the uncertainty of the portfolio returns is as important as determining the optimization model to be solved. In classical problems, the expected returns on assets are considered as problem parameters, estimated throughout historical data sets and assuming that the vector of returns on assets is multivariate-normally distributed. But financial information is not always completely available and, commonly, decisions are made under uncertainty. Then, a more realistic approach is to assume that the uncertainty of the future returns on assets can be quantified by means of fuzzy numbers, which allow the intro-

duction of the imperfect knowledge about the future market behaviour into the model [3, 15, 33]. Alternatively, the uncertainty of the future returns can be directly approximated using the historical returns on the portfolios [28, 31, 32]. However, note that the fuzzy representation of the uncertainty of the returns implies the use of a non-continuous, non-smooth, quasi-concave and quasi-convex function (a quantile function). Thus, in this case, the problem becomes more difficult to solve by means of classical optimization techniques given that we need to deal with non-linear and non-convex functions.

Once the portfolio selection model has been completely defined, a suitable multi-objective optimization technique able to provide Pareto optimal portfolios has to be used, but several issues have to be considered in this regard. On the one hand, under a fuzzy modelling framework to determine the uncertainty of the portfolio returns, the multi-objective portfolio selection problem is non-linear and non-convex, as already said. Besides, under the assumption of cardinality constraints, the portfolio selection model becomes a constrained mixed integer multi-objective optimization problem that is NP-hard, with both combinatorial and continuous optimization aspects [25]. To overcome these drawbacks, Evolutionary Multi-objective Optimization (EMO) algorithms [7, 8] can be used for successfully solving the portfolio selection problem. These algorithms operate with a population of individuals and attempt to search for a set of non-dominated solutions as close as possible to the Pareto optimal front (convergence) and, at the same time, which represent the entire Pareto optimal front (diversity). For a comprehensive literature review about the use of EMO algorithms in portfolio selection, see [18, 22, 28].

Another crucial aspect to reach a satisfactory Pareto optimal portfolio is the post-optimization decision making phase, in which the investor (the decision maker (DM)) must make an adequate decision to choose the portfolio which best fits her/his risk profile. To achieve this, several issues must be considered. Firstly, only a few Pareto optimal portfolios must be analysed by the DM in order not to overwhelm her/him, since comparing too many solutions may be difficult in the presence of three or more objectives. Secondly, asking preference information in a format as simple as possible is important to make the solution process more meaningful. Besides, if the DM feels that the portfolios obtained reflect well enough her/his wishes, (s)he is more motivated and (s)he is more likely to keep on looking for her/his most preferred Pareto optimal portfolio. To cope with these shortcomings, the so-called preference-based EMO algorithms [4, 6, 17] can be of great help in portfolio selection. They incorporate information about the preferences of the DM into the solution process and their main purpose is to focus the search for new solutions on the subset of Pareto optimal solutions which best fit these preferences. As a result, the DM only has to compare the trade-offs between solutions that are supposed to please her/him, avoiding to analyse undesirable ones and reducing the cognitive burden. Among the preference-based EMO algorithms, we can mention the *Weighting Achievement Scalarizing Function Algorithm* (WASF-GA) [27], which considers the information given by the DM in the form of desirable aspiration objective function values (which are the components of a so-called *reference point*). This is a natural way to express preferences which is not cognitively demanding. According to the reference point given, WASF-GA approximates a region of interest in the Pareto optimal front which contains potentially interesting Pareto optimal solutions for the DM.

With all of this, the purpose of this paper is twofold. Firstly, we propose a new credibility mean-absolute semi-deviation value-at-risk (MASdVaR) model for portfolio selection. We assume that the uncertainty of the portfolio returns is modelled by means of an LR-power fuzzy variable, which is built using a historical data set of the returns. The criteria considered in the MASdVaR model are the maximization of the credibility expected value of the return of the investment, and the minimization of two measures of the risk, the below-mean absolute semi-deviation and the fuzzy value-at-risk (VaR). The feasible portfolios are defined in the model by means of the budget con-

straint, a cardinality constraint to control the portfolio size and lower and upper bound constraints for the share invested in each asset to assure the diversification of the portfolio composition.

Secondly, taking into account the nature of the problem (non-linear, non-convex, mixed integer and NP-hard), we generate Pareto optimal portfolios of the MASdVaR model according to the profile of the investor by applying the preference-based EMO algorithm WASF-GA. Applications of evolutionary algorithms which handle preferences in portfolio selection can be found elsewhere, e.g. in [14, 20, 32, 37]. However, none of them directly use a preference-based EMO algorithm such as WASF-GA, which has not been specifically designed for portfolio selection and which allows to approximate a region of interest (a subset of Pareto optimal solutions) based on the preferences of DM, as we do in this paper. Regarding the preference information required in WASF-GA, the aspiration values for the objectives are calculated assuming a conservative profile for the investor. Additionally, to guarantee the feasibility of the portfolio solutions generated in WASF-GA, special care must be taken for handling the objectives and constraints along the solution process. Here, we have considered the mutation, crossover and reparation operators suggested in [28], which are designed ad-hoc for portfolio selection models which incorporate cardinality and lower and upper bound constraints, such as the MASdVaR model.

In a previous research [32], we proposed a cardinality constrained bi-objective optimization model in which the fuzzy VaR was not considered as a criterion as such. Instead, the fuzzy VaR was used in a post-optimization decision support stage applied once a set of Pareto optimal portfolios had been found by a genetic algorithm. Besides, the genetic algorithm applied in [32] did not take into account the investor preferences to generate these portfolios. Alternatively, in this paper, we analyse both the effect of considering the fuzzy VaR as a third objective function to be optimized, and also the consequences of including the preferences into the solution process of the genetic algorithm for directly generating Pareto optimal portfolios according to the expectations of the investor.

The rest of this work is structured as follows. Section 2 summarizes general definitions and concepts of multi-objective optimization and fuzzy theory, including a brief description of the WASF-GA algorithm. Next, the credibilistic MASdVaR portfolio selection model proposed in this paper is formulated in Section 3. We illustrate our proposal with a data set from the Spanish stock market and we analyse the numerical results obtained in Section 4. Finally, we conclude and summarize the most significant findings of this work in Section 5.

2. Background concepts

A *multi-objective optimization problem* can be defined by the following mathematical expression:

$$\begin{aligned} \text{minimize} \quad & \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \\ \text{subject to} \quad & \mathbf{x} \in S, \end{aligned} \tag{1}$$

where k ($k \geq 2$) objective functions f_i ($i = 1, \ldots, k$), defined from the *feasible set* $S \subset \mathbb{R}^n$ to $\mathbb{R}$, must be minimized simultaneously. A vector of decision variables $\mathbf{x} = (x_1, \ldots, x_n)^T$ is a *feasible solution* if it belongs to the feasible set S. Its image $\mathbf{z} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))^T$ is referred to as an *objective vector* and the set of all objective vectors, denoted as $Z = \mathbf{f}(S) \subset \mathbb{R}^k$, is the *feasible objective set*.

In most of the cases, it is impossible to find a feasible solution which simultaneously minimizes all the objective functions because of the conflict degree among them. Thus, the concept of Pareto optimality arises. Given $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^k$, we say that $\mathbf{z}$ *dominates* $\mathbf{z}'$ if $z_i \leq z_i'$ for all $i = 1, \ldots, k$ and $z_j < z_j'$ for at least one index j. When $\mathbf{z}$ and $\mathbf{z}'$ do not dominate each other, we say that they are *non-dominated*. Besides, a set $A \subset \mathbb{R}^k$ is said to be a *non-dominated set* if any pair of elements of A are non-dominated. Regarding to problem (1), a *Pareto optimal solution* can be defined as a

feasible solution $\mathbf{x} \in S$ for which there does not exist another $\mathbf{x}' \in S$ such that $\mathbf{f}(\mathbf{x}')$ dominates $\mathbf{f}(\mathbf{x})$. Usually, there are many Pareto optimal solutions and we refer to the set of all of them in the decision space as the *Pareto optimal set*, denoted by E, and its image in the objective space as the *Pareto optimal front*, denoted by $\mathbf{f}(E)$.

Since the Pareto optimal front contains more than one objective vector (obviously, if it was formed by a single vector, this vector would be the solution of problem (1)), it is useful to know the ranges of the objective vectors in the Pareto optimal front. The *ideal point* $\mathbf{z}^{\star} = (z_1^{\star}, \ldots, z_k^{\star})^T \in \mathbb{R}^k$ defines lower bounds for the objective function values and is obtained by optimizing each objective function individually over the feasible set, that is, $z_i^{\star} = \min_{\mathbf{x} \in S} f_i(\mathbf{x}) = \min_{\mathbf{x} \in E} f_i(\mathbf{x})$ for all $i = 1, \ldots, k$. The *nadir point* $\mathbf{z}^{\mathrm{nad}} = (z_1^{\mathrm{nad}}, \ldots, z_k^{\mathrm{nad}})^T \in \mathbb{R}^k$ is calculated as $z_i^{\mathrm{nad}} = \max_{\mathbf{x} \in E} f_i(\mathbf{x})$ for all $i = 1, \ldots, k$ and consists of upper bounds for the objectives. While the ideal point can be easily obtained, the nadir point is difficult to calculate because the set E is usually unknown. Different approaches for estimating it can be found in [9, 10].

In general, a *decision maker (DM)* is involved in the solution process by expressing her/his preferences, which can be elicited in different ways [23]. As previously said, one of them is by specifying desirable aspiration objective function values to form a so-called reference point. That is, a *reference point* is an objective vector $\mathbf{q} = (q_1, \ldots, q_k)^T \in \mathbb{R}^k$, where each q_i $(i = 1, \ldots, k)$ is an aspiration value for the objective function f_i given by the DM. A reference point is *achievable* if either $\mathbf{q} \in Z$ or if $\mathbf{q}$ is dominated by a Pareto optimal objective vector; otherwise, the reference point is said to be *unachievable*.

Given a reference point, an achievement scalarizing function (ASF) can be minimized over the feasible set in order to find the Pareto optimal solution which best fits this reference point [35]. ASFs are real-valued functions formulated using the k objective functions, the reference point $\mathbf{q}$ and a vector of positive weights $\mu = (\mu_1, \ldots, \mu_k)^T$ $(\mu_i > 0,$ for $i = 1, \ldots, k)$. One of the most used ASF was proposed in [35] and it is an extension of the L-infinity metric plus an augmentation term. An advantage of using this ASF is that the optimal solution which minimizes it over S is always a Pareto optimal solution of the original problem (1). Also, it is proven that every Pareto optimal solution of (1) (including non-supported solutions) can be generated by varying the reference point and/or the weight vector used in this ASF [29]. An overview about ASFs can be found in [24].

The MASdVaR model proposed in Section 3 is solved using a preference-based EMO algorithm. In general, *Evolutionary Multi-objective Optimization* (EMO) algorithms randomly generate a population of solutions and, at each generation, assign a fitness to each solution and apply selection, crossover, mutation and elitism operators to progressively converge towards the Pareto optimal front. Roughly speaking, we can say that the fitness assignment and the selection become crucial to quickly converge to the Pareto optimal front, the mutation and the crossover operators ensure the exploitation and the exploration of solutions, while the elitism operator prevents the best non-dominated solutions found along the generations from being discarded from the population. Among all of them, the EMO algorithms which introduce preference information into the solution process are aimed at directing the search for new non-dominated solutions towards a region of interest in the Pareto optimal front constituted by the Pareto optimal solutions which best suit these preferences.

In Section 4, the preference-based EMO algorithm WASF-GA [27] is used for generating non-dominated portfolios of the MASdVaR model. The main purpose of this algorithm is to approximate the region of interest of the Pareto optimal front defined by a reference point $\mathbf{q}$ given by a DM. Formally, according to [27], we can define the *region of interest of the Pareto optimal front associated to* $\mathbf{q}$ as follows. When $\mathbf{q}$ is achievable, the region of interest is the subset of Pareto optimal objective vectors which dominate $\mathbf{q}$, that is, the objective vectors $\mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E$ such that $f_i(\mathbf{x}) \leq q_i$, for every $i = 1, \ldots, k$. On the other hand, if $\mathbf{q}$ is unachievable, the region of interest is

formed by the Pareto optimal objective vectors which are dominated by $\mathbf{q}$, that is, the objective vectors $\mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E$ such that $f_i(\mathbf{x}) \geq q_i$, for every $i = 1, \ldots, k$. In this case, solutions lying in this region are likely to be more appealing for the DM than the ones outside it because, at them, the objective function values differ from the aspiration values as little as possible, although they do not improve any of them. The solutions outside this region may improve some of the aspiration values (and not all of them) but at the expense of a sacrifice in the rest of them, what may not be so attractive for the DM.

To approximate this region of interest, WASF-GA maintains a diverse set of non-dominated solutions by considering, on the one hand, a predefined set of weight vectors in $(0, 1)^k$ and, on the other hand, by minimizing at each generation the ASF proposed in [35] for the given reference point and these weight vectors. At each generation of WASF-GA, parents and offspring are classified into several fronts according to the values that each solution takes on the ASF, for the reference point used and for each one of the weight vectors in the predefined set. The lower the ASF values reached by a solution, the more this solution is highlighted. The solutions selected can be considered as the best solutions found at each generation for minimizing the ASF with respect to the each one of the weight vectors. From the practical point of view, we can say that the region of interest is approximated by projecting at each generation the reference point onto the Pareto optimal front, taking into account a set of well-spread search directions (defined by the weight vectors). Then, to preserve the diversity, the weight vectors must represent evenly distributed projection directions. For more details about the working procedure of WASF-GA, see [27].

In the rest of this paper, we consider membership functions associated to bounded LR-power fuzzy variables to approximate the uncertain portfolio returns. Let us recall some concepts of fuzzy set theory [11, 19, 38] which will be used hereafter. According to [16], a fuzzy number Q is said to be an *LR-type fuzzy number* if its membership function has the following form:

$$\mu_Q(y) = \begin{cases} L(\frac{A-y}{s_A}) & \text{if } -\infty < y \leq A, \\ 1 & \text{if } A \leq y \leq B, \\ R(\frac{y-B}{s_B}) & \text{if } B \leq y < +\infty, \end{cases} \tag{2}$$

where A and B (with $A \leq B$) represent lower and upper bounds of the core of Q, respectively, i.e. $[A, B] = \{y | \mu_Q(y) = 1\}$; s_A and s_B are the left and right spreads of Q, respectively; and $L, R : [0, +\infty) \to [0, 1]$ are reference functions which are non-increasing and upper semi-continuous with $\lim_{t \to +\infty} L(t) = \lim_{t \to +\infty} R(t) = 0$. Furthermore, Q is said to be a *bounded LR-type fuzzy number* if its reference functions are such that the support of Q is bounded, i.e. if there exist two real numbers a and b (with $a < b$) such that $\{y : \mu_Q(y) > 0\} \subset [a, b]$. Actually, we say that Q is an *LR-power fuzzy number* if its reference functions are of the type $L_\alpha(t) = 1 - t^\alpha$ and $R_\beta(t) = 1 - t^\beta$, for every $t \geq 0$, where α and β are two positive real values known as shape parameters. Note that any trapezoidal fuzzy number is a particular case of an LR-power fuzzy number with $\alpha = \beta = 1$. Additionally, if $A = B$ we obtain a triangular fuzzy number.

It is well-known that a fuzzy number Q induces a possibility distribution that matches with its membership function $\mu_Q(y)$ [38]. Since the possibility and necessity of every fuzzy event can be evaluated based on the possibility distribution associated to its membership function, every fuzzy number can be considered as a fuzzy variable ξ (in the sense of [19]), whose credibility distribution $\Phi_\xi : \mathbb{R} \to [0, 1]$ is given by:

$$\Phi_\xi(x) = \frac{1}{2}(\sup_{y \leq x} \mu_\xi(y) + 1 - \sup_{y > x} \mu_\xi(y)). \tag{3}$$

In this way, the fuzzy variable ξ is an *LR-type fuzzy variable* if its credibility distribution has a membership function μ_ξ as the one given in (2), which corresponds to an LR-type fuzzy number.

3. A new model for portfolio selection using the Value at Risk (VaR)

In most portfolio selection models, the expected return on assets, the covariance matrix and the upper moments are usually considered as known parameters estimated throughout a historical data set. Instead, we turn the problem around: we assume that the portfolio composition is known and we directly consider the historical returns on the given portfolio for quantifying the uncertainty of its future return. For this, we apply fuzzy set theory [11, 19, 38] in such a way that historical data information is used for building suitable membership functions for the return of the investment. This modelling approach has been previously used for ranking risky portfolios [1] and for obtaining Pareto optimal portfolios using multi-objective optimization approaches [2, 28, 32].

Therefore, throughout the paper, we use the credibility measure of a fuzzy event to quantify the uncertainty of the future return of the investment. To do so, we directly approximate the credibility distribution of the portfolio return (which is considered as a fuzzy variable) instead of aggregating the credibility distributions of the individual assets that compose the portfolio. However, the historical portfolio returns are evaluated by considering the returns on assets for each period of time. Note that quantifying the uncertainty of the portfolio return by means of a fuzzy variable and credibility distributions allows us to measure the percentage of loss on a given portfolio allocation by means of its fuzzy Value-at-Risk (VaR) [34], so we can calculate this function and introduce it into the model as a measure of the investment risk to be minimized.

3.1. *A credibility mean-absolute semi-deviation model with a fuzzy VaR criterion (MASdVaR)*

Let us consider a capital market with N financial assets offering uncertain rates of returns. An investor desires to know which is the optimal allocation of their wealth among the N assets, looking for the maximization of the expected return of the investment at the end of the period at the minimum risk. Let us denote a portfolio by $\mathbf{x} = (x_1, \ldots, x_N)^T$ in which the total wealth is allocated, where x_i $(i = 1, \ldots, N)$ represents the proportion of the total investment devoted to the asset i.

Each portfolio $\mathbf{x}$ must verify the budget constraint given as $\sum_{i=1}^{N} x_i = 1$ and the non-negative condition of every proportion, i.e. $x_i \geq 0$ for every $i = 1, \ldots, N$, when short selling is excluded. Also, lower and upper limits on the budget to be invested in each asset i can be imposed to assure the diversification of the investment. For this, constraints of the type $0 \leq l_i \leq x_i \leq u_i$ can be considered, where l_i and u_i denote the lower and upper bounds for the asset i, respectively $(i = 1, 2, \ldots, N)$. Additionally, a cardinality constraint can be incorporated into the model to control the number of assets that participate in the portfolio. The cardinality constraint is given by $h_l \leq c(\mathbf{x}) \leq h_u$, where $c(\mathbf{x}) = \text{rank}(\text{diag}(\mathbf{x}))$ denotes the rank of the diagonal matrix whose diagonal elements are the components of the vector $\mathbf{x}$ and h_l and h_u are two positive integer values. Thus, the cardinality constraint ensures that the number of assets that compose each portfolio $\mathbf{x}$ is always within the interval $[h_l, h_u]$. Furthermore, if $h_l = h_u$, the portfolios are forced to be always composed by the same number of assets. The function $c(\mathbf{x})$, which gives the number of positive proportions in the portfolio $\mathbf{x}$, is quasi-concave.

For a portfolio $\mathbf{x}$, let us consider the historical returns over T quotation periods, denoted by $\{r_t(\mathbf{x})\}_{t=1}^{T}$, whose sample percentiles are given by p_j, being j the order of the percentile. In our modelling approach, we assume that the uncertainty regarding the future return on $\mathbf{x}$ is approximated by a fuzzy variable $\xi_\mathbf{x}$, whose membership function and credibility distribution (given in equations (2) and (3)) are built using these sample percentiles p_j. To be more precise, the uncertain return on the portfolio $\mathbf{x}$ is modelled by means of a bounded LR-power fuzzy variable, denoted by $\xi_\mathbf{x} = (A, B, c, d)_{L_\alpha R_\beta}$, where $[A, B]$ is the core, c and d are the left and right spreads of

its membership function, while α and β are the positive shape parameters of the power reference functions L and R, respectively.

In our multi-objective optimization portfolio selection problem, the objective functions to be optimized are the credibility expected value of the return, the below-mean absolute semi-deviation and the fuzzy VaR$_{5\%}$ (that is, the 0.05-optimistic value of the fuzzy variable which represents the uncertainty on the percentage of loss in a given period of time). Obviously, the expected return value of the investment is to be maximized, and both the credibility below-mean absolute semi-deviation and the fuzzy VaR$_{5\%}$ are measures of the investment risk to be minimized. These two measures provide different information about the risk: the fuzzy VaR$_{5\%}$ represents the percentage of loss that will be lower than VaR$_{5\%}$ 95% of the time, while the below-mean absolute semi-deviation deals with the non-desired variability around the credibility expected value, i.e. $(\xi_{\mathbf{x}} - E(\xi_{\mathbf{x}}))^{-}$. According to [32], for every portfolio $\mathbf{x}$ whose uncertainty of its future return is modelled by the fuzzy variable $\xi_{\mathbf{x}}$, these credibility moments and optimistic value can be mathematically computed as follows:

- Credibility expected return value, denoted by $E(\xi_{\mathbf{x}})$:

$$E(\xi_{\mathbf{x}}) = \frac{A+B}{2} + \frac{d}{2}\frac{\beta}{\beta+1} - \frac{c}{2}\frac{\alpha}{\alpha+1}. \tag{4}$$

- Credibility below-mean absolute semi-deviation, denoted by MASd$(\xi_{\mathbf{x}})$:

$$\text{MASd}(\xi_{\mathbf{x}}) = \begin{cases} \frac{1}{2}\left(E(\xi_{\mathbf{x}}) - (A-c)\right) - \frac{1}{2}\frac{c}{\alpha+1}\left(1 - (\frac{A-E(\xi)}{c})^{\alpha+1}\right) & \text{if } A-c \leq E(\xi_{\mathbf{x}}) \leq A, \\ \frac{1}{2}\left(B - A + c\frac{\alpha}{\alpha+1} + d\frac{\beta}{\beta+1}\right) & \text{if } A \leq E(\xi_{\mathbf{x}}) \leq B, \\ \frac{1}{2}\left((B+d) - E(\xi_{\mathbf{x}})\right) - \frac{1}{2}\frac{d}{\beta+1}\left(1 - (\frac{E(\xi_{\mathbf{x}})-B}{d})^{\beta+1}\right) & \text{if } B \leq E(\xi_{\mathbf{x}}) \leq B+d. \end{cases} \tag{5}$$

- Fuzzy VaR$_{5\%}$, denoted by FVaR$_{5\%}(\xi_{\mathbf{x}})$:

$$\text{FVaR}_{5\%}(\xi_{\mathbf{x}}) = B + 0.9^{\frac{1}{\beta}}d. \tag{6}$$

Based on the aforementioned definitions, the cardinality constrained credibility mean-absolute semi-deviation value-at-risk (MASdVaR) model for the portfolio selection problem is formulated as follows:

$$\begin{aligned} (\textbf{MASdVaR}) \ \text{maximize} \quad & E(\xi_{\mathbf{x}}) \\ \text{minimize} \quad & \text{MASd}(\xi_{\mathbf{x}}) \\ \text{minimize} \quad & \text{FVaR}_{5\%}(\xi_{\mathbf{x}}) \\ \text{subject to} \quad & \sum_{i=1}^{N} x_i = 1, \\ & h_l \leq c(\mathbf{x}) \leq h_u, \\ & 0 \leq l_i \leq x_i \leq u_i, \ i = 1, 2, ..., N. \end{aligned} \tag{7}$$

In the rest of the paper, we assume that the cardinality constraint is formulated as $c(\mathbf{x}) = h$, for a given number $h \in \mathbb{N}$ (i.e. $h_l = h_u = h$). This means that every feasible portfolio of (7) is forced to invest in exactly h of the N available assets.

Because of the use of fuzzy variables to represent the uncertainty of the portfolio returns which depend on the sample percentiles, the MASdVaR model involves non-linear and non-convex functions. Besides, the introduction of the cardinality constraint with a quasi-concave function means that the model is a constrained mixed integer multi-objective optimization problem that is NP-hard [25]. Dealing with a non-linear and non-convex mixed integer NP-hard multi-objective

optimization problem is not straightforward and classical multi-objective optimization methods are not suitable. This is the reason why we have used an EMO algorithm to find Pareto optimal portfolios of the MASdVaR model, because they are able to handle problems with different types of variables and objectives, such as e.g. integer or binary variables and non-convex, non-differentiable or discontinuous functions.

4. Computational experiments

In this section, we solve the MASdVaR model (7) using the preference-based EMO algorithm WASF-GA [27]. Given that EMO algorithms cannot be directly applied in their standard form to constrained portfolio optimization problems, we must carefully define several issues for handling the objectives and constraints of the problem. Commonly, classical operators are applied to these problems [18], but a repair mechanism needs to be used in order to assure that the new portfolio solutions satisfy all the constraints. In the case of the MASdVaR model, the main drawback comes from the presence of the cardinality constraint, since new portfolios obtained by commonly used mutation and crossover operators are not assured to be composed by a limited number of assets. To deal with this, we have considered mutation, crossover and repair operators in WASF-GA which ensure to generate feasible portfolios of the MASdVaR model satisfying all the constraints.

In what follows, we briefly introduce the solution representation and operators used. Later, the case study and the parameters considered in the experiments are described and, finally, we discuss the results obtained.

4.1. *Solution representation*

We use a standard real-valued representation, i.e. a portfolio $\mathbf{x}$ is represented by a vector $\mathbf{x} = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$, where each x_i represents the proportion invested in the asset i, for $i = 1, \ldots, N$. This means that, if $x_i \neq 0$, the portfolio $\mathbf{x}$ invests in the asset i and the value of x_i indicates the proportion of the capital budget allocated to the asset i.

4.2. *Evolutionary operators*

To generate portfolios satisfying the budget, the upper and lower bound and the cardinality constraints, we have used the evolutionary operators proposed in [28]. These operations are specially designed for generating feasible portfolios of cardinality constrained problems such as the MASdVaR model.

- *Mutation operator.* The mutation operator suggested in [28] is a unary operator which, for a given portfolio, retrieves a mutated portfolio. The mutation operation is just a simple procedure for swapping the proportions of two different assets, which are randomly selected with a probability $P_m = 1/h$, where h is the number of assets that participates in the portfolio according to the cardinality constraint.
- *Crossover operator.* The crossover is a binary operator which generates two offspring portfolios from two given parent portfolios. Firstly, the offspring are initialized as empty portfolios and the assets which constitute them are added lately, inheriting h positive proportions from both parents. The inherited proportions from each parent are selected depending on the number of assets in common between the two parents. For further details, see [28].
- *Reparation operator.* Although the new portfolios generated with the above mentioned crossover operator satisfy the cardinality and the bound constraints, it is not guaranteed that they meet the budget constraint. For this reason, the reparation operator described in [28] is performed. After normalizing the positive proportions to satisfy the budget

constraint, a strategy is followed to readjust the proportions which do not meet the lower and the upper bounds.

4.3. *WASF-GA algorithm with the operators*

As explained in Section 2, the working procedure of WASF-GA is based on the classification of parents and offspring into several fronts. We have incorporated the aforementioned operators into WASF-GA as explained in Algorithm 1, which describes one generation t. Here, N_p denotes the population size and P_t and Q_t are the population of parents and offspring, respectively, at generation t.

Algorithm 1 Each generation t of WASF-GA

Require: The population P_t at generation t.
Ensure: The population P_{t+1} for the next generation $t + 1$.
  1: Set $Q_t = \emptyset$.
  2: **while** $\#(Q_t) < N_p$ **do**
  3:     **repeat**
  4:        Select two parents $\mathbf{x}^1$ and $\mathbf{x}^2$ from P_t by using binary tournament selection.
  5:        *Crossover*: using $\mathbf{x}^1$ and $\mathbf{x}^2$, generate two offspring portfolios $\mathbf{y}^1$ and $\mathbf{y}^2$ by means of the crossover operator proposed in[28].
  6:        *Reparation*: apply the reparation operator proposed in [28] to the portfolios $\mathbf{y}^1$ and $\mathbf{y}^2$ if required.
  7:        *Mutation*: perform the mutation operator suggested in [28] on $\mathbf{y}^1$ and $\mathbf{y}^2$ to produce two mutated portfolios $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$.
  8:        Insert $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$ into Q_t.
  9:     **until** $\#(Q_t) = N_p$
 10: **end while**
 11: Combine P_t and Q_t to form a population of size $2N_p$. Select the best N_p portfolios from the combined population to constitute the population P_{t+1} as it is usually done in WASF-GA [27].

4.4. *Case study and parameters setting*

Firstly, the case study we have used consists of a set of weekly returns on assets from the Spanish IBEX35 index, observed in $T = 165$ periods (weeks) from January 2013 till March 2016. The number of available assets for allocating the wealth in this period of time is $N = 33$ (although this index is constituted by 35 assets, there were two assets which were not included in the IBEX35 index through all the time window considered). For every $i = 1, \dots, 33$ and $t = 1, \dots, 165$, the sample return on the individual asset i at week t, denoted by r_{ti}, is calculated as follows:

$$r_{ti} = \frac{CP_i(t+1) - CP_i(t)}{CP_i(t)}, \tag{8}$$

where $CP_i(t)$ denotes the closing price of the asset i on Wednesday at week t. Thus, for $t = 1, \dots, 165$, the weekly return on each portfolio $\mathbf{x}$ for the week t is obtained as $r_t(\mathbf{x}) = \sum_{i=1}^{33} r_{ti} \cdot x_i$. Finally, we consider a bounded LR-power fuzzy variable $\xi_{\mathbf{x}} = (A, B, c, d)_{L_\alpha R_\beta}$ to represent the uncertainty of the future return on each portfolio $\mathbf{x}$ and we build its membership function by means of the sample percentiles of its weekly returns set, denoted by p_j, where j is the order of the percentiles. In particular, the core and the support of $\xi_{\mathbf{x}}$ are represented by the intervals $[p_{40}, p_{60}]$ (medium return values) and $[p_3, p_{97}]$ (the 3rd the 97th percentiles), respectively, while the shape parameters are obtained as $\alpha = \ln(0.5)/\ln(\frac{p_{40}-p_{20}}{c})$ and $\beta = \ln(0.5)/\ln(\frac{p_{80}-p_{60}}{d})$, assuming that the sample percentiles p_{20} and p_{80} have a 50% possibility of being realistic (they are obtained in such a way that the fuzzy and empirical quartiles coincide).

Tabla 1. Parameter settings

Parameter	Value
Population size	300
Number of evaluations	420000
Stopping criterion	Maximum number of evaluations
Crossover operator	$P_c = 0.90, \eta_c = 20$
Mutation operator	$P_m = 1/k, \eta_m = 20$

Regarding the MASdVaR model, we have assumed that the portfolios are composed by 9 assets (i.e. $h = 9$ and $c(\mathbf{x}) = 9$ is the cardinality constraint). The number of assets has been limited to 9 following the advice given in [5], which suggests that investors should not consider h-values above one third of the total number of assets because of dominance relationships. Besides, the upper and lower bounds considered for each asset are $l_i = 0.00$ and $u_i = 0.30$ ($i = 1, \ldots, 33$).

In relation to the computational implementation needed, we have considered the jMetal framework [12] which includes a Java implementation of WASF-GA[1]. In the experiment, we have selected the reference point $\mathbf{q} = (0.80, 1.40, 4.82)^T$ because, on the one hand, having an expected value for the return of 0.80 is very desirable and, on the other hand, the aspiration values considered for the MASd and the VaR objectives, 1.40 and 4.82 respectively, are both realistic and good initial values for both risk measures (based on previously observed results obtained in similar situations). Thus, this reference point represents an intermediate level of risk for a real investor which corresponds to an optimistic but conservative profile. Besides, 30 independent simulation runs of WASF-GA were performed to approximate the region of interest determined by this reference point. Table 1 summarizes the parameter setting used in these simulation runs. Note that the solutions found by the algorithm in all the runs were combined in order to select the non-dominated ones to built a non-dominated set which represents the region of interest as best as possible.

4.5. *Numerical results*

Once all the portfolios generated in the 30 runs of WASF-GA were combined, we obtained 369 non-dominated portfolio solutions approximating the region of interest associated to $\mathbf{q} = (0.80, 1.40, 4.82)^T$. Table 2 shows a numerical description of the characteristics of these non-dominated portfolios (as independent observations of the objective functions). Observe that the obtained portfolios have improved the aspiration values considered for the three objectives (remember that the expected value is to be maximized and the two risk measures are to be minimized). This means that the reference point used is achievable. Thus, the portfolios obtained achieve expected return values better than 0.80 and values for both measures of the risk lower than 1.40 and 4.82, respectively.

Figure 1 shows two bivariate scatter plots of the credibilistic MASd and VaR values versus the expected return values attained by the 369 non-dominated portfolios, respectively. In this figure, the solid diamond represents the reference point $\mathbf{q} = (0.80, 1.40, 4.82)^T$. The marginal box-plots of the achieved objective function values have been also included in this figure. Note that all these portfolios are non-dominated in a three-dimensional multi-objective optimization framework, although these two-dimensional plots may wrongly show a domination among the portfolios represented.

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[1]jMetal is an object-oriented Java-based framework for multi-objective optimization using meta-heuristic algorithms. It can be downloaded as open source at `http://jmetal.sourceforge.net/`, where the implementation of the WASF-GA algorithm is publicly available.

Tabla 2. Numerical description of the 369 non-dominated portfolios generated by WASF-GA

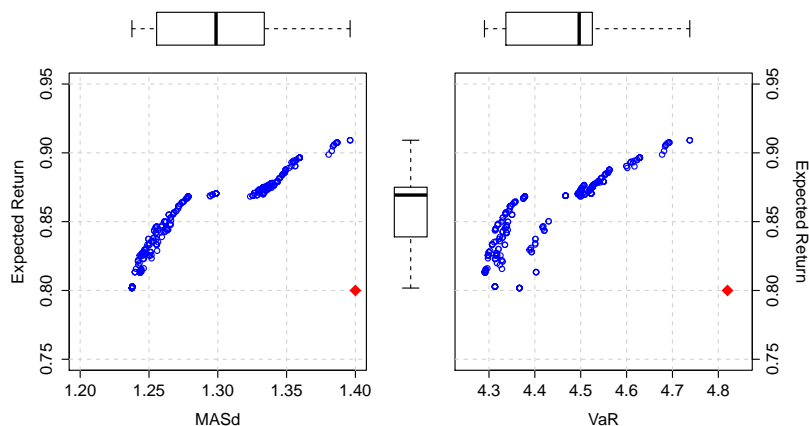|	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Expected return	0.802	0.839	0.869	0.860	0.875	0.909
MASd	1.238	1.256	1.299	1.301	1.334	1.396
VaR	4.290	4.337	4.497	4.453	4.525	4.738

Fig. 1. On the left, the scatter plot of MASd versus expected return with their marginal box-plots. On the right, the scatter plot of VaR versus expected return with their marginal box-plots.

Tabla 3. Characteristics of some specific portfolios

Portfolio	Expected return	MASd	VaR
1	0.802	1.238	4.366
2	0.803	1.238	4.313
3	0.813	1.243	4.290
4	0.909	1.396	4.738

From the point of view of the investor, each portfolio represents an investment proposal generated according to the reference point used. In Table 3, we can see the characteristics of four different portfolios, which are shown just to give an idea of the diversity of the obtained results from an investment perspective. Portfolios 1 and 2 are the ones with the lowest expected returns among all the non-dominated solutions and, accordingly, they have the lowest MASd values, but acceptable VaR values. These two portfolios could be recommended for a conservative investor. On the other hand, portfolio 3 attains the lowest VaR value and has quite small expected return and MASd values (both of them below their first quartiles). Finally, portfolio 4 represents the most risky option, since it reaches the highest MASd and VaR values, but this is the price to pay if the investor wants to have also the highest expected return value. Then, this portfolio could be recommended for an optimistic investor.

It is noteworthy that if we had used other aspiration values for the objective functions (i.e. a different reference point), representing a more risky or a more conservative investor profile, the non-dominated portfolio solutions generated by the WASF-GA algorithm would have approximated a different region of interest constituted by Pareto optimal portfolios fitting the profile considered. Actually, if the investor is too pessimistic or conservative (i.e. the reference point is achievable),

the portfolios generated are likely to achieve better values for the three objective functions that the expected ones, as in our experiment. But if the investor is too optimistic (i.e. the reference point is unachievable), there may not exist portfolio solutions attaining or improving the expectations of the investor although (s)he would be informed about the portfolios which best suit her/his investment aspirations. Thus, in practice, the reference point used in the experiment determines the type of portfolios to be generated.

5. Conclusions

In this work, we have suggested a new multi-objective optimization model for the portfolio selection problem. In the new model, which is called the credibility mean-absolute semi-deviation value-at-risk (MASdVaR) model, the uncertainty of the future return of the portfolios is modelled using LR-power fuzzy variables and the three criteria considered are the maximization of the credibility expected value of the return and the minimization of two risk measures, the below-mean absolute semi-deviation and the fuzzy value-at-risk. To define the feasible portfolios, the model includes a budget constraint, lower and upper bound constraints to limit the proportion invested in each asset, and a cardinality constraint to control the number of assets participating in each portfolio.

Given that the resulting MASdVaR model is non-linear, non-convex, mixed integer and NP-hard, we have used the preference-based evolutionary multi-objective optimization algorithm WASF-GA to generate non-dominated portfolios according to the investor profile, which has been specified by means of a reference point. The experiment carried out using a set of weekly returns on assets from the Spanish IBEX35 index has demonstrated that very promising portfolios can be obtained with the MASdVaR model, each of them reprenting an investment option fitting the investor profile considered. This shows that the WASF-GA algorithm, togehter with suitable genetic operators, has been able to generate non-dominated portfolios for the MASdVaR model. Furthermore, the introduction of the preferences into the algorithm has enabled the generation of portfolio solutions which directly correspond to the investor expectations, without the necessity of any post-optimization stage to filter the portfolios generated.

As future research lines, we plan to investigate how the portfolios generated would vary according to different investment profiles. Additionally, a comparison of the performance of WASF-GA against other preference-based EMO algorithms will let us confirm the goodness of the results obtained for the MASdVaR model.

Acknowledgements

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