A NOTE ON CONJUGACY CLASSES FOR MULTISTATE CELLULAR AUTOMATA

M. AKIYAMA, I. ARCAYA, N. ROMERO, AND J. SILVA

ABSTRACT. The main goal of this note is to classify by conjugacy classes the collection of r-cellular automata generated by the permutations of r fixed local rules.
RESUMEN. El objetivo de estas notas es clasificar por clases de conjugación la colección de r-autómatas celulares generada por las permutaciones de r reglas locales fijadas.
Keywords. Shift mapping. Local rules. Cellular Automata. r-Cellular Automata.
2000 Mathematics Subject Classification: 37B15.

1. INTRODUCTION

Cellular Automata (CA) are discrete dynamical systems acting on the *configuration space* $\mathcal{A}^{\mathbb{Z}}$ of all 1-dimensional sequences $x : \mathbb{Z} \to \mathcal{A}$, where $\mathcal{A}$ is any finite and nonempty set called *alphabet*, $\mathbb{Z}$ is the 1-dimensional integer lattice, and the *global transition maps* defining CA are given by the action of one *local rule* which determines the evolution, sequentially and synchronously, of the state $x(n)$ of the *cell* $n \in \mathbb{Z}$ in the *configuration* $x \in \mathcal{A}^{\mathbb{Z}}$ depending on the states of cells on a finite and uniform neighborhood. More explicitly, $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a *cellular automaton* if and only if there are: a finite and nonempty set $\mathbb{V}$ and a local rule $f : \mathcal{A}^{\mathbb{V}} \to \mathcal{A}$, where $\mathcal{A}^{\mathbb{V}}$ is the set of all functions from $\mathbb{V}$ to $\mathcal{A}$, such that for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$ it holds:

$$F(x)(n) = f(x|_{\mathbb{V}+n}), \tag{1}$$

where $x|_{\mathbb{V}+n} : \mathbb{V} \to \mathcal{A}$ is given by $x|_{\mathbb{V}+n}(k) = x(n+k)$ for all $k \in \mathbb{V}$. In other words, the state of cell n in the configuration $F(x)$ depends of the states of the cells $n+k$, $k \in \mathbb{V}$, in the configuration x.

On the configuration space $\mathcal{A}^{\mathbb{Z}}$ one can consider different topological structures in order to study dynamic properties for endomorphisms on $\mathcal{A}^{\mathbb{Z}}$:

I) **Product topology (or Cantor topology)**: is the finest topology such that, for each $n \in \mathbb{Z}$, the projection $\pi_n : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}$, $\pi_n(x) = x(n)$, is a continuous map. It is well known, even $\mathcal{A}$ any discrete topological space, that the family of cylinders $C(\mathbb{U}, h) = \{x \in \mathcal{A}^{\mathbb{Z}} : x|_{\mathbb{U}} = h\}$, where $\mathbb{U}$ is a finite and nonempty subset of $\mathbb{Z}$ and h is a function from $\mathbb{U}$ into $\mathcal{A}$, is a clopen basis for this topology; clopen means closed and open set. It is well know that the function $d_C : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \to [0, +\infty)$ given by $d_C(x, y) = 2^{-i}$ where $i = \inf\{|n| : n \in \mathbb{Z},\ x(n) \neq y(n)\}$ define a metric compatible with the product topology on $\mathcal{A}^{\mathbb{Z}}$. In particular, when the alphabet $\mathcal{A}$ is finite, this metric is called Cantor metric and $\mathcal{A}^{\mathbb{Z}}$ is a Cantor set; that is: compact, perfect and totally disconnected.

II) **Besicovitch topology and Weyl topology**: introduced in the context of symbolic dynamics to refine the study of discrete dynamical systems on $\mathcal{A}^{\mathbb{Z}}$. These topologies are defined by a shift invariant pseudometric; the first one measures the density of differences in central segments of two configurations; it is given by the formula

$$d_B(x,y) = \limsup_{\ell\to\infty} \frac{\#\{m \in \{-\ell,\cdots,\ell\} : x(m) \neq y(m)\}}{2\ell+1}.$$

Weyl's pseudometric measures the density of differences in arbitrary segments and it is given by the formula

$$d_W(x,y) = \lim_{\ell\to\infty} \left(\sup_{p\in\mathbb{Z}} \frac{\#\{m \in \{p,\cdots,p+\ell-1\} : x(m) \neq y(m)\}}{\ell} \right).$$

For more details about Besicovitch and Weyl topologies we remit the reader to [2].

For instance, consider $\mathcal{A}$ any discrete topological space which we assume finite. It is obvious that any local rule is a continuous function and CA are continuous transformations of the configuration space $\mathcal{A}^{\mathbb{Z}}$. Also it is simple to verify that CA commute with the *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ which is defined, for each $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, as $\sigma(x)(n) = x(n+1)$. Notice that shift map is a cellular automaton. As homeomorphism on $\mathcal{A}^{\mathbb{Z}}$ the shift map has an important impact in the developing of the dynamical systems theory. In a classical article, Hedlund [3] characterized CA in terms of shift map, in fact: every continuous transformation of $\mathcal{A}^{\mathbb{Z}}$ is a cellular automaton if and only if it commutes with the shift map on $\mathcal{A}^{\mathbb{Z}}$; see theorem 3.4 in [3]. This result is true only in the Cantor topology; in [2] there are examples of continuous transformations, in Besicovitch and Weyl topologies, commuting with σ and these maps are not CA.

In this work we deal with discrete dynamical systems on $\mathcal{A}^{\mathbb{Z}}$ where the global transition maps are defined in a similar way to CA but depending on a finite number of local rules. This kind of dynamical systems are called *multistate cellular automata*, or *r-cellular automata*, where $r \geq 1$ is the number of local rules employed to describe the global transition maps. This notion extends the classical definition of CA, see [1] and [4].

The main result of this note shows that under certain permutations on the set of local rules, the corresponding multistate cellular automaton are topologically conjugated; that property is independent in the above topological structure on $\mathcal{A}^{\mathbb{Z}}$. Recall that two continuous transformations $f : X \to X$ and $g : Y \to Y$ (X and Y topological spaces) are topologically conjugated if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. Notice that h carries on orbits of f into orbits of g; therefore, f and g have the same topological dynamics.

In the next section we introduce the concept of r-cellular automata and the corresponding version of Hedlund's theorem when the alphabet $\mathcal{A}$ is finite; also we show a couple of examples, each of them defined by the same set of local rules arranged in different ways. Finally we introduce the notion of conjugacy classes and prove the main result.

2. r-Cellular Automata and Conjugacy classes

Consider a finite alphabet $\mathcal{A}$ and a positive integer r. Let $\mathbb{V}_i$ be a finite and nonempty subset of $\mathbb{Z}$ and $f_i : \mathcal{A}^{\mathbb{V}_i} \to \mathcal{A}$ the local rule acting on $\mathbb{V}_i$ ($i = 0,\cdots,r-1$). By means of these local rules we will define a global transition map on $\mathcal{A}^{\mathbb{Z}}$.

Definition 2.1. Given local rules $f_i : \mathcal{A}^{\mathbb{V}_i} \to \mathcal{A}$ with $0 \leq i \leq r-1$, the *r-cellular automaton*, or *multistate cellular automaton*, generated by them is the transformation $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined, for each $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$, by

$$F(x)(m) = f_q\left(x|_{\mathbb{V}_q+m}\right), \tag{2}$$

where q is the integer m taken modulo r.

Clearly CA on $\mathcal{A}^{\mathbb{Z}}$ are 1-cellular automata. Observe that for any r-cellular automaton F the state temporal evolution, $\{F^n(x)(m) : n \geq 0\}$, of the configuration $x \in \mathcal{A}^{\mathbb{Z}}$ depends on a particular local rule which is given by the location of m in $\mathbb{Z}$. Moreover, it is easy to verify that F commutes with σ^r and it is a continuous on Cantor, Besicovitch and Weyl topologies. The following extension of Hedlund's theorem was proved in the context of product topology on $\mathcal{A}$, even when $\mathcal{A}$ is any discrete topological space, see [1].

Theorem 2.1. *Every continuous transformation $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ commuting with σ^r is a r-cellular automaton.*

Given a r-cellular automaton F, always it is possible to redefine it in such a way that the local rules act on the same finite neighborhood V of $\mathbb{Z}$, for example in an interval $[\ell, \ell+k] = \{\ell, \ell+1, \cdots, \ell+k\}$ ($\ell, k \in \mathbb{Z}$ with $k \geq 0$). Assume that F is the r-cellular automata induced by the local rules $f_i : \mathcal{A}^{V_i} \to \mathcal{A}$ ($0 \leq i < r$). Let ℓ and k be integers, with $k \geq 0$, such that each V_i is contained in the interval $[\ell, \ell+k]$. Define, for every $0 \leq i < r$, the function $\widetilde{f}_i : \mathcal{A}^{[\ell,\ell+k]} \to \mathcal{A}$ as $\widetilde{f}_i(h) = f_i(h|_{V_i})$, where $h|_{V_i}$ denotes the restriction of h to V_i. Thus, for every $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$, the r-cellular automaton $\widetilde{F}$, generated by these local rules and the uniform neighborhood $[\ell, \ell+k]$, satisfies:

$$\begin{aligned} \widetilde{F}(x)(m) &= \widetilde{f}_q\left(x|_{[\ell,\ell+k]+m}\right) = \widetilde{f}_q(x(\ell+m), \cdots, x(\ell+k+m)) \\ &= f_q\left(x|_{\mathbb{V}_q+m}\right) = F(x)(m). \end{aligned} \tag{3}$$

In other words, each r-cellular automaton is determined by a set of r functions $f_0, \cdots, f_{r-1} : \mathcal{A}^{k+1} \to \mathcal{A}$, for some nonnegative integer k, and can be written as in equation (3). However, one could change the order of the local rules by permutations of $\{0, \cdots, r-1\}$ and obtains, possibly, different r-cellular automata.

Example 2.1. Consider $\mathcal{A} = \mathbb{Z}_2 = \{0, 1\}$. Let $\varphi_0, \varphi_1, \varphi_2 : \mathcal{A}^3 \to \mathcal{A}$ be the local rules given by:

$$\varphi_0(a_{-1}, a_0, a_1) = a_0 + a_1 \pmod 2,\ \varphi_1(a_{-1}, a_0, a_1) = a_1$$
$$\text{and } \varphi_2(a_{-1}, a_0, a_1) = a_{-1} + a_0 \pmod 2.$$

In order to obtain 3-cellular automata from this fuctions one can rearrange these local rules according to some permutation of $\{0, 1, 2\}$. So, if $\mathbb{V} = [-1, 1]$, the local rule vector $(\varphi_0, \varphi_1, \varphi_2)$ defines the 3-cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given, for each $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, by

$$F(x)(n) = \begin{cases} \varphi_0(x|_{V+n}) = x(n) + x(n+1) \pmod 2, & \text{if } n = 0 \pmod 3 \\ \varphi_1(x|_{V+n}) = x(n+1), & \text{if } n = 1 \pmod 3 \\ \varphi_2(x|_{V+n}) = x(n-1) + x(n) \pmod 2, & \text{if } n = 2 \pmod 3 \end{cases}$$

While the local rule vector $(\varphi_0, \varphi_2, \varphi_1)$ generates the 3-cellular automaton G on $\mathcal{A}^{\mathbb{Z}}$ defined, for each $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, as

$$G(x)(n) = \begin{cases} \varphi_0\,(x|_{V+n}) = x(n) + x(n+1)\,(\text{mod } 2), & \text{if } n = 0\,(\text{mod } 3) \\ \varphi_2\,(x|_{V+n}) = x(n-1) + x(n)\,(\text{mod } 2), & \text{if } n = 1\,(\text{mod } 3) \\ \varphi_1\,(x|_{V+n}) = x(n+1), & \text{if } n = 2\,(\text{mod } 3) \end{cases}.$$

These 3-cellular automata have different topological dynamics when $\mathcal{A}^{\mathbb{Z}}$ is provided with the Cantor topology: they are not topologically conjugated; in other words, for every homeomorphism $h : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ on the Cantor topology $F \circ h \neq h \circ G$. This claim is a straightforward consequence of the fact that F has infinitely many fixed points; in fact, every configuration $x = (x(n))_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ satisfying

$$x(3\ell + 1) = x(3\ell + 2) = 0 \text{ for all } \ell \in \mathbb{Z}$$

is a fixed point for F. However, G has only one fixed point: the null configuration $o(n) = 0$ for every $n \in \mathbb{Z}$.

In Besicovitch and Weyl topologies can be constructed r-cellular automata with the same performance in preceding example.

Let $f_0, \cdots, f_{r-1} : \mathcal{A}^{k+1} \to \mathcal{A}$ be a set of local rules and let $[\ell, \ell + k]$ (ℓ, $k \in \mathbb{Z}$ with $k \geq 0$) be the set of integers between ℓ and $\ell + k$. Consider the group $\mathcal{S}_r$ of permutations of $\{0, \cdots, r-1\}$. It is clear that every $\tau \in \mathcal{S}_r$ arranges the set of local rules in a vector $(f_{\tau(0)}, \cdots, f_{\tau(r-1)})$, which we refer as *local vector rules*. Thus, for such a vector and the neighborhood $[\ell, \ell + k]$ we have the r-cellular automaton $F_\tau : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined, for each $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$, by

$$F_\tau(x)(m) = f_{\tau([m]_r)}(x(\ell + m), \cdots, x(\ell + k + m)), \tag{4}$$

where $[m]_r$ is the integer m taken module r.

Example 2.1 shows that different permutations on the set of local rules induce, possibly, different topological dynamics on $\mathcal{A}^{\mathbb{Z}}$ by means of the corresponding r-cellular automata. However, we will introduce a partition on $\mathcal{S}_r$, generated by an equivalence relation, which make possible to classify in conjugacy classes the collection of r-cellular automata given by (4) for every $\tau \in \mathcal{S}_r$. In fact, let σ be the cyclic permutation

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & r-2 & r-1 \\ 1 & 2 & \cdots & r-1 & 0 \end{pmatrix}.$$

For this special permutation we use the same nomenclature of the shift map because, in fact, it is a shift with periodic boundary conditions.

Definition 2.2. Given permutations $\eta, \tau \in \mathcal{S}_r$, we say that they belong to the same *cyclic class* if, and only if, there exists $0 \leq j < r$ such that $\tau = \eta \circ \sigma^j$.

Obviously this defines an equivalence relation on $\mathcal{S}_r$. Observe that for every $\tau \in S_r$, its cyclic class is the set of permutations $\{\tau, \tau\sigma, \cdots, \tau\sigma^{r-1}\}$; in particular, the cyclic class of σ is the subgroup $\{i, \sigma, \cdots, \sigma^{r-1}\}$, where i denote the identity on $\mathcal{S}_r$. It is clear that there is exactly $(r-1)!$ cyclic classes on $\mathcal{S}_r$.

The following Theorem is the main result of this note.

Theorem 2.2. *Given local rules $f_0, \cdots, f_{r-1} : \mathcal{A}^{k+1} \to \mathcal{A}$ and the neighborhood $[\ell, \ell+k] \subset \mathbb{Z}$; if $\tau, \eta \in \mathcal{S}_r$ belong to the same cyclic class, then the r-cellular automata F_τ and F_η, as defined in (4), are topologically conjugated.*

Proof. Assume $\tau = \eta \circ \sigma^j$ for some $0 \leq j < r$. It is well known that $\sigma^j : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, given by $\sigma^j(x)(m) = x(m+j)$ for all $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$, is a homeomorphism on $\mathcal{A}^{\mathbb{Z}}$ with Cantor, Besicovitch and Weyl topologies. On the other hand, for each $0 \leq j < r$ the permutation σ^j satisfies $\sigma^j(i) = [i+j]_r$ for all $0 \leq i < r$. Now we will show that the homeomorphism σ^j of $\mathcal{A}^{\mathbb{Z}}$ conjugates the r-cellular automata F_τ and F_η. In fact, for every $x \in \mathcal{A}^{\mathbb{Z}}$ and $m \in \mathbb{Z}$ we have:

$$\begin{aligned}
(F_\tau \circ \sigma^j)(x)(m) &= f_{\tau([m]_r)}(\sigma^j(x)(\ell+m), \cdots, \sigma^j(x)(\ell+k+m)) \\
&= f_{\tau([m]_r)}(x(\ell+m+j), \cdots, x(\ell+k+m+j)) \\
&= f_{\eta(\sigma^j([m]_r))}(x(\ell+m+j), \cdots, x(\ell+k+m+j)) \\
&= f_{\eta([m+j]_r)}(x(\ell+m+j), \cdots, x(\ell+k+m+j)) \\
&= F_\eta(x)(m+j) \\
&= (\sigma^j \circ F_\eta)(x)(m).
\end{aligned}$$

Thus F_τ and F_η have the same topological dynamics. □

The following corollary is an obvious consequence of the number of cyclic classes.

Corollary 2.1. *Fixed r local rules $f_0, \cdots, f_{r-1}$, the number of different topological dynamics given by (4) with $\tau \in \mathcal{S}_r$ has $(r-1)!$ as an upper bound.*

Next example shows that topological conjugation is only a necessary condition to belong to the same conjugacy class.

Example 2.2. Consider $\mathcal{A} = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, $\mathbb{V} = \{0\}$ and the local rules $f_0, f_1, f_2 : \mathcal{A} \to \mathcal{A}$ given by

$$f_0(a) = 2a \,(\text{mod } 5), \; f_1(a) = a + 1 \,(\text{mod } 5) \;\text{ and }\; f_2(a) = 2a + 1 \,(\text{mod } 5).$$

The 3-cellular automaton generated by the local rule vector (f_0, f_1, f_2) and the neighborhood $\mathbb{V}$ is defined, for every $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, by

$$F(x)(n) = \begin{cases} 2x(n) \,(\text{mod } 5), & \text{if } n = 0 \,(\text{mod } 3) \\ x(n) + 1 \,(\text{mod } 5), & \text{if } n = 1 \,(\text{mod } 3) \\ 2x(n) + 1 \,(\text{mod } 5), & \text{if } n = 2 \,(\text{mod } 3) \end{cases}.$$

Now we rearrange the local rule vector (f_0, f_1, f_2) according to the permutation $\tau = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$; notice that τ and $i = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ are not equivalent according to definition 2.2.

Clearly the 3-cellular automaton F_τ is defined, for each $x \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, as:

$$F_\tau(x)(n) = \begin{cases} x(n) + 1 \,(\text{mod } 5), & \text{if } n = 0 \,(\text{mod } 3) \\ 2x(n) \,(\text{mod } 5), & \text{if } n = 1 \,(\text{mod } 3) \\ 2x(n) + 1 \,(\text{mod } 5), & \text{if } n = 2 \,(\text{mod } 3) \end{cases}.$$

It is easy to verify that $h : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, given by

$$h(x)(n) = \begin{cases} x(n+1), & \text{if } n = 0 \,(\text{mod } 3) \\ x(n-1), & \text{if } n = 1 \,(\text{mod } 3) \\ x(n), & \text{if } n = 2 \,(\text{mod } 3) \end{cases}$$

defines a homeomorphism on $\mathcal{A}^{\mathbb{Z}}$ and $h \circ F = F_{\tau} \circ h$.

References

[1] I. Arcaya and N. Romero. On a Hedlund's theorem and place-dependent cellular automata. To appear in Divulgaciones Matemáticas.

[2] F. Blanchard, E. Formenti and P. Kurka. Cellular automata in Cantor, Besicovitch and Weyl topological spaces. *Complex Systems*, **11(2)** (1999.) 107-123.

[3] G. Hedlund. Endomorphisms and Automorphims of the shift dynamical systems. *Math. Sys. Th.* **3**, (1969) 320–375.

[4] N. Romero. Dinámica Topológica y Autómatas celulares. *Publicaciones del Postgrado de Matemáticas, UCV* (2006).

Universidad Centroccidental Lisandro Alvarado. Departamento de Matemática. Decanato de Ciencias y Tecnología. Apartado Postal 400. Barquisimeto, Venezuela.
E-mail address: minoru_ak@yahoo.com

Universidad Central de Venezuela. Facultad de Ciencias. Caracas, Venezuela.
E-mail address: ignacia.arcaya@gmail.com

Universidad Centroccidental Lisandro Alvarado. Departamento de Matemática. Decanato de Ciencias y Tecnología. Apartado Postal 400. Barquisimeto, Venezuela.
E-mail address: nromero@uicm.ucla.edu.ve

Universidad Centroccidental Lisandro Alvarado. Departamento de Matemática. Decanato de Ciencias y Tecnología. Apartado Postal 400. Barquisimeto, Venezuela.
E-mail address: jsilva@uicm.ucla.edu.ve