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Nest and Complete Accumulation Point Compactness of the Product of Topological Spaces

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Abstract

Based on the definition of nest compactness (i.e., the intersection of a nest of nonempty closed sets is nonempty) we show that the product of the two nest compact topological spaces is nest compact, and, this without invoking the compactness of the product of two compact topological spaces based on the classical definition of compactness (i.e., every open cover has a finit subcover). The same is done based on the definition of complete accumulation point compactness. The latter, by Remark 5, extends easily to the infinite products of topological spaces.

In what follows, by a ***nest*** we mean a family of sets *well ordered* by the reverse inclusion $\supseteq$. Moreover, we call a topological space K *nest compact* iff every nest of nonempty closed sets of K has a nonempty intersection. Furthermore, we call a topological space K *classically compact* iff every cover of K by its open sets has a finite subcover.

We recall that the product topology $T_1 \times T_2$ of two topological spaces T_1 and T_2 is defined in terms of its open basis, i.e., the set of all $u \times v$'s where u is an open set of T_1 and v is an open set of T_2. Thus, in dealing with a product topology $T_1 \times T_2$ the use of the basic open sets $u \times v$'s of $T_1 \times T_2$ is "*inevitable*".

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We prove bellow that the product $K_1\times K_2$ of two nest compact topological spaces K_1 and K_2 is nest compact. We prove this without ever using the classical compactness of $K_1\times K_2$. However, the abovementioned "inevitability" manifests itself on two occasions where we invoke the classical compactness of each of the nest compact topological spaces K_1 and K_2 (but, as mentioned, not of their product $K_1\times K_2$).

Let us observe however, that our abovementioned invokation is justified by Lemma 1 below [cf. 1, p. 163] which is proved here perhaps in the shortest and simplest way.

Lemma 1. *If K is a nest compact topological space then K is classically compact.*

Proof. Assume on the contrary and let $(V_i)_{i\in H}$ be an open cover of K of the smallest cardinality H such that $(V_i)_{i\in H}$ has no finite subcover. Therefore H is an infinite cardinal. Consequently, the well ordered (by inclusion $\subseteq$) family $W=\{V_0,(V_0\cup V_1),\dots,(V_0\cup V_1\cup\dots\cup V_r),\dots\}$ with $r\in H$, is a family of *proper* open sets of K wich cover K. But then the family $\{V_0',(V_0'\cup V_1'),\dots,(V_0'\cup V_1'\cup\dots\cup V_r'),\dots\}$ of the complements of the elements of W is a nest of nonempty closed sets of K with an empty intersection, wich is a contradiction. Thus, our assumption is false and the Lemma is proved. ∎

Remark 1. *The converse de Lemma 1 is also valid. However, we do not need this fact.*

Theorem 1. *Let K_1 and K_2 be nest compact topological spaces. Then the product topological space $K_1\times K_2$ is also nest compact.*

Proof. Let $(C_i)_{i\in D}$ be a nest of nonempty closed sets C_i of $K_1\times K_2$. Let us assume on the contrary that $\cap_{i\in D}C_i$ is empty.

Let $x\in K_1$. Clearly, $\{x\}\times K_2$ is homeomorphic to K_2, thus $\{x\}\times K_2$ is nest compact. We claim that $\{x\}\times K_2$ is disjoint from at least one term, say, C_k of $(C_i)_{i\in D}$. This is so because otherwise $((\{x\}\times K_2)\cap C_i)_{i\in D}$ would be a nest of nonempty closed sets of $\{x\}\times K_2$ and therefore would have a nonempty intersection, which, in turn would imply that $\cap_{i\in D}C_i$ has a nonmepty intersection, contrading our assumption.

Since $\{x\}\times K_2$ is disjoint form the closed subset C_k of $K_1\times K_2$, there exists an open set V of $K_1\times K_2$ such that $\{x\}\times K_2\subseteq V$ and V is disjoint from

C_k. But then V is a union of basic open sets $u_i \times v_j$ of $K_1 \times K_2$ (where u_i is an open set of K_1 and v_j is an open set of K_2). However, by Lemma 1, since $\{x\} \times K_2$ is classically compact (here is our first invocation of Lemma 1) a finite number of basic open sets, say, $u_h \times v_t, \ldots, u_m \times v_n$ already cover $\{x\} \times K_2$. Thus, the open tube $u(x) = (u_h \cap \cdots \cap u_m) \times K_2$ of $K_1 \times K_2$ is disjoint from the term C_k of $(C_i)_{i \in D}$, where, of course, $x \in (u_h \cap \cdots \cap u_m)$.

Therefore, for every $x \in K_1$ there exists an open tube $u(x) \times K_2$ such that $u(x) \times K_2$ is disjoint from some term of $(C_i)_{i \in D}$. Clearly, the $u(x)$'s form an open cover of K_1. However, by Lemma 1, since K_1 is classically compact (here is our second invocation of Lemma 1) a finit number of open sets, say, $u(x_p), \ldots, u(x_q)$ already cover K_1. Hence,

$$u(x_p) \times K_2 \cup \cdots \cup u(x_q) \times K_2 = K_1 \times K_2. \tag{1}$$

However, each open tube in (1) is disjoint from a term of the nest $(C_i)_{i \in D}$ and therefore, from (1) it follows, that the intersection of corresponding finite number of terms of $(C_i)_{i \in D}$ must be empty. This contradicts the fact that $(C_i)_{i \in D}$ is a nest of nonempty closed sets of $K_1 \times K_2$ (and therefore no finite number of terms of $(C_i)_{i \in D}$ can have an empty intersection). Thus, our assumption is false and the Theorem is proved. ∎

Remark 2. *The converse the Theorem 1 is also valid, i.e., if $K_1 \times K_2$ is nest compact then each K_1 and K_2 is also nest compact. This follows from the fact that if $(S_i)_{i \in E}$ is a nest of nonempty closed sets of, say, K_1 then $(S_i \times K_2)_{i \in E}$ is a nest of nonempty closed sets of $K_1 \times K_2$ and therefore $\cap_{i \in E}(S_i \times K_2)$ as well as $\cap_{i \in E} S_i$ is nonempty.*

Next, let us recall [cf. 1, p. 163] that in a topological space T, a point p is called a *complete accumulation point* of an infinite sequence $(s_i)_{i \in D}$ (where D is an infinite cardinal) iff every neighborhood of p contains D terms of $(s_i)_{i \in D}$. Morevoer, we call a topological space A *complete accumulation point compact* (or CAP–*compact*, for short) iff every infinite sequence in A has a complete accumulation point.

We prove below that the product $A_1 \times A_2$ of two CAP–compact topological spaces A_1 and A_2 is CAP–compact. Again, nowhere in our proof do we use the classcical compactness of $A_1 \times A_2$. Again, however, as mentioned earlier, the use of classical compactness of A_1 or A_2 in the proof is inevitable. In fact, as our proof (of Theorem 2) shows, here we invoke only once the classical compactness of only A_1. We observe again that our invokation is justified by Lemma 2 below [cf. 1, p.163] wich we prove in a short and simple way.

As mentioned in Remark 5, the proof of Theorem 2 extends most conveniently to the case of a infinite product of CAP–compact topological spaces.

Lemma 2. *If A is a CAP–compact topological space then A is classically compact.*

Proof. Assume on the contrary and let $(V_i)_{i \in H}$ be an open cover of A of the smallest cardinality H such that $(V_i)_{i \in H}$ has no finite subcover. Therefore H is an infinite cardinality. Without loss of generality we may assume that $W = \{V_0, (V_0 \cup V_1), \ldots, (V_0 \cup V_1 \cup \cdots \cup V_r), \ldots\}$ with $r \in H$ is a family of distinct open subsets of A wich cover A. But then $(s_i)_{i \in H}$ with $s_i \in (V_{i+1} - V_i)$ is an infinite sequence en A without a complete accumulation point since W covers A. But this contradicts the hypotesis that A is CAP–compact. Thus our assumption is false and the Lemma is proved. ∎

Remark 3. *The converse the Lemma 2 is also valid. However, we don not need this fact.*

Theorem 2. *Let A_1 and A_2 be CAP–compact topological spaces. Then the product topological space $A_1 \times A_2$ is also CAP–compact.*

Proof. Let $((a_i, b_i))_{i \in D}$ be a sequence in $A_1 \times A_2$ where D is an infinite cardinal. Then $(a_i)_{i \in D}$ is an infinite sequence in A_1. Since A_1 is CAP–compact, $(a_i)_{i \in D}$ has a complete accumulation point a in A_1. Hence, if $u(a)$ is a neighborhood of a in A_1, we have

$$u(a) \text{ contains } D \text{ terms of } (a_i)_{i \in D}. \tag{2}$$

We claim that there exists a point b of A_2 such that (a, b) is a complete accumulation point of $((a_i, b_i))_{i \in D}$ in $A_1 \times A_2$ which would imply that $A_1 \times A_2$ is CAP–compact, as required.

Let us assume the contrary. Thus, for every point y of A_2 there exists basic open sets $u_y \times v_y$ of $A_1 \times A_2$ such that $(a, b) \in (u_y \times v_y)$ and such that for every $y \in A_2$ we have

$$u_y \times v_y \text{ contains less than } D \text{ terms of } ((a_i, b_i))_{i \in D}. \tag{3}$$

Clearly, $(u_y \times v_y)_{y \in A_2}$ is an open cover of $\{a\} \times A_2$ which is CAP–compact since it is homeomorphic to A_2. However, by Lemma 2, since A_2 is CAP–compact (here is our one and only invokation of Lemma 2) a finite number of open sets, say, $u_h \times v_t, \ldots, u_m \times v_n$ already cover $\{a\} \times A_2$. Obviously, $u(a) = u_h \cap \cdots \cap u_m$ is a neighborhood of a in A_1. Thus, by (2) we have

$$u(a) \times A_2 \text{ has } D \text{ terms of } \left((a_i, b_i)\right)_{i \in D}. \tag{4}$$

On the other hand, $u(a) \times A_2 \subseteq (u_h \times v_t) \cup \cdots \cup (u_m \times v_n)$ and therefore by (3) we see that $u(a) \times A_2$ has D terms of $((a_i, b_i))_{i \in D}$, which contradicts (4). Thus, our assumption is false and the Theorem is proved. ∎

Remark 4. *The converse of Theorem 2 is also valid, i.e., if $A_1 \times A_2$ is CAP–compact then each A_1 and A_2 is also CAP–compact. This follows from the fact that if $(a_i)_{i \in E}$ is an inifinite sequence in, say, A_1 and b is a point in A_2 then $((a_i, b))_{i \in E}$ is an infinite sequence in $A_1 \times A_2$ and therefore has a complete accumulation point (a, b) in $A_1 \times A_2$. Clearly, a is a complete accumulation point of $(a_i)_{i \in E}$ as required.*

Remark 5. *The proof of Theorem 2 can be extended most conveniently to the proof of the CAP–compactness of an infinite product $\prod_{j \in W} A_j$ of CAP-comapct topological spaces A_j where W is an infinite cardinal. This is done as follows.*
Let $S = \left(\left(a_0^i, a_i^1, a_i^2, \ldots, a_i^j, \ldots\right)\right)_{i \in D}$ with $j \in W$ be a sequence in A where D is an infinite cardinal. Then the existence of a complete accumulation point $p = (p^0, p^1, p^2, \ldots, p^j, \ldots)$ with $j \in W$ of S in A is proved (based on Theorem 2) using transfinite induction by requiring that the j–th entry p^j of p satisfy the following condition:
For every finite subset F of ordinals $\leq j$ and every neighborhood $u\left(p^k\right)$ of p^k in A^k the open tube

$$\prod_{j \in W} B_j \quad \text{where} \begin{cases} B_j = u\left(p^k\right), & \text{if } j \in F, \\ B_j = A_j, & \text{if } j \notin F \end{cases}$$

contains D terms of S.

References

[1] Kelley J. L, *General Topology*, D. Van Nostrand Co. New York, 1986.