Pasting and Reversing Operations over some Vector Spaces

Operaciones Pegar y Reversar sobre algunos espacios vectoriales

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Abstract. Pasting and Reversing operations have been used successfully over the set of integer numbers, simple permutations, rings and recently over a generalized vector product. In this paper, these operations are defined from a natural way to be applied over vector spaces. In particular we study Pasting and Reversing over vectors, matrices and we rewrite some properties for polynomials as vector space. Finally we present some properties concerning to palindromic and antipalindromic vector subspaces.

Keywords: Antipalindromic matrix, antipalindromic vector, palindromic matrix, palindromic vector, Pasting, Reversing.

Resumen. Operaciones Pegar y Reversar han sido aplicadas satisfactoriamente sobre el conjunto de números enteros, permutaciones simples, anillos y recientemente sobre un producto vectorial generalizado. En este artículo, estas operaciones son definidas de una manera natural para ser aplicadas sobre espacios vectoriales. En particular estudiamos Pegar y Reversar sobre vectores y matrices. También reescribimos algunas propiedades de los polinomios, vistos como espacio vectorial. Finalmente presentamos algunas propiedades de los denominados subespacios vectoriales palíndromes y antipalíndromes.

Palabras claves: Matrices palíndromes, matrices antipalíndromes, Pegar, Reversar, vectores palíndromes, vectores antipalíndromes.

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1. Introduction

Pasting and *Reversing* are natural processes that people do day after day, we *paste* two objects when we put them together as one object, and we *reverse* one object when we reflect it over a symmetry axis. We can apply these processes over words, thus, *Pasting* of *lumber* with *jack* is *lumberjack*, while *Reversing* of *lumber* is *rebmul.* A celebrate phrase of Albert Einstein is *I prefer pi*, in which *Reversing* of this phrase is itself and for instance

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this is a *palindromic* phrase, another palindromic phrases can be found in `http://www.palindromelist.net`.

Similarly to palindromic phrases, we can think in palindromic poetries, where each line can be palindromic or the hole poetry is palindromic. The following poems can be found at
`http://www.trauerfreuart.de/palindrome-poems.htm`

Deific- A poem

Same deficit sale: doom mood. Elastic if edemas?
Loops secreting in a doom mood. An igniter: cesspool.
Set agony care till in a doom mood. An illiteracy: no gates.
Senile fileting: I am, God, doom mood. Dogma: ignite lifelines.
Straws? Send a snowfield in a doom mood. An idle, if won sadness warts.
Me, opacified.

Put in us - sun it up

Put in rubies, I won't be demandable.
Balderdash: sure fire bottle fill-in.
Raw, put in urn action, I'm odd.
Local law: put in ruts. Awareness
elates pure gnawed limekiln. Us:
sunlike, mildew, anger, upset.
A lessen era was: turn it up! Wall,
a cold domino: it can run it up.
Warn: ill I felt to be rife. Rush! Sad.
Red label, bad name, debt nowise.
I burn it up

Space caps

Seed net: tabard. No citadel like sun is
but spirit. Sense can embargo to get on.
Still amiss: a pyro-memoir, an ecstasy.
A detail, if fades, paler, tall, affined dusk.
Row no risks, asks ironwork, sudden -
if fall at relapsed affiliate - days at scenario:
memory, pass! I'm all. It's not ego to grab
menaces. Nest. I rip stubs in use, killed at icon. Drab attendees.

One mathematical theory to express these processes as operations was developed by the first author in [2, 1], followed recently by [3, 4, 5].

In [1] is introduced the concept *Pasting* of positive integers to obtain their squares as well their squares roots. Five years later, in [2], are defined in a general way the concepts *Pasting* and *Reversing* to obtain genealogies of *simple permutations* in the right block of *Sarkovskii order* which contains the powers of two. Two years later, in [4], were applied *Pasting* and *Reversing*, as well *palindromicity* and *antipalindromicity*, over the ring of polynomials, differential rings and mathematical games incoming from M. Tahan's book *The man who counted*. Another approaches for *reversed* polynomials, *palindromic* polynomials and *antipalindromic* polynomials can be found in [6, 11]. One year later, in [3], is applied *Reversing* over matrices to study a generalized vector product, in particular were studied relationships between *palindromicity* and *antipalindromicity* with such generalized vector product. Finally, in the preprint [5] were applied *Pasting* and *Reversing* over *simple permutations* with mixed order $4n+2$, following [2].

The aim of this paper is to study *Pasting* and *Reversing*, as well *palindromicity* and *antipalindromicity*, over vector spaces (vectors, matrices, polynomials, etc.). Some properties are analyzed for vectors, matrices and polynomials as vector spaces, in particular we prove that $W_a \subset V$ (set of antipalindromic vectors of V) and $W_p \subset V$ (set of palindromic vectors of V) are vector subspaces of a vector space V. Recall that V is the direct sum of W_1 and W_2, denoted by $V = W_1 \oplus W_2$, whenever $V = \text{span}\{W_1 \cup W_2\}$ and $W_1 \cap W_2 = \{0\}$. Therefore, $V = W_a \oplus W_p$ and in consequence $\dim(V) = \dim(W_a) + \dim(W_p)$.

The reader does not need a high mathematical level to understand this paper, is enough with a basic knowledge of linear algebra and matrix theory, see for example the books given in references [7, 10]. Finally, as *butterfly effect*, we hope that the results and approaches presented here can be used and implemented in the teaching of basic linear algebra for undergraduate level.

2. Pasting and Reversing over Vectors

In this section we study Pasting and Reversing over vectors through basic definitions and properties. We consider the field K and the vector space $V = K^n$. Here we study Pasting and Reversing over vectors using the basic definitions and properties of vectors. In this way, any student beginner of linear algebra can understand the results presented. We start giving the definition of Reversing.

Definition 2.1. Let be $v = (v_1, v_2, \dots, v_n) \in K^n$. Reversing of v, denoted by $\widetilde{v}$, is given by $\widetilde{v} = (v_n, v_{n-1}, \dots, v_1)$.

Definition 2.1 leads us to the following proposition.

Proposition 2.2. *Consider v and $\widetilde{v}$ as in Definition 2.1. The following statements hold:*

(i) $\tilde{\tilde{v}} = v$.

(ii) $\widetilde{av + bw} = a\widetilde{v} + b\widetilde{w}$, *being* $a, b \in K$ *and* $v, w \in V$.

(iii) $v \cdot w = \widetilde{v} \cdot \widetilde{w}$.

(iv) $\widetilde{(v \times w)} = \widetilde{w} \times \widetilde{v}$ *for all* $v, w \in K^3$.

Proof. (i), (ii) and (iii) follow from the definition.

(iv) Consider $v, w \in K^3$, where $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$. The vector product between v and w is given by

$$v \times w = \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right),$$

by Definition 2.1 we have that

$$\widetilde{v \times w} = \left(\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \right).$$

Now, by properties of determinants (interchanging rows and columns) we obtain

$$\left(\begin{vmatrix} w_2 & w_1 \\ v_2 & v_1 \end{vmatrix}, - \begin{vmatrix} w_3 & w_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} w_3 & w_2 \\ v_3 & v_2 \end{vmatrix} \right) = \begin{vmatrix} e_1 & e_2 & e_3 \\ w_3 & w_2 & w_1 \\ v_3 & v_2 & v_3 \end{vmatrix},$$

therefore $\widetilde{v \times w} = \widetilde{w} \times \widetilde{v}$.

□

Definition 2.3. The vectors v and w are called palindromic vector and antipalindromic vector respectively whether they satisfy $\widetilde{v} = v$ and $\widetilde{w} = -w$.

The proof of following proposition is a routine exercise.

Proposition 2.4. *The following statements hold.*

(i) *The sum of two palindromic vectors belonging to* K^n *is a palindromic vector belonging to* K^n.

(ii) *The sum of two antipalindromic vectors belonging to* K^n *is an antipalindromic vector belonging to* K^n.

(iii) *The vector product of two palindromic vectors belonging to* K^3 *is an antipalindromic vector belonging to* K^3.

(iv) *The vector product of two antipalindromic vectors belonging to* K^3 *is the vector* $(0, 0, 0)$.

(v) *The vector product of one palindromic vector belonging to* K^3 *with one antipalindromic vector belonging to* K^3 *is a palindromic vector belonging to* K^3.

Remark 2.5. In [3] were studied, in a more general way, the vector product for palindromic and antipalindromic vectors. There was used a generalized vector product and were obtained some results involving the palindromicity and antipalindromicity of vectors. For completeness we present in Section 4 such results with proofs in detail.

Now we proceed to introduce the concept of Pasting over vectors.

Definition 2.6. Consider $v \in K^n$ and $w \in K^m$, then $v \diamond w$ is given by

$$(v_1, v_2, \dots, v_n) \diamond (w_1, w_2, \dots, w_m) = (v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_m).$$

Proposition 2.7. *If* $V = K^n$ *and* $W = K^m$, *then* $V \diamond W \cong K^{n+m}$.

Proof. Let $B_n = \{b_1, b_2, \dots, b_n\}$ and $B_m = \{c_1, c_2, \dots, c_m\}$ basis of K^n and K^m respectively. Due to $v \in K^n$ and $w \in K^n$, we have by Definition 2.6 that $v \diamond w \in K^{n+m}$, then there exists a basis $B_{n+m} = \{d_1, d_2, \dots, d_{n+m}\}$ belonging to K^{n+m}, therefore $K^n \diamond K^m \cong K^{n+m}$. □

Corollary 2.8. $dim(V \diamond W) = dimV + dimW$.

Proposition 2.9. *The following statements hold.*

(i) $\widetilde{v \diamond w} = \tilde{w} \diamond \tilde{v}$.

(ii) $(v \diamond w) \diamond z = v \diamond (w \diamond z)$.

Proof. The proof is left as an exercise to the reader. □

Proposition 2.10. *Let* V *be a vector space. Consider* W_p *and* W_a *as the sets of palindromic and antipalindromic vectors of* V *respectively. The following statements hold.*

(i) W_p *is a vector subspace of* V.

(ii) $\dim W_p = \lceil \frac{n}{2} \rceil$.

(iii) W_a *is a vector subspace of* V.

(iv) $\dim W_a = \lfloor \frac{n}{2} \rfloor$.

(v) $V = W_p \oplus W_a$.

(vi) $\forall v \in V$, $\exists (w_p, w_a) \in W_p \times W_a$ *such that* $v = w_p + w_a$.

Proof. (i) Set $a, b \in K$. Since $v, w \in W_p$, we have $v = \tilde{v}$ and $w = \tilde{w}$. By Proposition 2.2 $\widetilde{av + bw} = a\tilde{v} + b\tilde{w} = av + bw \in W_p$, in consequence, W_p is a vector subspace of V.

(ii) We analyze the cases when n is even and also when n is odd.

(a) Consider $V = K^n$ and we start assuming that $n = 2k$. If $v \in W_p$, then

$$(v_1, v_2, \dots, v_{2k-1}, v_{2k}) = (v_{2k}, v_{2k-1}, \dots, v_2, v_1),$$

then, we have that

$$\begin{array}{rcl} v_1 & = & v_{2k} \\ v_2 & = & v_{2k-1} \\ & \vdots & \\ v_k & = & v_{k+1}, \end{array}$$

which lead us to $v = (v_1, v_2, \dots, v_k, v_k, \dots, v_2, v_1)$. In this way we write the vector v as follows:

$$v = v_1(1, 0, \dots, 0, 0, \dots, 1) + \dots + v_k(0, 0, \dots, 1, 1, \dots, 0, 0), \quad v_i \in K.$$

The set of vectors of the previous linear combination are palindromic and linearly independent vectors, therefore they are a basis for W_p and in consequence

$$\dim W_p = k = \left\lceil \frac{2k}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

(b) Consider $V = K^n$ and now we assume that $n = 2k - 1$. If $v \in W_p$, then

$$(v_1, v_2, \dots, v_{2k-2}, v_{2k-1}) = (v_{2k-1}, v_{2k-2}, \dots, v_2, v_1).$$

Thus, we have that

$$\begin{array}{rcl} v_1 & = & v_{2k-1} \\ v_2 & = & v_{2k-2} \\ & \vdots & \\ v_{k-1} & = & v_{k+1}, \end{array}$$

that is, $k - 1$ pairs plus the fixed component v_k. This lead us to express the vector v as follows

$$v = (v_1, v_2, \dots, v_{k-1}, v_k, v_{k-1}, \dots, v_2, v_1)$$

and then we have that

$$v = v_1(1, 0, \dots, 0, 0, \dots, 1) + \dots + v_k(0, 0, \dots, 0, 1, 0, \dots, 0, 0), \quad v_i \in K.$$

The set of vectors of the previous linear combination are palindromic and linearly independent vectors, hence they are a basis for W_p and in consequence

$$\dim W_p = k = \left\lceil \frac{2k-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

In this way, we have proved that for all $n \in \mathbb{Z}^+$, $\dim W_p = \lceil \frac{n}{2} \rceil$.

(iv) We analyze the cases when n is even as well when n is odd.

(a) Consider $V = K^n$ and we can start assuming that $n = 2k$. If $v \in W_a$, then

$$(v_1, v_2, \dots, v_{2k-1}, v_{2k}) = -(v_{2k}, v_{2k-1}, \dots, v_2, v_1),$$

therefore, we have that

$$\begin{array}{rcl} v_1 & = & -v_{2k} \\ v_2 & = & -v_{2k-1} \\ & \vdots & \\ v_k & = & -v_{k+1}, \end{array}$$

which lead us to

$$v = (v_1, v_2, \dots, v_k, -v_k, \dots, -v_2, -v_1).$$

This implies that

$$v = v_1(1, 0, \dots, 0, 0, \dots, -1) + \dots + v_k(0, 0, \dots, 1, -1, \dots, 0, 0), \quad v_i \in K.$$

The set of vectors of the previous linear combination are antipalindromic and linearly independent vectors, thus they are a basis for W_a and in consequence

$$\dim W_a = k = \left\lfloor \frac{2k}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.$$

(b) Consider $V = K^n$ and now we suppose that $n = 2k - 1$. If $v \in W_a$, then

$$(v_1, v_2, \dots, v_{2k-2}, v_{2k-1}) = -(v_{2k-1}, v_{2k-2}, \dots, v_2, v_1).$$

Thus, we obtain that

$$\begin{array}{rcl} v_1 & = & -v_{2k-1} \\ v_2 & = & -v_{2k-2} \\ & \vdots & \\ v_{k-1} & = & -v_{k+1}, \end{array}$$

that is, $k - 1$ pairs plus the fixed component $v_k = 0$. This lead us to express the vector v as follows:

$$v = (v_1, v_2, \dots, v_{k-1}, 0, -v_{k-1}, \dots, -v_2, -v_1)$$

and for instance we have that

$$v = v_1(1, 0, \dots, 0, 0, \dots, -1) + \dots + v_{k-1}(0, 0, \dots, 1, 0, -1, \dots, 0, 0), \quad v_i \in K.$$

The set of vectors of the previous linear combination are antipalindromic and linearly independent vectors, for instance they are a basis for W_a and in consequence

$$\dim W_a = k - 1 = \left\lfloor \frac{2k-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.$$

In this way we have proved that for all $n \in \mathbb{Z}^+$, $\dim W_a = \left\lfloor \frac{n}{2} \right\rfloor$.

(v) Since $W_p \cap W_a = \{\mathbf{0}\}$ and $\dim W_p + \dim W_a = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n = \dim V$, we have $W_p \oplus W_a = V$.

(vi) Consider $v \in V$, we can observe that

$$w_p = \frac{v + \widetilde{v}}{2}$$

is a palindromic vector. In the same way we can observe that

$$w_a = \frac{v - \widetilde{v}}{2}$$

is an antipalindromic vector and for instance $v = w_p + w_a$, $\forall v \in V$.

□

3. Pasting and Reversing over Polynomials

In [4] we studied Pasting and Reversing over polynomials from an different approach, we studied these operations focusing on the ring structure for polynomials. In this section we rewrite some properties of Pasting and Reversing over polynomials, but considering to the polynomials as a vector space. Thus, we apply the previous results for vectors, which we gave in Section 2.

Along this section we consider $(K_n[x], +, \cdot)$ as the vector space of the polynomials of degree less than or equal to n over the field K. This vector space is isomorphic to $(K^{n+1}, +, \cdot)$. In this context we do not impose conditions over the polynomials just like the conditions given in [4], for example, we do not need that $x \nmid P(x)$. The following result summarizes the properties given in Section 2 for polynomials.

Proposition 3.1. *Consider $P \in K_n[x]$, $Q \in K_m[x]$ and $R \in K_s[x]$, the following statements hold.*

(i) $\tilde{\tilde{P}} = P$.

(ii) $\widetilde{P \diamond Q} = \tilde{Q} \diamond \tilde{P}$.

(iii) $(P \diamond Q) \diamond R = P \diamond (Q \diamond R)$.

(iv) $a\widetilde{P + b}Q = a\widetilde{P} + b\widetilde{Q}$, *being* $a, b \in K$ *and* $P, Q \in K_n[x]$.

(v) The sum of two palindromic polynomials of degree n is a palindromic polynomial of degree n.

(vi) The sum of two antipalindromic polynomials of degree n is an antipalindromic polynomial of degree n.

(vii) If $V = K_n[x]$ *and* $W = K_m[x]$*, then* $V \diamond W = K_{n+m+1}[x]$.

(viii) W_p *is vector subspace of* $K_n[x]$*, being* W_p *the set of palindromic polynomials of degree* $K_n[x]$.

(ix) $\dim W_p = \lceil \frac{n+1}{2} \rceil$.

(x) W_a *is a vector subspace of* $K_n[x]$*, being* W_a *the set of antipalindromic polynomials of* $K_n[x]$.

(xi) $\dim W_a = \lfloor \frac{n+1}{2} \rfloor$.

(xii) $K_n[x] = W_p \oplus W_a$.

(xiii) $\forall P \in K_n[x]$, $\exists (Q_p, Q_a) \in W_p \times W_a$ *such that* $P = Q_p + Q_a$.

Proof. Owing to $(K_n[x], +, \cdot) \cong (K^{n+1}, +, \cdot)$ as vector spaces we apply $\widetilde{v}$ and $\diamond$ over the polynomials as vectors. The proof is done using properties of Pasting and Reversing proved in Section 1. □

Remark 3.2. As we can see, this section is a rewriting of Section 2 without new results for polynomials as vector space, only we suggest the proofs based on the definition and properties of $\widetilde{v}$. Another interesting thing of this section is that we recover some results given in [4, 11].

4. Pasting and Reversing over Matrices

In this section we consider the vector space $\mathcal{M}_{n\times m}$ (matrices of size $n \times m$ with elements belonging to K) which is isomorphic to K^{nm}. We present here different approaches for Pasting and Reversing.

4.1. Pasting and Reversing by rows or columns

We can see any matrix as a row vector of its column vectors, as well as a column vector of its row vectors. Thus, to matrices we can introduce *Pasting and Reversing* by rows and columns respectively. Let us denote by $\widetilde{A}_r$ Reversing of the row vectors $v_i \in K^n$ of A and, by $\widetilde{A}_c$ Reversing of the column vectors $c_j \in K^m$ of A, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Then,

$$\widetilde{A}_r = \begin{pmatrix} \widetilde{v_1} \\ \widetilde{v_2} \\ \vdots \\ \widetilde{v_n} \end{pmatrix}, \quad \widetilde{A}_c = \begin{pmatrix} \widetilde{c_1} & \widetilde{c_2} & \cdots & \widetilde{c_m} \end{pmatrix}.$$

Owing to $\widetilde{v}_i = v_i \widetilde{I}_m$ and $\widetilde{c}_i = \widetilde{I}_n c_j$ for $1 \leq i \leq n$, $1 \leq j \leq m$ we obtain that $\widetilde{A}_r = A\widetilde{I}_m$ and $\widetilde{A}_c = \widetilde{I}_n A$. Therefore we can define palindromicity and antipalindromicity by rows and columns respectively.

Now, we can assume $A \in \mathcal{M}_{n\times m}(K)$, $B \in \mathcal{M}_{q\times m}(K)$ and $C \in \mathcal{M}_{n\times p}(K)$ given as follows.

$$A = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & \cdots & f_m \end{pmatrix}, \quad v_i \in K^m,\, f_j^T \in K^n,\, 1 \leq i \leq n,\, 1 \leq j \leq m,$$

$$B = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_q \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_m \end{pmatrix}, \quad s_i \in K^m,\, g_j^T \in K^q,\, 1 \leq i \leq q,\, 1 \leq j \leq m,$$

$$C = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & \cdots & h_p \end{pmatrix}, \quad w_i \in K^p,\, h_j^T \in K^n,\, 1 \leq i \leq n,\, 1 \leq j \leq p.$$

As we can see, in agreement with Section 2, we transformed the column vectors f_j, g_j and h_j in the form of row vectors through the transposition of matrices (f_j^T, g_j^T and h_j^T are row vectors). Thus, we can define both *Pasting by rows* (denoted by $\diamond_r$) over the matrices A and C and *Pasting by columns* (denoted by $\diamond_c$) over the matrices A and B as follows.

$$A \diamond_r C = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad z_i = v_i \diamond w_i, \quad A \diamond_c B = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}, \quad y_i^T = f_i^T \diamond g_i^T.$$

From now on we paste column vectors directly without the use of trasposition of vectors. Thus, Pasting of column vectors f_i and g_i is $f_i \diamond g_i$. Therefore, $f_i^T \diamond g_i^T = (f_i \diamond g_i)^T$.

Proposition 4.1. *Consider matrices A, B and C under the previous assumptions. The following statements hold.*

(i) $\widetilde{\widetilde{A}}_r = A$, $\widetilde{\widetilde{A}}_c = A$.

(ii) $\widetilde{(A \diamond_r B)}_r = (\widetilde{B}_r) \diamond_r (\widetilde{A}_r)$, $\widetilde{(A \diamond_c B)}_c = (\widetilde{B}_c) \diamond_c (\widetilde{A}_c)$.

(iii) $(A \diamond_r B) \diamond_r C = A \diamond_r (B \diamond_r C)$, $(A \diamond_c B) \diamond_c C = A \diamond_c (B \diamond_c C)$.

(iv) $\widetilde{(\alpha A + \beta B)}_r = \alpha \widetilde{A}_r + \beta \widetilde{B}_r$, $\widetilde{(\alpha A + \beta B)}_c = \alpha \widetilde{A}_c + \beta \widetilde{B}_c$, $\alpha, \beta \in K$.

(v) *If* $V = \mathcal{M}_{n\times m}(K)$ *and* $W = \mathcal{M}_{n\times p}(K)$, *then* $V \diamond W = \mathcal{M}_{n\times(m+p)}(K)$. *In the same way, if* $T = \mathcal{M}_{n\times m}(K)$ *and* $S = \mathcal{M}_{l\times m}(K)$, *then* $T \diamond S = \mathcal{M}_{(n+l)\times m}(K)$.

(vi) *Let* W_p^r *and* W_p^c *be the set of palindromic matrices by rows and columns of* $\mathcal{M}_{n\times m}(K)$ *respectively, then the sets* W_p^r *and* W_p^c *are vector subspaces of* $\mathcal{M}_{n\times m}(K)$.

(vii) $\dim W_p^r = n \left\lceil \frac{m}{2} \right\rceil$, $\dim W_p^c = m \left\lceil \frac{n}{2} \right\rceil$.

(viii) *Let* W_a^r *and* W_a^c *be the set of antipalindromic matrices by rows and columns of* $\mathcal{M}_{n\times m}(K)$ *respectively, then* W_a^r *and* W_a^c *are vector subspaces of* $\mathcal{M}_{n\times m}(K)$.

(ix) $\dim W_a^r = n \left\lfloor \frac{m}{2} \right\rfloor$, $\dim W_a^c = m \left\lfloor \frac{n}{2} \right\rfloor$.

(x) *The sum of two palindromic matrices by rows (resp. by columns) of the same vector space is a palindromic matrix by rows (resp. by columns).*

(xi) *The sum of two antipalindromic matrices by rows (resp. by columns) of the same vector space is an antipalindromic matrix by rows (resp. by columns).*

(xii) $\mathcal{M}_{n\times m}(K) = W_p^r \oplus W_a^r = W_p^c \oplus W_a^c$.

(xiii) $\forall A \in \mathcal{M}_{n\times m}(K), \exists (A_p^r, A_a^r, A_p^c, A_a^c) \in W_p^r \times W_a^r \times W_p^c \times W_a^c$ *such that* $A = A_p^r + A_a^r = A_p^c + A_a^c$.

(xiv) $A \diamond_r B = A((I_n \diamond_c \mathbf{0}_{(n-m)\times m}) \diamond_r \mathbf{0}_{n\times p}) + \mathbf{0}_{n\times m} \diamond_r B$, $A \in \mathcal{M}_{n\times m}(K)$, $B \in \mathcal{M}_{n\times p}(K)$,
$A \diamond_c B = A((I_n \diamond_r \mathbf{0}_{n\times(m-q)}) \diamond_c \mathbf{0}_{n\times p})) + \mathbf{0}_{n\times m} \diamond_c B$, $A \in \mathcal{M}_{n\times m}(K)$, $B \in \mathcal{M}_{p\times m}(K)$.

Proof. From (i) to (xiii) we proceed as in the proofs of Section 2 using the properties of $\widetilde{v}$. (xiv) is consequence of the definition of Pasting by rows and columns. □

Remark 4.2. There are a lot of mathematical software in where Pasting of matrices is very easy, for example, in Matlab Pasting by rows is very easy: $[A, B]$, as well by columns $[A; B]$, however we can build our own program using our approach given in the previous proposition, following the same structure of Pasting of polynomials as in [4]. Thus, we paste matrices by rows and columns using the item (xiv) in Proposition 4.1. The interested reader may proof the statements of this paper concerning to Pasting using such equations.

The following proposition summarizes some properties derived from Pasting and Reversing by rows and columns with respect to classical matrix operations.

Proposition 4.3. *The following statements hold.*

(i) $(\widetilde{A}_r)^T = \widetilde{(A^T)}_c$, $(\widetilde{A}_c)^T = \widetilde{(A^T)}_r$.

(ii) $(A \diamond_c B)^T = A^T \diamond_r B^T$, $(A \diamond_r B)^T = A^T \diamond_c B^T$.

(iii) $\widetilde{(AB)}_r = A(\widetilde{B}_r)$, $\widetilde{(AB)}_c = (\widetilde{A}_c)B$.

(iv) $\det(\widetilde{A}_r) = \det(\widetilde{A}_c) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det A$.

(v) $(\widetilde{A}_c)^{-1} = \widetilde{(A^{-1})}_r$, $(\widetilde{A}_r)^{-1} = \widetilde{(A^{-1})}_c$.

(vi) *The product of two palindromic matrices by rows (resp. by columns) is a palindromic matrix by rows (resp. by columns).*

(vii) The product of two antipalindromic matrices by rows (resp. by columns) is an antipalindromic matrix by rows (resp. by columns).

(viii) $AB \neq \mathbf{0}$ *is a palindromic matrix by rows (resp.* $AB \neq \mathbf{0}$ *is a palindromic matrix by columns) if and only if* B *is a palindromic matrix by rows (resp.* A *is a palindromic matrix by columns).*

(ix) $AB \neq \mathbf{0}$ *is an antipalindromic matrix by rows (resp.* $AB \neq \mathbf{0}$ *is an antipalindromic matrix by columns) if and only if* B *is an antipalindromic matrix by rows (resp.* A *is an antipalindromic matrix by columns).*

Proof. In each item we include the proof of the first claim, the other one is similar.

(i) We see that $\tilde{A}_r = A\widetilde{I}_n = (\widetilde{I}_n^T A^T)^T = (\widetilde{I}_n A^T)^T = (\widetilde{A^T}_c)^T$ and then $\widetilde{A^T}_c = (\widetilde{A}_r)^T$.

(ii) Assume $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{n\times p}(K)$. Let v_i and w_i be the row vectors of A and B respectively. Thus $v_i \diamond w_i$, $i = 1, \dots, n$, are the row vectors of $A \diamond_r B$, then $(v_i \diamond w_i)^T = v_i^T \diamond w_i^T$ are the column vectors of $(A \diamond_r B)^T = A^T \diamond_c B^T$.

(iii) Consider $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{m\times p}(K)$, therefore $\widetilde{(AB)}_c =$
$\widetilde{I}_n(AB) = (\widetilde{I}_n A)B = (\widetilde{A}_c)B$.

(iv) Consider $A \in \mathcal{M}_{n\times n}(K)$, then we obtain $\det(\widetilde{A}_c) = \det(\widetilde{I}_n A) = \det(\widetilde{I}_n)\det A = \det(A)\det(\widetilde{I}_n) = \det(A\widetilde{I}_n) = \det(\widetilde{A}_r)$. Now, it is follows by induction that we can transform $\widetilde{I}_n$ into I_n throught $\lfloor \frac{n}{2} \rfloor$ elementary operations, therefore, $\det(\widetilde{I}_n) = (-1)^{\lfloor \frac{n}{2} \rfloor}$.

(v) Assume $A \in \mathcal{M}_{n\times n}(K)$, with $\det A \neq 0$. Therefore, $(\widetilde{A}_r)^{-1} = (A\widetilde{I}_n)^{-1} = \widetilde{I}_n^{-1}A^{-1} = \widetilde{I}_n A^{-1} = \widetilde{(A^{-1})}_c$.

(vi) Assume $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{m\times p}(K)$, such that $\widetilde{A}_r = A$ and $\widetilde{B}_r = B$, therefore $\widetilde{(AB)}_r = A(\widetilde{B}_r) = AB$.

(vii) Assume $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{m\times p}(K)$, such that $\widetilde{A}_r = -A$ and $\widetilde{B}_r = -B$, therefore $\widetilde{(AB)}_r = A(\widetilde{B}_r) = -AB$.

(viii) Assume $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{m\times p}(K)$, with $AB \neq \mathbf{0}$. By previous item we see that if A is palindromic by rows (resp. if B is palindromic by columns), then AB is palindromic by rows (resp. then AB is palindromic by columns). Now, suppose that $\widetilde{(AB)}_c = AB \neq \mathbf{0}$, then we have $(\widetilde{A}_c)B = (\widetilde{I}_n A)B = AB$, which implies that $\widetilde{A}_c = A$.

(ix) Assume $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{m\times p}(K)$, with $AB \neq \mathbf{0}$. By previous item we see that if A is antipalindromic by rows (resp. if B is antipalindromic by columns), then AB is antipalindromic by rows (resp. then AB is antipalindromic by columns). Now, suppose that $\widetilde{(AB)}_c = -AB \neq \mathbf{0}$, then $(\widetilde{A}_c)B = (\widetilde{I}_n A)B = -AB$, which implies that $\widetilde{A}_c = -A$.

□

Now, in a natural way, we can introduce the sets

$$\mathsf{W}_{pp} := W_p^r \cap W_p^c,\ \mathsf{W}_{pa} := W_p^r \cap W_a^c,\ \mathsf{W}_{ap} := W_a^r \cap W_p^c \text{ and } \mathsf{W}_{aa} := W_a^r \cap W_a^c.$$

The sets W_{pp} and W_{aa} correspond to the set of *double palindromic matrices* and the set of *double antipalindromic matrices* respectively.

Proposition 4.4. *The following statements hold.*

(i) W_{pp}, W_{pa}, W_{ap} *and* W_{aa} *are vector subspaces of* $\mathcal{M}_{n\times m}(K)$.

(ii) $\dim \mathsf{W}_{pp} = \lceil \frac{n}{2} \rceil \lceil \frac{m}{2} \rceil$, $\dim \mathsf{W}_{pa} = \lceil \frac{n}{2} \rceil \lfloor \frac{m}{2} \rfloor$, $\dim \mathsf{W}_{ap} = \lfloor \frac{n}{2} \rfloor \lceil \frac{m}{2} \rceil$ *and* $\dim \mathsf{W}_{aa} = \lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor$.

(iii) $\mathcal{M}_{n\times m}(K) = \mathsf{W}_{pp} \oplus \mathsf{W}_{pa} \oplus \mathsf{W}_{ap} \oplus \mathsf{W}_{aa}$.

(iv) $\forall A \in \mathcal{M}_{n\times m}(K), \exists (A_{pp}, A_{pa}, A_{ap}, A_{aa}) \in \mathsf{W}_{pp} \times \mathsf{W}_{pa} \times \mathsf{W}_{ap} \times \mathsf{W}_{aa}$ *such that* $A = A_{pp} + A_{pa} + A_{ap} + A_{aa}$.

Proof. The intersection of vector subspaces is a vector subspace. The rest is obtained through elementary properties of vector subspaces and by the nature of W_{pp}, W_{pa}, W_{ap} and W_{aa}. □

Finally, according to Remark 2.5, we present the results concerning to the relationship between Reversing and the generalized vector product of $n-1$ vectors of K^n, which are given in [3, §3].

Let $M_1 = (m_{11}, m_{12}, \ldots, m_{1n}), \ldots, M_{n-1} = (m_{n-1,1}, a_{n-1,2}, \ldots, m_{n-1,n})$, be $n-1$ vectors belonging to K^n. It is known that the generalized vector product of these vectors is given by

$$\times (M_1,\ M_2,\ \ldots,\ M_{n-1}) = \sum_{k=1}^{n} (-1)^{1+k} \det \left(M^{(k)} \right) e_k,$$

where e_k is the k-th element of the canonical basis for K^n, and $M^{(k)}$ is the square matrix obtained after deleting of the k-th column of the matrix $M = (m_{ij}) \in \mathcal{M}_{(n-1)\times n}(K)$, for more information see [3, 10]. Therefore, the matrix $M^{(k)}$ is a square matrix of size $(n-1)\times(n-1)$ and is given by

$$M^{(k)} = \left(m_{i,j}^{(k)} \right) = \begin{cases} (m_{i,j}) & \text{if } j < k \\ (m_{i,j+1}) & \text{if } j \geq k \end{cases}, \quad M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \end{pmatrix}.$$

Proposition 4.5. *Consider the matrix* $M = (m_{ij}) \in \mathcal{M}_{(n-1)\times n}(K)$, *then* $\left(\widetilde{M^{(k)}}\right)_r = M^{(n-k+1)} \widetilde{I}_{n-1}$, *for* $1 \leq k \leq n$.

Proof. We know that $\widetilde{M}_r = M\widetilde{I}_n = (m_{i,n-j+1})$, $1 \leq j \leq n$. Therefore

$$\widetilde{M^{(k)}}_r = \widetilde{\left(m_{i,j}^{(k)} \right)}_r = \begin{cases} (m_{i,n-j+1}) & \text{if} j < k, \\ (m_{i,n-(j+1)+1}) & \text{if } j \geq k. \end{cases}$$

On the other hand,

$$M^{(n-k+1)} \widetilde{I}_{n-1} = \left(m_{i,j}^{(n-k+1)} \right) \widetilde{I}_{n-1} = \begin{cases} (m_{i,j})\, \widetilde{I}_{n-1} & \text{if } j < n-k+1 \\ (m_{i,j+1})\, \widetilde{I}_{n-1} & \text{if } j \geq n-k+1 \end{cases}$$

$$= \left(m_{i,(n-1)-j+1}^{(n-k+1)} \right) = \left(m_{i,n-j}^{(n-k+1)} \right) = \begin{cases} \left(m_{i,(n-j)}\right) & \text{if } n-j < n-k+1 \\ \left(m_{i,(n-j)+1}\right) & \text{if } n-j \geq n-k+1 \end{cases}$$

$$= \begin{cases} \left(m_{i,(n-j)}\right) & \text{if } j > k-1 \\ \left(m_{i,(n-j)+1}\right) & \text{if } j \leq k-1 \end{cases} = \begin{cases} (m_{i,n-j}) & \text{if } j \geq k \\ (m_{i,n-j+1}) & \text{if } j < k \end{cases},$$

therefore $\left(\widetilde{M^{(k)}}\right)_r = M^{(n-k+1)} \widetilde{I}_{n-1}$, for $1 \leq k \leq n$. □

Proposition 4.6. *Consider the vectors* $M_i \in K^n$*, where* $1 \leq i \leq n-1$*. Then*

$$\times\left(\widetilde{M_1}_r, \widetilde{M_2}_r, \ldots, \widetilde{M_{n-1}}_r\right) = (-1)^{\left\lceil \frac{3n}{2} \right\rceil}\left(\times(M_1, \widetilde{M_2, \ldots}, M_{n-1})\right)_r.$$

Proof. For convenience we write

$$\mathfrak{M} = \times\left(\widetilde{M_1}_r, \widetilde{M_2}_r, \ldots, \widetilde{M_{n-1}}_r\right).$$

Now, applying the generalized vector product we obtain

$$\begin{aligned}
\mathfrak{M} &= \sum_{k=1}^{n}(-1)^{k+1}\det\left(\widetilde{M^{(k)}}_r\right)e_k \\
&= \sum_{k=1}^{n}(-1)^{k+1}\det\left(M^{(n-k+1)}\widetilde{I}_{n-1}\right)e_k \\
&= \sum_{k=1}^{n}(-1)^{k+1}\det\left(M^{(n-k+1)}\right)\det\left(\widetilde{I}_{n-1}\right)e_k \\
&= \det\left(\widetilde{I}_{n-1}\right)\sum_{k=1}^{n}(-1)^{n-k}\det\left(M^{(k)}\right)e_{n-k+1} \\
&= (-1)^{n+1}\det\left(\widetilde{I}_{n-1}\right)\sum_{k=1}^{n}(-1)^{k+1}\det\left(M^{(k)}\right)e_{n-k+1} \\
&= (-1)^{n+1}\det\left(\widetilde{I}_{n-1}\right)\left(\sum_{k=1}^{n}(-1)^{k+1}\det\left(M^{(k)}\right)e_k\right)\widetilde{I}_n \\
&= (-1)^{n+1}\det\left(\widetilde{I}_{n-1}\right)\left(\times\left(M_1,\ \widetilde{M_2,\ \ldots},\ M_{n-1}\right)\right)_r
\end{aligned}$$

therefore,

$$\begin{aligned}
\mathfrak{M} &= (-1)^{n+1}(-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor}\left(\times\left(M_1,\ \widetilde{M_2,\ \ldots},\ M_{n-1}\right)\right)_r \\
&= \begin{cases} (-1)^{\frac{3n}{2}}\left(\times\left(M_1,\ \widetilde{M_2,\ \ldots},\ M_{n-1}\right)\right)_r, n = 2k \\ (-1)^{\frac{3n+1}{2}}\left(\times\left(M_1,\ \widetilde{M_2,\ \ldots},\ M_{n-1}\right)\right)_r, n = 2k-1 \end{cases} \\
&= (-1)^{\left\lfloor \frac{3n}{2} \right\rfloor}\left(\times\left(M_1,\ \widetilde{M_2,\ \ldots},\ M_{n-1}\right)\right)_r.
\end{aligned}$$

Thus we conclude the proof. □

Remark 4.7. If M is a palindromic matrix by rows, then the minors $M^{(k)}$ have at least $\left\lfloor \dfrac{n}{2} \right\rfloor - 1$ pair of equal columns. This implies that for $n \geq 4$, the minors have at least one pair of equal columns and therefore $\det\left(M^{(k)}\right) = 0$ for all $1 \leq k \leq n$, which leads us to

$$\times\left(M_1,\ M_2,\ \ldots,\ M_{n-1}\right) = \mathbf{0} \in K^n.$$

This means that the generalized vector product of $(n-1)$ palindromic vectors belonging to K^n is interesting whenever $1 \leq n \leq 3$. The same result is obtained when we assume M as an antipalindromic matrix by rows, so we recover the previous results given in Section 2 with respect to the vector product. Moreover, some rows of M can be palindromic vectors, while the rest can be antipalindromic vectors, in this way, we can obtain similar results.

4.2. Pasting and Reversing simultaneously by rows and columns

Following Section 2, we can consider the matrices

$$A = (v_{11}, \dots, v_{1m}, \dots, v_{n1}, \dots, v_{nm}), \quad B = (w_{11}, \dots, w_{1q}, \dots, w_{p1}, \dots w_{pq})$$

as vectors. To avoid confusion in this section, we use $\widehat{A}$ instead of $\widetilde{A}$ to denote Reversing of A. Thus, we can see, in a natural way, that

$$\widehat{A} = A\widehat{I}_{nm} = (v_{nm}, \dots, v_{n1}, \dots, v_{1m}, \dots, v_{11})$$

and also for $n = p$ or $m = q$ (exclusively) that

$$A \diamond B = (v_{11}, \dots, v_{1m}, \dots, v_{n1}, \dots v_{nm}, w_{11}, \dots, w_{1q}, \dots, w_{p1}, \dots, w_{pq}).$$

We come back to express $\widehat{A}$ and $A \diamond B$ in term of matrices instead of vectors, i.e.,

$$\widehat{A} = \begin{pmatrix} v_{nm} & \dots & v_{n1} \\ \vdots & & \\ v_{1m} & \dots & v_{11} \end{pmatrix}, \; A \diamond B = \begin{cases} \begin{pmatrix} v_{11} & \dots & v_{1m} & w_{11} & \dots & w_{1q} \\ \vdots & & & & & \\ v_{n1} & \dots & v_{nm} & w_{p1} & \dots & w_{pq} \end{pmatrix}, & \begin{matrix} n = p \\ m \neq q \end{matrix} \\ \begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \\ v_{n1} & \dots & v_{nm} \\ w_{11} & \dots & w_{1p} \\ \vdots & & \\ w_{p1} & \dots & w_{pq} \end{pmatrix}, & \begin{matrix} n \neq p \\ m = q \end{matrix} \end{cases}$$

We say that any matrix P, with the conditions established above, is a palindromic matrix whether $\widehat{P} = P$. In the same way, we say that any matrix A, with the conditions established above, is an antipalindromic matrix whether $\widehat{A} = -A$. Note that $\widehat{(I_n)} = I_n \widehat{I}_{n^2}$, where I_n is written as vector. Thus, we arrive to the following elementary result.

Lemma 4.8. *Consider $M \in \mathcal{M}_{n\times m}(K)$. Then*

$$\widehat{A} = \begin{pmatrix} \widetilde{v_n} \\ \vdots \\ \widetilde{v_1} \end{pmatrix}, \quad \textit{where} \quad A = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Proof. It is followed by definition of Reversing in matrices seen as vectors. □

The following proposition summarizes the previous results for matrices as vectors.

Proposition 4.9. *Consider $A \in M_{n\times m}(K)$, $B \in M_{p\times q}(K)$ and $C \in M_{r\times s}(K)$ satisfying the conditions established above. The following statements hold.*

(i) $\hat{\hat{A}} = A$.

(ii) $\widehat{(A \diamond B)} = \widehat{B} \diamond \widehat{A}$.

(iii) $(A \diamond B) \diamond C = A \diamond (B \diamond C)$.

(iv) $\widehat{(bA + cB)} = b\widehat{A} + c\widehat{B}$ *where* $b, c \in K$ *and* $A, B \in \mathcal{M}_{n\times m}(K)$.

(v) *If* $V = \mathcal{M}_{n\times m}(K)$ *and* $W = \mathcal{M}_{p\times q}(K)$, *then* $V \diamond W = \mathcal{M}_{r\times s}(K)$, *where either* $n = p = r$, $m \neq q$ *and* $s = m + q$, *or* $n \neq p$, $m = q$ *and* $r = n + p$.

(vi) *Let* W_p *be the set of palindromic matrices of* $\mathcal{M}_{n\times m}(K)$, *then* W_p *is a vector subspace of* $\mathcal{M}_{n\times m}(K)$.

(vii) $\dim W_p = \left\lceil \frac{nm}{2} \right\rceil$.

(viii) *Let* W_a *be the set of antipalindromic matrices of* $\mathcal{M}_{n\times m}(K)$, *then* W_a *is a vector subspace of* $\mathcal{M}_{n\times m}(K)$.

(ix) $\dim W_a = \left\lfloor \frac{nm}{2} \right\rfloor$.

(x) *The sum of two palindromic matrices of the same vector space is a palindromic matrix.*

(xi) *The sum of two antipalindromic matrices of the same vector space is an antipalindromic matrix.*

(xii) $\mathcal{M}_{n\times m}(K) = W_p \oplus W_a$.

(xiii) $\forall A \in \mathcal{M}_{n\times m}(K), \exists (A_p, A_a) \in W_p \times W_a$ *such that* $A = A_p + A_a$.

Proof. Proceed as in the proofs of Section 2 using Lemma 4.8. □

The following result shows the relationship of Reversing with matrices classical operations.

Proposition 4.10. *The following statements hold.*

(i) $\widehat{I_n} = I_n$.

(ii) $\widehat{A} = \widetilde{\left(\widetilde{A}_r\right)_c} = \widetilde{\left(\widetilde{A}_c\right)_r}$.

(iii) $\widehat{(AB)} = \widehat{(A)}\widehat{(B)}$.

(iv) $(\widehat{A})^{-1} = \widehat{(A^{-1})}$.

(v) $\det(\widehat{A}) = \det A$.

(vi) $\mathrm{Tr}(\widehat{A}) = \mathrm{Tr} A$.

(vii) $\widehat{A^T} = (\widehat{A})^T$.

(viii) *The product of two palindromic matrices is a palindromic matrix.*

(ix) *The product of two antipalindromic matrices is a palindromic matrix.*

(x) *The product of one palindromic matrix with one antipalindromic matrix is an antipalindromic matrix.*

Proof. (i) Due to I_n can be seen as the vector

$$(1, 0, \dots, 0, 0, 1, \dots, 0, \dots, 0, \dots 0, 1) \in K^{n^2},$$

then

$$\widehat{I_n} = I_n \widehat{I}_{n^2} = (1, 0, \dots, 0, 0, 1, \dots, 0, \dots, 0, \dots 0, 1) = I_n.$$

(ii) Assumming $A = (v_1, \dots, v_n)^T = (c_1, \dots, c_n)$, we arrive to

$$\widetilde{(\widetilde{A}_r)}_c = \widetilde{(A\widetilde{I}_m)}_c = \widetilde{I}_n(A\widetilde{I}_m) = \widetilde{I}_n(\widetilde{v}_1, \dots, \widetilde{v}_n)^T = (\widetilde{v}_n, \dots, \widetilde{v}_1)^T.$$

By Lemma 4.8 we conclude $\widetilde{(\widetilde{A}_r)}_c = \widehat{A}$.

(iii) Assume $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{m\times r}(K)$. Thus, $AB \in \mathcal{M}_{n\times r}(K)$, $\widehat{A} \in \mathcal{M}_{n\times m}$, $\widehat{B} \in \mathcal{M}_{m\times r}$ and $\widehat{AB} \in \mathcal{M}_{n\times r}$. Suppose that $A = [a_{ij}]_{n\times m}$, $B = [b_{ij}]_{m\times r}$, $AB = C = [c_{ij}]_{n\times r}$ and $\widehat{C} = [d_{ij}]_{n\times r}$, thus

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}, \quad d_{ij} = c_{(n+1-i)(r+1-j)} = \sum_{k=1}^{m} a_{(n+1-i)k}b_{k(r+1-j)},$$

which implies that $\widehat{C} = \widehat{A}\widehat{B}$ and then $\widehat{(AB)} = \widehat{(A)}\widehat{(B)}$.

(iv) Assume $A \in \mathcal{M}_{n\times n}(K)$, being $\det A \neq 0$. Therefore

$$\widehat{(AA^{-1})} = \widehat{(A)}\widehat{(A^{-1})} = \widehat{(I_n)} = I_n.$$

(v) Assume $A \in \mathcal{M}_{n\times n}(K)$. Due to $\widehat{A}$ is obtained throughout $2k$ elementary operations of A, interchanging k rows and interchanging k columns, then $\det(\widehat{A}) = (-1)^{2k}\det A = \det A$.

(vi) Assume $A = [a_{ij}]_{n\times n} \in \mathcal{M}_{n\times n}(K)$. Thus

$$\mathrm{Tr}\widehat{A} = \sum_{i=1}^{n} a_{(n+1-i)(n+1-i)} = a_{nn} + a_{(n-1)(n-1)} + \cdots + a_{22} + a_{11} = \sum_{k=1}^{n} a_{kk} = \mathrm{Tr}A.$$

(vii) Assume $A = [a_{ij}]_{n\times m} \in \mathcal{M}_{n\times m}(K)$ and $\widehat{A^T} = [d_{ij}]_{m\times n} \in \mathcal{M}_{m\times n}(K)$. We see that $d_{ij} = a_{(m+1-j)(n+1-i)} = c_{ji}$, where $(\widehat{A})^T = [c_{ji}]_{m\times n}$ hence $\widehat{A^T} = (\widehat{A})^T$.

(viii) Assume $\widehat{A} = A$ and $\widehat{B} = B$. Therefore, $\widehat{AB} = \widehat{A}\widehat{B} = AB$.

(ix) Assume $\widehat{A} = -A$ and $\widehat{B} = -B$. Therefore, $\widehat{AB} = \widehat{A}\widehat{B} = AB$.

(x) Assume $\widehat{A} = -A$ and $\widehat{B} = B$. Therefore, $\widehat{AB} = \widehat{A}\widehat{B} = -AB$.

□

At this point, we have considered Pasting over an special case of matrices. As we can see, it can be confused when both matrices have the same size, how can we paste them? Another natural question is: how can we paste to matrices whenever $n \neq p$ and $m \neq q$? To avoid this difficulty we introduce *Pasting by blocks*, which will be denoted by $\diamond_b$. Consider matrices $A \in \mathcal{M}_{n\times m}(K)$ and $B \in \mathcal{M}_{r\times s}(K)$, Pasting by blocks of A with B is given by

$$A \diamond_b B := \begin{pmatrix} A & \mathbf{0}_{n\times s} \\ \mathbf{0}_{r\times m} & B \end{pmatrix} \in \mathcal{M}_{(n+r)\times(m+s)}(K).$$

It is well known that Pasting by blocks corresponds to a particular case of *block matrices*, also called *partitioned matrices*, see [8, 9]. The following result, although is known from block matrices point of view, is consequence of Proposition 4.9 and Proposition 4.10 considering $\widehat{\ }$ as above and $\diamond_b$ instead of $\diamond$.

Proposition 4.11. *Consider the matrices $A \in M_{n\times m}(K)$, $B \in M_{p\times q}(K)$ and $C \in M_{r\times s}(K)$, the following statements hold.*

(i) $\widehat{(A \diamond_b B)} = \widehat{(B)} \diamond_b \widehat{(A)}$.

(ii) $(A \diamond_b B) \diamond_b C = A \diamond_b (B \diamond_b C)$.

(iii) If $V = \mathcal{M}_{n\times m}(K)$ and $W = \mathcal{M}_{p\times q}(K)$, then $V \diamond_b W = \mathcal{M}_{r\times s}(K)$, where $r = n + p$ and $s = m + q$.

(iv) $(A \diamond_b B)^T = A^T \diamond_b B^T$.

(v) $\det(A \diamond_b B) = \det A \det B$.

(vi) $\mathrm{Tr}(A \diamond_b B) = \mathrm{Tr}A + \mathrm{Tr}B$.

(vii) $(A \diamond_b B)^{-1} = A^{-1} \diamond_b B^{-1}$.

Proof. (i) and (ii) follow directly from the definition of Pasting by blocks, and Reversing and Pasting of vectors. (iii) is due to $A \diamond_b B = A \oplus B$. (iv) to (vii) are known properties of block matrices. □

Final Remarks

In this paper we solved one question proposed in [4], which relates Pasting and Reversing with vector spaces and basic matrix theory. Although the paper was motivated through poetries and lines, we studied the mathematical structure of Pasting and Reversing giving the proofs of each statement proposed by us. We considered some extensions to the definition of Reversing such as palindromic and antipalindromic vectors and matrices. However, we insist that this paper is only an starting point to develop and solve theories and problems with a higher mathematical level, see for example [2, 5] where Pasting and Reversing were applied over simple permutations and combinatorial dynamics. We hope that the material presented here can be useful for the interested reader to start his own research project around this subject.

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References

[1] P.B. Acosta-Humánez, *La operación pegamiento y el cuadrado de números naturales*, Civilizar **4** (2003), 85–97.

[2] ______, *Genealogía de permutaciones simples de orden una potencia de dos*, Revista Colombiana de Matemáticas **42** (2008), 1–14.

[3] P.B. Acosta-Humánez, M. Aranda, and R. Núnez, *Some remarks on a generalized vector product*, Integración **29** (2011), 151–162.

[4] P.B. Acosta-Humánez, A. Chuquen, and A. Rodríguez, *Pasting and reversing operations over some rings*, Boletín de Matemáticas **17** (2010), 143–164.

[5] P.B. Acosta-Humánez and O.E. Martínez, *Simple permutations with order* $4n+2$*. part i*, Preprint arXiv:1012.2076. (2012), 1–17.

[6] F. Brafman, *A set generating functions of the bessel polynomials*, American Mathematical Society **4** (1953), 275–277.

[7] H. Eves, *Elementary matrix theory*, 1980.

[8] M. Hall, *Combinatorial theory*, Second edition, 1983.

[9] P Lancaster and M Tismenetsky, *The theory of matrices*, Second edition, 1985.

[10] S. Lang, *Linear algebra*, Third edition, 2010.

[11] I Markovsky and S. Rao, *Palindromic polynomials, time-reversible systems, and conserved quantities*, Proceedings 16th Mediterranean Conference on Control and Automation, Congress Centre, Ajaccio, France, June 25–27, 2008 (2008), 125–130.