Periodic travelling waves of moderate amplitude

Ondas viajeras periódicas de amplitud moderada

Alex Manuel Montes[1,a]

Abstract. In this paper, using a variational approach, we show the existence of periodic travelling waves for a nonlinear dispersive equation that emerges in the study of the evolution free surface for waves of moderate amplitude in the shallow water regime.

Keywords: Water waves, Moderate amplitude, Periodic travelling waves, Pass mountain theorem.

Resumen. En este trabajo, usando un enfoque variacional, mostramos la existencia de ondas viajeras periódicas para una ecuación no lineal de tipo dispersivo que surge en el estudio de la evolución de ondas de agua de amplitud moderada en el regimen de aguas poco profundas.

Palabras claves: Ondas de agua, Amplitud moderada, Ondas viajeras periódicas, Teorema de paso de montaña.

Mathematics Subject Classification: 35Q35, 35Q51, 76B25.

Recibido: enero de 2014 Aceptado: junio de 2014

1. Introduction

The focus of the present work is the one-dimensional nonlinear equation

$$u_t - u_{xxt} + u_x - au_{xxx} + \alpha uu_x = \lambda(uu_{xxx} + 2u_x u_{xx}), \tag{1}$$

modeling the evolution of the free surface for waves of moderate amplitude in the shallow water regime. As it is well known, models for dispersive and nonlinear water waves with small or moderate amplitude in finite depth are derived from the full water wave problem through an approximation process, under the imposition of some restrictions on the parameters that affect the propagation of gravity water waves, as the nonlinearity (amplitude parameter) and the dispersion (shallowness parameter). A typical example is the Korteweg-de Vries (KdV) equation [7] which models shallow water waves propagation:

$$u_t + u_x + u_{xxx} + \alpha uu_x = 0.$$

In this equation, the steeping effect of the nonlinearity, represented by uu_x, and the effect of dispersion, represented by u_{xxx}, are in balance with each other. More recently, it has been noticed by Benjamin, Bona and Mahony [2] that

[1]Departamento de Matemáticas, Universidad del Cauca, Popayán, Colombia
[a]amontes@unicauca.edu.co

the (KdV) equation belongs to a wider class of equations which provide an approximation of the exact water wave equations of the same accuracy as the (KdV) equation:

$$u_t + u_x + u_{xxx} - u_{xxt} + \alpha u u_x = 0.$$

Since the (KdV) and (BBM) type equations do not model breaking waves, several model equations were proposed to capture this phenomenon. In particular we recall the Camassa-Holm (CH) equation [3]

$$u_t + u_x - u_{xxt} + \alpha u u_x = \lambda(u u_{xxx} + 2 u_x u_{xx}),$$

that arises as a model describing the evolution of the horizontal fluid velocity at a certain depth within the regime of shallow water waves of moderate amplitude. In terms of the two fundamental parameters μ (shallowness parameter) and ϵ (amplitude parameter), the shallow water regime of waves of small amplitude (proper to KdV and BBM) is characterized by $\mu \ll 1$ and $\epsilon = O(\mu)$, while the regime of shallow water waves of moderate amplitude (proper to CH) corresponds to

$$\mu \ll 1, \quad \epsilon = O(\sqrt{\mu}). \tag{2}$$

In a recent paper, A. Constantin and D. Lannes in [4] showed that the correct generalization of the (KdV), (BBM) and (CH) equations under the scaling (2) is provided by the class of equations (1). This equations capture stronger nonlinear effects than the classical nonlinear dispersive Benjamin-Bona-Mahony and Korteweg-de Vries equations. In particular, they accommodate wave breaking, a fundamental phenomenon in the theory of water waves.

For the equation (1) we distinguish the following results. By perform a phase plane analysis, A. Geyer in [1] showed the existence of solitary waves which propagate with velocity $c > 1$. A. Geyer *et al.* in [6], using the approach of Grillakis, Shatah and Strauss, showed the stability of solitary wave solutions with speed of wave $c > 1$. A. Constantin *et al.* in [4], using Kato theory, showed that the initial value problem associated to (1) is locally well-posed in the Sobolev space $H^s(\mathbb{R}), s > \frac{5}{2}$ and N. Duruk in [5] improved this result, establishing the local well-posed in the Sobolev space $H^s(\mathbb{R}), s > \frac{3}{2}$.

In this paper, when $a, \alpha, \lambda > 0$ we establish the existence of periodic travelling waves of (1) for wave velocity $c > c_0$ with $c_0 = \max\{1, a\}$ and the mean zero property. We will get the result by using a variational approach for which solitary waves corresponding to a critical point of a suitable action functional. The paper is organized as follows. In Section 2 we include some preliminaries and in Section 3 we characterize periodic travelling solitary waves variationally as critical points of an action functional. Then we prove the existence of periodic travelling waves for the equation (1) by using the mountain pass Theorem.

2. Preliminaries and main result

In this section we present some definitions and results that are used in this paper. Also we present our main theorem. Here X is a Hilbert space, $\|\cdot\|_X$ denotes the norm, $<, >_X$ its inner product and X' represents the dual space.

Set $\Omega \subset \mathbb{R}$ and $L^p(\Omega), 1 \leq p \leq \infty$, denotes the usual Lebesgue space. Given $T > 0$, the Sobolev space $W^1_{per} = W^1_{per}[0,T]$ of periodic functions with period T is defined in the following standard way. Let $C^\infty_{per}([0,T])$ be the space of smooth functions which are periodic with period T and have compact support in $[0,T]$ and define

$$X_T = \left\{ \psi|_{[0,T]} \;:\; \psi \in C^\infty_{per}([0,T]) \right\}.$$

We define the Sobolev space $W^1_{per}[0,T]$ as the closure of X_T with respect to the norm given by

$$\|v\|_{W^1_{per}} = \left[\int_0^T \left(v^2 + (v')^2 \right) dx \right]^{1/2}.$$

Then we have that the space $W^1_{per}[0,T]$ is a Hilbert space with the inner product defined as

$$\langle v, w \rangle_{W^1_{per}} = \int_0^T (vw + v'w')\, dx.$$

Now, we define the Sobolev space of periodic functions with the mean zero property. $H^1_{per} = H^1_{per}[0,T]$ denotes the closed subspace of $W^1_{per}[0,T]$ given by

$$H^1_{per}[0,T] = \left\{ v \in W^1_{per}[0,T] \;:\; \int_0^L v\, dx = 0 \right\}.$$

In this work we show the existence of periodic travelling waves in the space $H^1_{per}[0,T]$.

Next, we present the following minimax type theorem.

Lemma 2.1 (Mountain pass lemma)**.** *Let X be a Hilbert space, $\varphi \in C^2(X, \mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\|_X > r$ and*

$$b = \inf_{\|u\|_X = r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

Then, given $n \in \mathbb{N}$, there is $u_n \in X$ such that

$$\varphi(u_n) \to d, \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{in } X',$$

where

$$d = \inf_{\pi \in \Pi} \max_{t \in [0,1]} \varphi(\pi(t)), \quad \Pi = \left\{ \pi \in C([0,1], X) : \pi(0) = 0, \quad \pi(1) = e \right\}.$$

Finally, the following theorem is our main result.

Theorem 2.2. *For $a, \alpha, \lambda > 0$ and $c > \max\{1, a\}$, the equation (1) admits travelling wave solutions $u(x,y) = v(x - ct, y)$ in the space $H^1_{per}[0,T]$.*

3. Existence of periodic travelling waves

In this section we assume $a, \alpha, \lambda > 0$. A periodic travelling wave for the equation (1) is a solution of the form $u(x,t) = v(z)$ with $z = x - ct$ periodic. Indeed, when this ansatz is substituted into (1) there appears the ordinary differential equation

$$(1-c)v' + (c-a)v''' + \alpha v v' - \lambda \left(v v''' + 2v'v'' \right) = 0,$$

which, using integration and the periodicity of v, yields

$$(c-1)v - (c-a)v'' - \frac{\alpha}{2} v^2 + \frac{\lambda}{2} \left((v')^2 + 2 v v'' \right) = A, \tag{3}$$

where A is a constant of integration. Note that A must be different from zero due to the assumption on the mean property in $[0, L]$. We can see that solutions v of (3) are critical points of the functional J_c given by

$$J_c(v) = I_c(v) + G(v),$$

where the functionals I_c and G are defined on the space H^1_{per} by

$$I_c(v) = \frac{1}{2} \int_0^T \left((c-1)v^2 + (c-a)(v')^2 \right) dx,$$

$$G(v) = -\frac{1}{2} \int_0^T \left(\frac{\alpha}{3} v^3 + \lambda v (v')^2 \right) dx.$$

First we have that $I_c, G, J_c \in C^2(H^1_{per}, \mathbb{R})$ and its derivatives in v in the direction of w are given by

$$\langle I_c'(v), w \rangle = \int_0^T \left((c-1) v w + (c-a) v' w' \right) dx,$$

$$\langle G'(v), w \rangle = -\frac{1}{2} \int_0^T \left(\alpha v^2 w + \lambda \left((v')^2 w + 2 v v' w' \right) \right) dx.$$

As a consequence of this, after integration by parts, we conclude that

$$J_c'(v) = (c-1)v - (c-a)v'' - \frac{\alpha}{2} v^2 + \frac{\lambda}{2} \left((v')^2 + 2 v v'' \right),$$

meaning that a critical point v of the functional J_c in a space having the mean zero property satisfies the travelling wave equation (3). In fact, let w with the mean zero property, then

$$\langle J_c'(v), w \rangle = 0 = A \int_0^T w dx = \langle A, w \rangle .$$

In particular, we have that

$$\langle J_c'(v), v \rangle = 2 I_c(v) + 3 G(v) = 2 J_c(v) + G(v). \tag{4}$$

Thus on any critical point $v \in H^1_{per}$, we have that

$$J_c(v) = \frac{1}{3} I_c(v), \quad J_c(v) = -\frac{1}{2} G(v), \quad I_c(v) = -\frac{3}{2} G(v). \tag{5}$$

Hereafter, we will say that weak solutions for (3) are critical points of the functional J_c. Next, it is easy to show the following results on properties of I_c and G. We will write

$$L_T^\infty = L^\infty\,[0,T]\,.$$

Lemma 3.1. *The functional I_c is well-defined on H^1_{per}. In addition; for $c > \max\{1, a\}$, we have that $I_c(v) \geq 0$. Moreover, there are some positive constants $C_1(a,c) < C_2(a,c)$ such that*

$$C_1\|v\|^2_{H^1_{per}} \leq I_c(v) \leq C_2\|v\|^2_{H^1_{per}}. \tag{6}$$

Lemma 3.2. *The functional G is well-defined on H^1_{per}. Moreover, there is a positive constant $C = C(\alpha, \lambda)$ such that*

$$|G(v)| \leq C\|v\|^3_{H^1_{per}}. \tag{7}$$

Proof. Since the embedding $H^1_{per} \hookrightarrow L^\infty_T$ is continuous we have that there is $C > 0$ such that

$$\int_{-T}^{T} |v||v'|^2 dx \leq \|v\|_{L^\infty_T} \int_{-T}^{T} |v'|^2 dx \leq C\|v\|^3_{H^1_{per}}.$$

In a similar fashion we see that $\int_{\mathbb{R}} |v|^3 dx \leq C\|v\|^3_{H^1_{per}}$. Then the result follows. □

Our approach to show the existence of a non trivial critical point for J_c is to use the mountain pass lemma without the Palais-Smale condition (M. Willem [8]) to build a Palais-Smale sequence for J_c for a minimax value and use a embedding result to obtain a critical point for J_c as a weak limit of such Palais-Smale sequence. First we establish an important result for our analysis, which is related to the characterization of "vanishing" sequences in $H^1_{per}[0,T]$.

Proposition 3.3. *If $(v_n)_n$ is a bounded sequence in H^1_{per} such that*

$$\lim_{n\to\infty} \|v_n\|_{L^\infty_T} = 0.$$

Then we have that

$$\lim_{n\to\infty} \int_0^T v_n^3\, dx = \lim_{n\to\infty} \int_0^T v_n (v_n')^2\, dx = 0. \tag{8}$$

Proof. The Hölder inequality implies that

$$\begin{aligned}\int_0^T |v_n|^3 dx + \int_0^T |v_n||v_n'|^2\, dx &\leq \|v_n\|_{L^\infty_T} \left(\int_0^T |v_n|^2 dx + \int_0^T |v_n'|^2 dx \right) \\ &= \|v_n\|_{L^\infty_T} \|v_n\|^2_{H^1_{per}}.\end{aligned}$$

Thus, under the assumptions of the lemma we obtain the result. □

In the following proposition we verify the mountain pass lemma's hypotheses (Lemma 2.1) and we build a Palais-Smale sequence for J_c.

Proposition 3.4. *Let $a, \alpha, \lambda > 0$ and $c > \max\{1, a\}$. Then*

(i) There exists $\rho > 0$ small enough such that

$$b(c) := \inf_{\|z\|_{H^1_{per}}=\rho} J_c(z) > 0.$$

(ii) There is $e \in H^1(\mathbb{R})$ such that $\|e\|_{H^1_{per}} \geq \rho$ and $J_c(e) \leq 0$.

(iii) If $d(c)$ is defined as

$$d(c) = \inf_{\pi\in\Pi} \max_{t\in[0,1]} J_c(\pi(t)),\ \Pi = \{\pi \in C([0,1]), H^1_{per} : \pi(0) = 0,\ \pi(1) = e\},$$

then $d(c) \geq b(c)$ and there is a sequence $(v_n)_n$ in H^1_{per} such that

$$J_c(v_n) \to d, \quad J'_c(v_n) \to 0 \ \ \text{in} \ \ \left(H^1_{per}\right)'. \tag{9}$$

Proof. From inequalities (6)-(7), we have for any $v \in H^1_{per}$ that

$$\begin{aligned} J_c(v) &\geq C_1(a,c)\|v\|^2_{H^1_{per}} - C(\alpha,\lambda)\|v\|^3_{H^1_{per}} \\ &\geq \left(C_1(a,c) - C(\alpha,\lambda)\|v\|_{H^1_{per}}\right)\|v\|^2_{H^1_{per}}. \end{aligned}$$

Then for $\rho > 0$ small enough such that

$$C_1 - \rho C > 0, \tag{10}$$

we conclude for $\|v\|_{H^1_{per}} = \rho$ that

$$J_c(v) \geq (C_1 - \rho C)\,\rho^2 := \delta > 0.$$

In particular, we have that

$$b(c) = \inf_{\|z\|_{H^1_{per}}=\rho} J_c(z) \geq \delta > 0. \tag{11}$$

Now, if $C_0^\infty([0,T])$ denotes the space of smooth compactly support functions with zero mean value, it is not hard to prove that there exists $v_0 \in C_0^\infty([0,T])$ such that $\int_{\mathbb{R}} v_0^3 dx$ and $\int_{\mathbb{R}} v_0(v_0')^2 dx$ are positive quantities. Hence, for any $t \in \mathbb{R}$ we see that

$$J_c(tv_0) = t^2\left(I_c(v_0) - \frac{t}{2}\int_0^T \left(\frac{\alpha}{3}v_0^3 + \lambda v_0(v_0')^2\right)dx\right).$$

Using the hypotheses we have that

$$\lim_{t\to\infty} J_c(tv_0) = -\infty,$$

because $0 \leq I_c(v_0) \leq C_2(c)\|v_0\|^2_{H^1_{per}}$. So that, there is $t_0 > 0$ such that $e = t_0 v_0 \in H^1_{per}$ satisfies that

$$t_0\|v_0\|_{H^1_{per}} = \|e\|_{H^1_{per}} > \rho$$

and that $J_c(e) \leq J_c(0) = 0$. The third part follows by applying Lemma 2.1. □

Proof of Theorem 2.2

We will see that $d(c)$ (see Proposition 3.4) is in fact a critical value of J_c. Let $(v_n)_n$ be the sequence in H^1_{per} given by Proposition 3.4. First note from (11) that $d(c) \geq b(c) \geq \delta$. Using the definition of J_c and (4) we have that

$$I_c(v_n) = 3J_c(v_n) - \langle J_c'(v_n), v_n \rangle .$$

But from (6) we conclude for n large enough that

$$C_1(a,c)\|v_n\|^2_{H^1_{per}} \leq I(v_n) \leq 3(d(c)+1) + \|v_n\|_{H^1_{per}}.$$

Then we have shown that $(v_n)_n$ is a bounded sequence in H^1_{per}. We claim that

$$\delta^* = \overline{\lim_{n\to\infty}} \|v_n\|_{L^\infty_T} > 0.$$

Suppose that

$$\overline{\lim_{n\to\infty}} \|v_n\|_{L^\infty_T} = 0.$$

Hence, from Proposition 3.3 we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} v_n^3 \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} v_n (v_n')^2 \, dx = \lim_{n\to\infty} G(v_n) = 0.$$

Then, we have from (4) and (11) that

$$\begin{aligned} 0 < \delta \leq d(c) &= J_c(v_n) - \frac{1}{2} \langle J_c'(v_n), v_n \rangle + o(1) \\ &\leq -\frac{1}{2} G(v_n) + o(1) \\ &\leq o(1), \end{aligned}$$

but this is a contradiction. Thus, there is a subsequence of $(v_n)_n$, which is denoted the same way, such that

$$\|v_n\|_{L^\infty_T} \geq \frac{\delta^*}{2}. \tag{12}$$

Now, since $(v_n)_n$ is a bounded sequence in H^1_{per} for some subsequence of $(v_n)_n$, denoted the same way, and for some $v \in H^1_{per}$ we have that

$$v_n \rightharpoonup v, \quad \text{as } n \to \infty \quad (\text{weakly in } H^1_{per}).$$

Since the embedding $H^1_{per} \hookrightarrow L^\infty_T$ is compact we see that

$$v_n \to v \ \text{ in } L^\infty_T.$$

Note that $v \neq 0$ because using (12) we have that

$$\|v\|_{L^\infty_T} = \lim_{n\to\infty} \|v_n\|_{L^\infty_T} \geq \frac{\delta^*}{2}.$$

Moreover, if $W \in C_0^\infty([0,T])$, then we have that

$$\begin{aligned}\langle I_c'(v), W\rangle &= \int_0^T ((c-1)vW + (c-a)v'W')\,dx \\ &= \lim_{n\to\infty}\int_0^T \left((c-1)\tilde{v}_n W + (c-a)\left(\tilde{v}_n\right)' W'\right)dx \\ &= \lim_{n\to\infty}\langle I_c'(\tilde{v}_n), W\rangle .\end{aligned}$$

Now (taking a subsequence, if necessary) noting that

$$(\tilde{v}_n)^2 \rightharpoonup v^2, \quad (\tilde{v}_n')^2 \rightharpoonup (v')^2, \quad \tilde{v}_n(\tilde{v}_n') \rightharpoonup vv' \quad \text{in} \quad L^2_{loc}[0,T],$$

we have that

$$\int_0^T (\tilde{v}_n)^2 W\,dx \to \int_0^T v^2 W\,dx, \qquad \int_0^T (\tilde{v}_n')^2 W\,dx \to \int_0^T (v')^2 W dx$$

and

$$\int_0^T \tilde{v}_n(\tilde{v}_n')W'dx \to \int_0^T vv'W'dx.$$

Then we conclude that

$$\langle G'(v), W\rangle = \lim_{n\to\infty}\langle G'(\tilde{v}_n), W\rangle, \quad \langle J_c'(v), W\rangle = \lim_{n\to\infty}\langle J_c'(\tilde{v}_n), W\rangle = 0.$$

If $W \in H^1(\mathbb{R})$, by using density, there is $W_k \in C_0^\infty([0,T])$ such that $W_k \to W$ in H^1_{per}. Hence,

$$\begin{aligned}|\langle J_c'(v), W\rangle| &\le |\langle J_c'(v), W - W_k\rangle| + |\langle J_c'(v), W_k\rangle| \\ &\le \|J_c'(v)\|_{(H^1_{per})'}\|W - W_k\|_{H^1_{per}} + |\langle J_c'(v), W_k\rangle| \to 0.\end{aligned}$$

Thus, we have already established that $J_c'(v) = 0$. In other words, v is a nontrivial solution for problem (3).

Acknowledgements

A. M. Montes was supported by Universidad del Cauca (Colombia) under the project I.D. 3982.

References

[1] Geyer A., *Solitary Traveling Water Waves of Moderate Amplitude*, Journal of Nonlinear Mathematical Physics **19** (2012), 12.

[2] T. Benjamin, J. Bona, and J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philosophical Transactions of the Royal Society Lond. A **272** (1972), no. 1220, 47–78.

[3] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Physical Review Letters **71** (1993), 1661–1665.

[4] A. Constantin and D. Lannes, *The Hydrodynamical Relevance of the Camassa-Holm and Degasperis-Procesi Equations*, Archive for Rational Mechanics and Analysis **192** (2009), no. 1, 165–186.

[5] N. Duruk Mutlubas, *On the Cauchy problem for a model equation for shallow water waves of moderate amplitude*, Nonlinear Analysis: Real World Applications **14** (2013), no. 5, 2022–2026.

[6] N. Duruk Mutlubas and Geyer A., *Orbital stability of solitary waves of moderate amplitude in shallow water*, Journal of Differential Equations **255** (2013), no. 2, 254–263.

[7] D. Korteweg and G. de Vries, *On the chance of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Journal of Differential Equations **39** (1895), 422–443.

[8] M. Willem, *Minimax theorems*, Birkhuser, Boston, 1996.