Existencia de solución en un modelo de actividad eléctrica de tipo monodominio para un ventrículo
Existence of global solutions in a model of electrical activity of the monodomain type for a ventricle

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Resumen

Introducción: Se formula un modelo de monodominio de actividad eléctrica en un ventrículo aislado. Este modelo se escribe como una EDP de tipo reacción difusión acoplada a una EDO, se utiliza el modelo de Rogers-Mculloch para representar la actividad eléctrica a través de la membrana celular.

Método: Se proponen definiciones de solución débil y fuerte respectivamente para el problema de Cauchy variacional asociado al modelo de monodominio. Se propone una sucesión de soluciones aproximadas de tipo Faedo-Galerkin.

Resultados: Se demuestra que la sucesión de soluciones aproximadas converge a una solución débil según la definición que se propone. Finalmente, se obtiene que la solución débil es también una solución fuerte.

Conclusión: El modelo de monodominio de actividad eléctrica en un ventrículo aislado que se propone tiene solución débil en un sentido apropiado. Además, esta solución débil también es una solución fuerte.

Abstract

Introduction: A monodomain model of electrical activity for an isolated ventricle is formulated. This model is written as a reaction diffusion PDE coupled to an ODE, The Rogers-Mculloch model is used to represent the electrical activity through the cell membrane.

Method: We give a definition of weak and strong solution of the variational Cauchy problem associated to the monodomain model. A sequence of approximate solutions of Faedo-Galerkin type is proposed.

Results: It is shown that the sequence of approximate solutions converge to a weak solution according to the proposed definition. Finally, we have that this weak solution is also a strong solution.

Conclusion: The monodomain model of electrical activity in an isolated ventricle that is proposed has a weak solution in an appropriate sense. In addition, this weak solution is also a strong solution.

Keywords: monodomain; bidomain; reaction-diffusion; Faedo-Galerkin

Introduction

The bidomain model represents an active myocardium on a macroscopic scale by relating membrane ionic current, membrane potential, and extracellular potential (Henriquez 1993). Created in 1969 (Schmidt 1969), (Clerc 1976) and first developed formally in 1978 (Tung 1978), (Miller 1978, I), the bidomain model was initially used to derive forward models, which compute extracellular and body-surface potentials from given membrane potentials (Miller 1978, I),

(Gulrajani 1983), (Miller 1978, II) and (Gulrajani 1998). Later, the bidomain model was used to link multiple membrane models together to form a bidomain reaction-diffusion (R-D) model (Barr 1984), (Roth 1991), which simulates propagating activation based on no other premises than those of the membrane model, those of the bidomain model, and Maxwell's equations. Other mathematical derivations of the macroscopic bidomain type models directly from the microscopic properties of tissue and using asymptotic and homogenization methods along with basic physical principles are presented in (Neu 1993), (Ambrosio 2000) and (Pennacchio 2005).

Monodomain R-D models, conceived as a simplification of the R-D bidomain models, with advantages both for mathematical analysis and computation, were actually developed before the first bidomain R-D models, and few papers have compared monodomain with bidomain results. Those that did, have shown small differences (Vigmond 2002), and monodomain simulations have provided realistic results (Leon 1991), (Hren 1997), (Huiskamp 1998), (Bernus 2002), (Trudel 2004) and (Berenfeld 1996). In (Potse 2006) has been investigated the impact of the monodomain assumption on simulated propagation in an isolated human heart, by comparing results with a bidomain model. They have shown that differences between the two models were extremely small, even if extracellular potentials were influenced considerably by fluid-filled cavities. All properties of the membrane potentials and extracellular potentials simulated by the bidomain model have been accurately reproduced by the monodomain model with a small difference in propagation velocity between both models, even in abnormal cases with the Na conductivity (Bernus 2002) reduced to 1=10 of its normal value, and have arrived at the same conclusions. The difference between the results that may be obtained with one or another model are small enough to be ignored for most applications, with the exception of simulations involving applied external currents or in the presence adjacent fluid on within, although these effects seem to be ignorable on the scale of a human heart. A formal derivation of the monodomain equation as we present here can be found in (Sundnes 2006). There are few references in the literature dealing with the proof of the well-posedness of the bidomain model. The most important seem to be Colli-Franzone and Savarés paper (Colli 2002), Veneroni's technical report (Veneroni 2009) and Y. Bourgault, Y. Coudière and C. Pierre's paper (Bourgault 2009). In (Colli 2002), global existence in time and uniqueness for the solution of the bidomain model is proven, although their approach applies to particular cases of ionic models, typically of the form $f(u,w) = k(u) + \alpha w$ and $g(u,w) = \beta u + \gamma w$, where $k \in C^1(\mathbb{R})$ satisfies $\inf_\mathbb{R} k' > -\infty$. In practice a common ionic

model reading this form is the cubic-like FitzHugh-Nagumo model (Fitzhugh 1961), which, although it is important for qualitatively understanding of the action potential propagation, its applicability to myocardial excitable cells is limited (Keener 1998), (Panfilov 1997).However, from the results of (Colli 2002) is not possible to conclude the existence of solution for other simple two variable ionic models widely used in the literature for modelling myocardial cells, such as the Aliev-Panfilov (Aliev 1996) and MacCulloch (Rogers 1994) models. In (Veneroni 2009), Colli-Franzone and Savarés results have been extended to more general and more realistic ionic models, namely those taking the form of the Luo and Rudy I model (Luo 1991), this result still does not include the Aliev-Panfilov and MacCulloch models. In reference (Bourgault 2009), global in time weak solutions are obtained for ionic models reading as a single ODE with polynomial nonlinearities. These ionic models include the FitzHugh-Nagumo model (Fitzhugh 1961) and simple models more adapted to myocardial cells, such as the Aliev-Panfilov (Aliev 1996) and Rogers-MacCulloch (Rogers 1994) models.

In this paper, we give a definition of weak solution of the variational Cauchy problem and, from this one, we give a definition of strong solution. We aim to obtain the existence of a global weak solution for a monodomain R-D model when applied to a ventricle isolated from the torso in absence of blood on within, which is activated through the endocardium by a Purkinje current and for simpler ionic models reading as a single ODE with polynomial nonlinearities. Also, it is proved that this weak solution is strong in the sense of the given definition. We will consider a bounded subset $\Omega \in \mathbb{R}^3$ simulating an isolated ventricle surrounded by an insulating medium. The boundary $\partial\Omega$ of the spatial region is formed by two disjoint components; the component Γ_0 imulating the epicardium and the component Γ_1 simulating the endocardium. The way Ω is electrically stimulated is by means Purkinje fibers, which directly stimulate only the inner wall Γ_1 then the excitable nature of the tissue allows this stimulus to propagate by Ω. We will assume that the ventricle is isolated from the heart and torso, that is to say that Γ_0 is in contact with an electrically insulating medium. We will use the monodomain model and the Rogers-McCulloch model for ion currents through the cell membrane, in this way and for the above considerations this model can be written as one parabolic PDE with boundary conditions, coupled to a ODE, and some initial data:

$$\frac{\partial u}{\partial t} + f(u,w) - \nabla \cdot (\sigma \nabla u) = 0, \qquad\qquad (t,x) \in (0,\infty) \times \Omega, \qquad\qquad (1)$$

$$\frac{\partial w}{\partial t} + g(u,w) = 0, \qquad\qquad (t,x) \in (0,\infty) \times \Omega, \qquad\qquad (2)$$

$$(\sigma \nabla u) \cdot \eta = 0, \qquad\qquad (t,x) \in (0,\infty) \times \Gamma_0, \qquad\qquad (3)$$

$$(\sigma \nabla u) \cdot \eta = s(t)\varphi(x), \qquad\qquad (t,x) \in (0,\infty) \times \Gamma_1, \qquad\qquad (4)$$

$$u(0,x) = u_0(x), \qquad w(0,x) = w_0(x), \qquad\qquad x \in \Omega. \qquad\qquad (5)$$

The unknowns are the scalar functions $u(t,x)$ and $w(t,x)$ which are the membrane potential and an auxiliary variable without physiological interpretation called the recovery variable, respectvely. We denote by η the unit normal to $\partial\Omega$ out of Ω. The anisotropic properties of the tissue are included in the model by the conductivity tensor $\sigma(x)$. The functions $f(u,w)$ and $g(u,w)$ crrespond to the flow of ions through the cell membrane. The function $s: (0,+\infty) \to \mathbb{R}$ represents the electrical activation of the endocardium by means of Purkinje fibers. The function $\varphi: \Omega \to \mathbb{R}$ represents the activation spatial density. Because we consider that Ω is surrounded by an insulating medium, there is no current flowing out of Ω, this is expressed in the boundary condition (3).

The specific assumptions we will make about (1) - (5) are as follows:

(h1) Ω has Lipschitz boundary $\partial\Omega$.

(h2) $\sigma(x)$ is a symmetric matrix, function of the spatial variable $x \in \Omega$, with coefficients in $L^\infty(\Omega)$ and such that there are positive constants m and M such that

$$0 < m\,|\xi|^2 \le \xi^t \sigma(x)\xi \le M\,|\xi|^2 < \infty, \quad \forall \xi \in \mathbb{R}^3, \qquad\qquad (6)$$

Is met for almost all $x \in \Omega$.

(h3) $s \in L^\infty(0,+\infty)$.

(h4) $\varphi \in L^2(\Gamma_1)$.

(h5) $f(u,w)$ y $g(u,w)$ stands for Rogers-McCulloch ionic model,

$$f(u, w) = a_1(u - u_{rest})(u - u_{th})(u - u_{peak}) + a_2(u - u_{rest})w, \tag{7}$$

$$g(u, w) = b(-u + u_{rest} + c_3 w). \tag{8}$$

(h6) $u_0, w_0 \in L^2(\Omega)$.

It is convenient to establish some notations that we will follow throughout this work. For convenience, we will denote $V = H^1(\Omega)$ and $H = L^2(\Omega)$ since we will make constant use of these spaces. It is important to note that in the context of this work the following inclusions are fulfilled for $2 \leq p \leq 6$

$$V \subset L^p(\Omega) \subset H \equiv H' \subset L^{p'}(\Omega) \subset V', \tag{9}$$

Note that only H is identified with its dual space. In particular, we will consider $p = 4$ from here on. As usual, p' denotes a positive number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Let X be a Banach space of integrable functions over Ω, we define the subspace

$$X/\mathbb{R} = \left\{ u \in X \mid \int_\Omega u = 0 \right\} \subset X,$$

Which is a Banach space with the norm induced by X. For any $u \in X$, we denote

$$[u] = u - (1/|\Omega|) \int_\Omega u,$$

Thus $[u] \in X/\mathbb{R}$.

This paper is organized as follows. The spaces $L^q(0, T; X)$ are the functional setting we will work in, so in section 2.1 the definition of this spaces along with some important facts about them are presented. In section 2.2 some preliminary results are established, mainly related to the diffusion term $\nabla(\sigma \nabla u)$ and with the model for the ionic current f and g. In section 2.3 we state

the definition of weak and strong solution, and enunciate some results that allow us to find a relation between them. The existence will be shown in sections 3.1 and 4.1.

Method

$L^q(0, T; X)$ spaces

Let X be a Banach space, we denote by $L^q(0, T; X)$ the space of the functions $t \rightarrow f(t)$ of $[0, T] \rightarrow X$ that are measurable with values in X such that

$$\|f\|_{L^q(0,T;X)} = \left(\int_0^T \|f(t)\|_X^q \, dt \right)^{1/q} < \infty, \tag{10}$$

with this norm $L^q(0, T; X)$ is complete. Observe that

$$L^q(0, T; L^q(\Omega)) = L^q(Q_T),$$

where $Q_T = [0, T] \times \Omega$.

It is necessary to give a definition of the derivative of an element of $L^q(0, T; X)$, for this we will consider the space of distributions on $[0, T]$ with values in X , see (Lions 1969, 7).

Definition 1. *We define* $\mathcal{D}'(0, T; X)$*, the space of distributions on* $[0, T]$ *with values in X, as*

$$\mathcal{D}'(0, T; X) = \mathcal{L}(\mathcal{D}(0, T); X), \tag{11}$$

where $\mathcal{D}(0, T)$ *is the set of infinitely differentiable functions of compact support in* $(0, T)$*.*

If $f \in \mathcal{D}'(0,T;X)$ we can define its derivative in the sense of distributions as $\frac{\partial f}{\partial t} \in$ $\mathcal{D}'(0,T;X)$ given by

$$\frac{\partial f}{\partial t}(\phi) = -f\left(\frac{d\phi}{dt}\right), \quad \forall \phi \in \mathcal{D}(0,T). \tag{12}$$

If $f \in L^q(0,T;X)$ it corresponds a distribution $\hat{f}$ in $\mathcal{D}'(0,T;X)$ defined as follows

$$\hat{f}(\phi) = \int_0^T f(t)\phi(t)dt \in X, \quad \forall \phi \in \mathcal{D}(0,T). \tag{13}$$

In this way, we can define the derivative in the sense of distributions of a function $f \in$

$$\frac{\partial \hat{f}}{\partial t}(\phi) = -\int_0^T f(t)\phi' dt, \quad \forall \phi \in \mathcal{D}(0,T).$$

$L^q(0,T;X)$ as

Theorem 1. *Let Q_T a bounded open in $\mathbb{R} \times \mathbb{R}^N$, f_n and f functions in $L^q(Q_T)$, $1 < q < \infty$, such that*

$$\|f_n\|_{L^q(Q_T)} \leq C, \qquad f_n \to f \text{ c.p. en } Q_T,$$

for a certain constant $C > 0$, then,

$$f_n \to f \text{ weakly in } L^p(Q_T).$$

Proof. (Lions 1969, lema 1.3, p. 12).

For the chain of inclusions (9) and the fact that the immersion $V \to H$ is compact we can enunciate the following result, which is a particular case of a classic compactness result, see (Lions 1969, th. 5.1, p.58).

Theorem 2. *We define for T finite and $0 < q_i < \infty, i = 0, 1,$*

$$W^{1,q_0,q_1}(0,T;V,V') = \left\{ v \mid v \in L^{q_0}(0,T;V), \quad v' = \frac{dv}{dt} \in L^{q_1}(0,T;V') \right\}, \quad (14)$$

endowed with the norm $\|v\|_{L^{p_0}(0,T;V)} + \|v'\|_{L^{p_1}(0,T;V')}$. Then, $W^{1,q_0,q_1}(0,T; V,V')$ is a Banach space and $W^{1,q_0,q_1}(0,T; V,V') \subset L^{q_0}(0,T;H)$. The immersion of $W^{1,q_0,q_1}(0,T; V,V')$ in $L^{q_0}(0,T;H)$ is compact.

Proposition 1. *Let $u \in L^{q_0}(0,T; V)$ with $q_0 \geq 2$, then, $u \in W^{1,q_0,q_1}(0,T; V,V')$, for some $q_1 \geq 2$, if and only if there exist a function $\tilde{u} \in L^{q_1}(0,T; V')$ that satisfies*

$$-\int_0^T (u,v)\phi' = \int_0^T \langle \tilde{u}, v \rangle_{V' \times V} \, \phi \qquad \forall \phi \in \mathcal{D}(0,T), \forall v \in V,$$

where $(\cdot, \cdot)$ represents the scalar product in H, and $\langle \tilde{u}, v \rangle_{V' \times V}$ represents the evaluation of functional $\tilde{u}$ in u. That is, $\tilde{u}$ is the distributional derivative of u, and is the only function in $L^{q_1}(0,T; V')$, that satisfies

$$\frac{d}{dt}(u,v) = \langle \tilde{u}, v \rangle_{V' \times V}, \quad \text{for all } v \in V.$$

From now on, we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{V' \times V}$.

Theorem 3. *If $f \in L^q(0,T;X)$ and $\partial_t f \in L^q(0,T;X)$ $(1 \leq q \leq \infty)$, then, f is continuous almost everywhere from $(0,T)$ to X.*

Proof. (Lions 1969, lema 1.2, p. 7).

Preliminaries

Definition 2. *For all $u, v \in V \times V$ we define the bilinear form*

$$a(u, v) = \int_\Omega \sigma \nabla[u] \cdot \nabla[v].$$

(15)

Proposition 2. *The bilinear form $a(\cdot, \cdot)$ is symmetric, continuous and coercitive in V,*

$$|a(u, v)| \leq M \|u\|_V \|v\|_V, \qquad \forall u, v \in V,$$

(16)

$$\alpha \|u\|_V^2 \leq a(u, u) + \alpha \|u\|_H^2, \qquad \forall u \in V,$$

(17)

with $\alpha, M > 0$. There is a growing sequence $0 = \lambda_0 < \cdots < \lambda_i < \cdots \in \mathbb{R}$ and there is an orthonormal basis of H formed by eigenvectors $\{\psi_i\}_{i \in \mathbb{N}}$ such that, $\psi_i \in V$ y

$$\forall v \in V, \quad a(\psi_i, v) = \lambda_i(\psi_i, v).$$

(18)

Proof. The symmetry of $a(\cdot, \cdot)$ is immediate consequence of the symmetry of σ. By (h2) we have that σ is uniformly elliptic and symmetric, then satisfies the following inequality

$$0 < m |\nabla u|^2 \leq \sigma \nabla u \cdot \nabla u,$$

then, integrating over Ω and adding $m\|u\|_H^2$ on both sides of the inequality we get

$$m \left(\int_\Omega \|u\|_H^2 + |\nabla u|^2 \right) \leq a(u, u) + m \|u\|_H^2 ,$$

which shows (17), the continuity of $a(\cdot, \cdot)$ is also a consequence of (6). The existence of egenvalues and eigenvectors is obtained by a classical result, see (Raviart 1992, thm 6.2-1 y rem. 6.2-2, p. 137-138), taking into account that $\lambda_0 = 0$ because the bilinear form $a(\cdot, \cdot)$ is canceled only for constant functions.

It is important to note that the properties of the bilinear form $a(\cdot, \cdot)$ allow to introduce an operator in a natural way.

Definition 3. *By the previous lemma, the hypotheses of the Lax-Milgram theorem for the bilinear form $a(\cdot, \cdot)$ are fulfilled and therefore there is an operator $A: V \to V'$ injective and continuous with continuous inverse such that*

$$a(u, v) = \langle Au, v \rangle . \tag{19}$$

If v is a function defined on Ω we denote its trace to the boundary $\partial\Omega$ also as v, its meaning will always be clear from the context.

Proposition 3. *If $\varphi \in L^2(\Gamma_1)$ then for $v \in V$ the function*

$$v \longmapsto \hat{\varphi}(v) = \int_{\Gamma_1} \varphi v,$$

defines a linear and continuous functional. This is, we have $\hat{\varphi} \in V'$.

We will denote

$$f(u, w) = f_1(u) + f_2(u)w, \qquad g(u, w) = g_1(u) + g_2 w, \tag{20}$$

with

$$f_1(u) = a_1(u - u_{rest})(u - u_{th})(u - u_{peak}) = a_1 u^3 - \alpha_2 u^2 + \alpha_1 u - \alpha_0,$$

$\alpha_0 = a_1 u_{rest} u_{th} u_{peak}$

$\alpha_1 = a_1(u_{rest} u_{peak} + u_{th} u_{peak} + u_{rest} u_{th}),$

$\alpha_2 = a_1(u_{peak} + u_{rest} + u_{th}),$

$f_2(u) = a_2(u - u_{rest}),$

$g_1(u) = -bu + bu_{rest},$

$g_2 = bc_3.$

Proposition 4. *For $p = 4$, there are constants $c_i \geq 0, i = 1, ..., 6$, such that for all $u \in \mathbb{R}$ the following inequalities hold.*

$$|f_1(u)| \leq c_1 + c_2 |u|^{p-1}, \qquad |f_2(u)| \leq c_3 + c_4 |u|^{p/2-1}, \qquad |g_1(u)| \leq c_5 + c_6 |u|^{p/2}.$$

Proof. Due to Young's inequality the following estimates are met

$$|u|^2 \leq \frac{2|u|^3}{3} + \frac{1}{3}, \quad |u| \leq \frac{|u|^3}{3} + \frac{2}{3}, \quad |u| \leq \frac{|u|^2}{2} + \frac{1}{2}.$$

Then,

$$|f_1(u)| \leq a_1 |u|^3 + \alpha_2 \left(\frac{2|u|^3}{3} + \frac{1}{3} \right) + \alpha_1 \left(\frac{|u|^3}{3} + \frac{2}{3} \right) + \alpha_0$$

$$= \frac{\alpha_2}{3} + \frac{2\alpha_1}{3} + \alpha_0 + \left(a_1 + \frac{2\alpha_2}{3} + \frac{\alpha_1}{3} \right) |u|^3$$

$$|f_2(u)| = |a_2(u - u_{rest})| \leq a_2 u_{rest} + a_2 |u|$$

$$|g_1(u)| = |-bu + bu_{rest}| \leq bu_{rest} + b|u|$$

$$\leq bu_{rest} + b \left(\frac{|u|^2}{2} + \frac{1}{2} \right) = bu_{rest} + \frac{b}{2} + \frac{b}{2} |u|^2.$$

Proposition 5. *For $= 4$, there are $a, \lambda > 0, \mu, c \geq 0$ such that for all $(u, w) \in \mathbb{R}$ we have*

$$\lambda u f(u, w) + w g(u, w) \geq a |u|^p - \mu \left(\lambda |u|^2 + |w|^2 \right) - c. \tag{21}$$

Proof. By direct calculation from (20) we have

$$\lambda u f(u, w) + w g(u, w) = \lambda a_1 u^4 - \lambda \alpha_2 u^3 + \lambda \alpha_1 u^2 - \lambda \alpha_0 u + \lambda a_2 u^2 w$$
$$+ \lambda a_2 u_{rest} u w - b u w + b u_{rest} w + b c_3 w^2.$$

On the other hand, from Young's inequality we have

$$|\alpha_2 u^3| \leq \frac{3}{4} (\theta |u|^3)^{4/3} + \frac{1}{4} \left(\frac{\alpha_2}{\theta} \right)^4, \qquad |u^2 w| \leq \frac{1}{2} (\beta |u|^2)^2 + \frac{1}{2} \left(\frac{|w|}{\beta} \right)^2,$$

$$|\alpha_0 u| \leq \frac{|u|^2}{2} + \frac{\alpha_0^2}{2}, \qquad |uw| \leq \frac{|u|^2}{2} + \frac{|w|^2}{2}, \qquad |u_{rest} w| \leq \frac{|w|^2}{2} + \frac{u_{rest}^2}{2}.$$

Then,

$$\lambda u f(u, w) + w g(u, w) \geq$$
$$\left(\lambda a_1 - \lambda \frac{3}{4} \theta^{4/3} - \lambda a_2 \frac{\beta^2}{2} \right) |u|^4 - \frac{\lambda}{4} \left(\frac{\alpha_2}{\theta} \right)^4 - \frac{\lambda a_2}{2} \left(\frac{|w|}{\beta} \right)^2$$
$$- \lambda \frac{|u|^2}{2} - \lambda \frac{\alpha_0^2}{2} - \lambda a_2 u_{rest} \frac{|u|^2}{2} - \lambda a_2 u_{rest} \frac{|w|^2}{2} - b \frac{|u|^2}{2} - b \frac{|w|^2}{2}$$
$$- b \frac{|w|^2}{2} - b \frac{u_{rest}^2}{2}$$
$$= \left(\lambda a_1 - \lambda \frac{3}{4} \theta^{4/3} - \lambda a_2 \frac{\beta^2}{2} \right) |u|^4 - \left(\frac{\lambda}{2} + \frac{\lambda a_2 u_{rest}}{2} + \frac{b}{2} \right) |u|^2$$
$$- \left(\frac{\lambda a_2}{2\beta^2} + \frac{\lambda a_2 u_{rest}}{2} + \frac{b}{2} + \frac{b}{2} \right) |w|^2 - \left(\frac{\lambda}{4} \left(\frac{\alpha_2}{\theta} \right)^4 + \lambda \frac{\alpha_0^2}{2} + b \frac{u_{rest}^2}{2} \right).$$

To continue, it is necessary to extract a common term from the coefficients corresponding to $|u|^2$ and $|w|^2$, for this we can write $\frac{1}{2} = \rho \gamma$, with $\rho < 1$ y $\gamma = \frac{1}{2\rho} > 2$,

$$\lambda u f(u,w) + w g(u,w) \geq$$

$$\lambda\left(a_1 - \frac{3}{4}\theta^{4/3} - a_2\frac{\beta^2}{2}\right)|u|^4 - \lambda\gamma\left(\rho + \rho a_2 u_{rest} + \rho\frac{b}{\lambda}\right)|u|^2$$

$$- \gamma\left(\rho\frac{\lambda a_2}{\beta^2} + \rho\lambda a_2 u_{rest} + 2\rho b\right)|w|^2 - \left(\frac{\lambda}{4}\left(\frac{\alpha_2}{\theta}\right)^4 + \lambda\frac{\alpha_0^2}{2} + b\frac{u_{rest}^2}{2}\right).$$

To conclude it is necessary to verify that θ, β and ρ can be chosen so that

$$a_1 - \frac{3}{4}\theta^{4/3} - a_2\frac{\beta^2}{2} > 0,$$

$$\rho + \rho a_2 u_{rest} + \rho\frac{b}{\lambda} \leq 1,$$

$$\rho\frac{\lambda a_2}{\beta^2} + \rho\lambda a_2 u_{rest} + 2\rho b \leq 1,$$

which is fulfilled for

$$\frac{3}{4}\theta^{4/3} = \frac{a_1}{2}, \qquad \frac{\beta^2}{2} = \frac{1}{4},$$

obviously, we can find a ρ small enough to meet such conditions. We have $\mu = \gamma, \lambda > 0$ arbitrary, $a = \frac{\lambda a_1}{4}$ and

$$c = \frac{\lambda}{4}\left(\frac{\alpha_2}{\theta}\right)^4 + \lambda\frac{\alpha_0^2}{2} + b\frac{u_{rest}^2}{2}.$$

Proposition 6. *Let $u \in L^p(\Omega)$ and $w \in H$, Then $f(u,w) \in L^{p'}(\Omega)$ and $g(u,w) \in H$. In addition, the following inequalities are met*

$$\|f(u,w)\|_{L^{p'}(\Omega)} \le A_1 |\Omega|^{1/p'} + A_2 \|u\|_{L^p(\Omega)}^{p/p'} + A_3 \|u\|_H^{2/p'},$$
$$\|g(u,w)\|_H \le B_1 |\Omega|^{1/2} + B_2 \|u\|_{L^p(\Omega)}^{p/2} + B_3 \|u\|_H,$$

where $A_i \ge 0, i = 0, \ldots, 3$, y $B_i \ge 0$, $i = 0, \ldots, 3$, are constants that depend only on $c_i, i = 1, \ldots, 6$ and p.

Proof. Let $(u,w) \in \mathbb{R}^2$, by proposition 4 we have

$$|f(u,w)| \le c_1 + c_2|u|^{p-1} + c_3|w| + c_4|w||u|^{p/2-1},$$
$$g(u,w) \le B_1 + B_2 |u|^{p/2} + B_3 |w|,$$

with $B_1 = c_5, B2 = c_6$ y $B_3 = |g_2|$. On the other hand, by Young's inequality, with $\beta = \frac{2}{p'} > 1$ and $\frac{1}{\beta} + \frac{1}{\beta'} = 1$, we have

$$|w||u|^{p/2-1} \le \frac{|w|^\beta}{\beta} + \frac{|u|^{p/2-1}\beta'}{\beta'},$$

then, because $\left(\frac{p}{2} - 1\right)\beta' = \left(\frac{p}{2} - 1\right)2\frac{p-1}{p-2} = p - 1$ we have

$$|f(u,w)| \le c_1 + \left(c_2 + \frac{c_4}{\beta'}\right)|u|^{p-1} + c_3|w| + \frac{c_4}{\beta}|w|^\beta,$$

then, once more by Young's inequality $|w| \le \frac{|w|^\beta}{\beta} + \frac{1}{\beta}$, therefore we can find constants A_1, A_2 y A_3 such that

$$|f(u,w)| \le A_1 + A_2|u|^{p-1} + A_3|w|^\beta.$$

If $(u, v) \in L^p(\Omega) \times H$, by direct calculation and taking into account that $(p-1)p' = p$, $\beta p' = 2$ we have

$$\|f(u, v)\|_{L^{p'}(\Omega)} \leq \|A_1 + A_2|u|^{p-1} + A_3|w|^\beta\|_{L^{p'}(\Omega)}$$
$$\leq A_1|\Omega|^{1/p'} + A_2\|u\|_{L^p(\Omega)}^{p/p'} + A_3\|w\|_H^{2/p'}.$$

In a similar way

$$\|g(u, w)\|_H \leq \|B_1 + B_2|u|^{p/2} + B_3|w|\|_H$$
$$\leq B_1|\Omega|^{1/2} + B_2\|u\|_{L^p(\Omega)}^{p/2} + B_3\|w\|_H.$$

Definition of weak and strong solution

This section establishes the definition of the solution that will be obtained in section 3.1 for the model (1)-(5) of a ventricle. Also, we define strong solution and give a result of selectivity of the weak solution. It will be necessary to consider the weak formulation both in time and space. In order to give a bit of context to this definition we will start by considering the variational formulation in the spatial variable of the original model,

$$\int_\Omega \frac{\partial}{\partial t} u(t)v + \int_\Omega f(u(t), w(t))v + a(u(t), v) = s(t) \int_{\Gamma_1} \varphi v, \qquad \forall v \in V, \qquad (22)$$

$$\int_\Omega \frac{\partial w}{\partial t} z + \int_\Omega g(u(t), w(t))z = 0, \qquad \forall z \in H, \qquad (23)$$

$$u(0, x) = u_0(x), \quad w(0, x) = w_0(x), \qquad x \in \Omega, \qquad (24)$$

in this way it will be natural to introduce a succession of approximate solutions through a discretization of the space in which we will look for the solution. This procedure is known as the Faedo-Galerkin method.

We will denote as V_m the linear space generated by $\{\psi_0, \psi_1, ..., \psi_m\}$, where the functions $\psi_i, i = 0, ..., m$, are eigenfunction of the bilinear form $a(\cdot, \cdot)$ as established in the proposition 2. Note that $V_m \subset V$. For each m, we consider the variational problem restricted to the space V_m, that is, instead of v and z we take $\psi_i, i = 0, ..., m$, and approximate $u(t)$ and $w(t)$ by $u_m(t)$ and $w_m(t)$ respectively, with

$$u_m(t) = \sum_{i=0}^{m} u_{im}(t)\psi_i \in V_m, \qquad w_m(t) = \sum_{i=0}^{m} w_{im}(t)\psi_i \in V_m. \tag{25}$$

By means of these substitutions we obtain from (22)-(24) the following system

$$\frac{d}{dt} u_{im}(t) + \int_\Omega f(u_m(t), w_m(t))\psi_i + \lambda_i u_{im}(t) = s(t) \int_{\Gamma_1} \varphi \psi_i, \tag{26}$$

$$\frac{d}{dt} w_{im}(t) + \int_\Omega g(u_m(t), w_m(t))\psi_i = 0, \tag{27}$$

$$u_m(0) = u_{m0}, \quad w_m(0) = w_{m0}, \tag{28}$$

for $i = 0, ..., m$.

Definition 4. (Weak Solution). *Let $\tau > 0$ and the functions $u : t \in [0, \tau) \mapsto u(t) \in H$, $w : t \in [0, \tau) \mapsto w(t) \in H$. We say that (u, w) is a weak solution of the varitional formulation of the problem (1)-(4) if for any $T \in (0, \tau)$,*

1. *$u : [0, T] \mapsto H$ and $w : [0, T] \mapsto H$ are continuous.*

2. *For almost all $t \in (0, \tau)$, we have $u(t) \in V$, also $u \in L^p(Q_T) \cap L^2(0, T; V)$ and $w \in L^2(Q_T)$, with $Q_T = (0, T) \times \Omega$.*

In addition, the functions u and w satisfy

$$\frac{d}{dt}(u(t), v) + a(u(t), v) + \int_\Omega f(u(t), w(t))v = s(t) \langle \hat{\varphi}, v \rangle, \qquad \forall v \in V, \qquad (29)$$

$$\frac{d}{dt}(w(t), z) + \int_\Omega g(u(t), w(t))z = 0, \qquad \forall z \in H, \qquad (30)$$

where equality is considered in $\mathcal{D}'(0,T)$.

If, furthermore, given u_0, w_0 in H, u, w are weak solutions that satisfy

$$u(0) = u_0, \quad w(0) = w_0, \quad in \ H,$$

then we call u, w a weak solution of variational Cauchy problem associated to (1)-(5).

Remark 1. *The derivatives that appear in the first terms of the equations (29) and (30) refer to derivatives in the sense of distributions, that is, for $\phi \in \mathcal{D}(0,T)$ we have*

$$\int_0^T \frac{d}{dt}(u(t), v)\phi = - \int_0^T (u(t), v)\phi'.$$

Now, we can give a definition of strong solution for the variational formulation. Suppose that, u, w are weak solutions, in the sense of definition 4, and furthermore, $u \in W^{1,2,p'}(0,T;V',V)$ and $w \in W^{1,2,2}(0,T;H,H)$, then the equation (29) means that

$$-\int_0^T (u, v)\phi' + \int_0^T a(u, v)\phi + \int_0^T \int_\Omega f(u, w)v\phi = \int_0^T \langle s(\tau)\hat{\varphi}, v \rangle \phi, \ \text{for all } \phi \in \mathcal{D}(0,T),$$

thus, by proposition 1, it has

$$\int_0^T \left\langle \frac{du}{dt}, v \right\rangle \phi + \int_0^T \langle Au, v \rangle \phi + \int_0^T \langle f(u, w), v \rangle \phi = \int_0^T \langle s(\tau)\hat{\varphi}, v \rangle \phi,$$

which implies that

$$\int_0^T \left\langle \frac{du}{dt} + Au + f(u,w) - s(\tau)\hat{\varphi}, v \right\rangle \phi d\tau = 0, \quad \text{for all } \phi \in \mathcal{D}(0,T), v \in V.$$

From the above it follows that,

$$\frac{du}{dt} + Au + f(u,w) = s(t)\hat{\varphi}, \quad \text{for a.a. } t \in [0,T], \tag{31}$$

which holds in V'. In a similar for it is possible to prove that

$$\frac{dw}{dt} = g(u,w), \quad \text{for a.a. } t \in [0,T], \tag{32}$$

is fulfilled in H.

Definition 5. (Strong Solution). *Let be $u \in W^{1,2,p'}(0,T;V,V')$ and $w \in W^{1,2,2}(0,T;H,H)$ we call u,w strong solutions of the variational formulation problem (1)-(4), if they satisfy the equation (31)-(32) in V' and H, respectively.*

If, besides,

$$u(0) = u_0, \quad w(0) = w_0, \quad \text{in } H,$$

for u_0, w_0 given, we say that u,w are strong solutions of variational Cauchy problem associated to (1)-(5).

Results

Existence of global solution

The main result of this section is the following theorem.

Theorem 4. (Existence of weak solution). *Under the hypotheses (h1)-(h5) plus*

(h6') the sequences um0, wm0 are bounded in H,

the system (1)- (4) has a weak solution (u, w) in the sense of the definition 4 with $\tau = +\infty$.

    The demonstration is developed in the following two subsections,

- a sequence of approximate solutions u_m, w_m is defined,

- then, it is verified that the approximate solutions converge to a function that satisfies the definition 4.

Existence of approximate solutions

The next lemma states that the approximate solutions u_m, w_m are defined for all $t > 0$, other important estimates are also established to demonstrate later that the succession of approximate solutions converges to a solution. The following norms will be used.

$$\|\cdot\|_{L^P(Q_T)\cap L^2(0,T;V)} = \max\left(\|\cdot\|_{L^P(Q_T)}, \|\cdot\|_{L^2(0,T;V)}\right).$$

$$\|\cdot\|_{L^{P'}(Q_T)+L^2(0,T;V')} = \inf_{u=u_1+u_2}\left(\|u_1\|_{L^{P'}(Q_T)} + \|u_2\|_{L^2(0,T;V')}\right).$$

Lemma 1. *The Cauchy problem (26) - (28) has solution for all $t > 0$. In addition, there are*

constants $C_i > 0, i = 1, ... ,4$, such that for all $T > 0$. The following estimates are met a priori

$$\lambda \|u_m(t)\|_H^2 + \|w_m(t)\|_H^2 \leq C_1, \quad \forall t \in [0, T], \tag{33}$$

$$\|u_m\|_{L^p(Q_T) \cap L^2(0,T;V)} \leq C_2, \tag{34}$$

$$\|u'_m\|_{L^{p'}(Q_T) + L^2(0,T;V')} \leq C_3, \tag{35}$$

$$\|w'_m\|_{L^2(Q_T)} \leq C_4, \tag{36}$$

where $u'_m = \sum_{i=0}^m u'_{im}\psi_i$ and $w'_m = \sum_{i=0}^m w'_{im}\psi_i$ are the derivatives of the functions $u_m :$ $[0, T] \longmapsto V$ and $w_m : [0, T] \longmapsto H$.

Proof. Note that the integrals in (26) and (27) are well defined, in deed, as $u_m(t) \in V \subset L^p(\Omega)$ and $w_m(t) \in H$ we have from proposition 6 that $f(u_m(t), w_m(t)) \in L^{p'}(\Omega) \subset V'$ and $g(u_m, w_m) \in H$, then because $\psi_i \in V \subset L^p(\Omega)$ and $\psi_i \in H$ we have

$$\int_\Omega f(u_m(t), w_m(t))\psi_i = \langle f(u_m(t), w_m(t)), \psi_i \rangle,$$

$$\int_\Omega g(u_m(t), w_m(t))\psi_i = \langle g(u_m(t), w_m(t)), \psi_i \rangle.$$

The terms in (26) and (27) are continuous as functions of $u_{im}(t)$ and $w_{im}(t)$, then the initial value problem formed by (26) - (27) with initial conditions (28) has a unique maximal solution defined for $t \in [0, t_m)$ with u_{im} and w_{im} in C^1, for each initial condition u_{0m}, w_{0m}, (by Cauchy-Peano theorem).

If (u_m, w_m) is not a global solution, this is $t_m < 1$, then it is not bounded in $[0, t_m)$. Suppose that (u_m, w_m) is a maximal solution of (26)-(28). Multiplying (26) by λu_{im}, (27) by w_{im} and adding on $i = 0, ... , m$ we get

$$\lambda \sum_{i=0}^m \left(u_{im}(t)\frac{d}{dt}u_{im}(t) + \lambda\lambda_i u_{im}(t)u_{im}(t) \right.$$

$$\left. + \int_\Omega \lambda f(u_m(t), w_m(t))u_{im}(t)\psi_i \right) = \lambda s(t)\left\langle \hat{\varphi}, \sum_{i=0}^m u_{im}(t)\psi_i \right\rangle, \tag{37}$$

$$\sum_{i=0}^m \left(w_{im}(t)\frac{d}{dt}w_{im}(t) + \int_\Omega g(u_m(t), w_m(t))w_{im}(t)\psi_i \right) = 0. \tag{38}$$

Note that for being $\{\psi_i\}$ an orthonormal set we have

$$\frac{d}{dt}\|u_m\|_H^2 = 2\sum_{i=0}^{m} u_{im}(t)\frac{d}{dt}u_{jm}(t), \quad a(u_m, u_m) = \sum_{i=0}^{m} \lambda_i u_{im}(t)u_{jm}(t).$$

Then, by the previous observations, adding (37) and (38) we have for all $t \in [0, t_m)$

$$\frac{1}{2}\frac{d}{dt}\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) + \lambda a(u_m(t), u_m(t))$$
$$+ \int_\Omega (\lambda f(u_m(t), w_m(t))u_m + g(u_m(t), w_m(t))w_m) = \lambda s(t)\langle \hat{\varphi}, u_m(t)\rangle. \tag{39}$$

On the other hand, note that for being $a(\cdot, \cdot)$ coercitive, see (17), we have

$$\lambda\alpha\left(\|u\|_V^2 - \|u\|_H^2\right) \leq \lambda a(u_m(t), u_m(t)). \tag{40}$$

Also, from proposition 5, by integrating both sides of (21) on Ω we get

$$\left(a\int_\Omega |u_m(t)|^p\right) - \mu(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2) - c|\Omega|$$
$$\leq \int_\Omega \lambda f(u_m(t), w_m(t))u_m(t) + g(u_m(t), w_m(t))w_m(t). \tag{41}$$

Then, adding (40) and (41) we get

$$\lambda\alpha\left(\|u_m(t)\|_V^2 - \|u_m(t)\|_H^2\right) + \left(a\int_\Omega |u_m(t)|^p\right) - \mu(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2) - c|\Omega|$$
$$\leq \lambda a(u_m(t), u_m(t)) + \int_\Omega \lambda f(u_m(t), w_m(t))u_m(t) + g(u_m(t), w_m(t))w_m(t).$$

Adding $\frac{1}{2}\frac{d}{dt}(\lambda\|u_m(t)\|_H^2 + \|u_m(t)\|_H^2)$ on both sides of the previous inequality we get from (39) the following

$$\frac{1}{2}\frac{d}{dt}\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) + \lambda\alpha\left(\|u_m(t)\|_V^2 - \|u_m(t)\|_H^2\right)$$
$$+ \left(a\int_\Omega |u_m(t)|^p\right) - \mu\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) - c|\Omega| \leq \lambda|s(t)|\,\|\hat{\varphi}\|_{V'}\,\|u_m(t)\|_V .$$

Then, reorganizing terms and adding $\alpha\|w_m(t)\|_H^2$ to the right side of the previous inequality we get

$$\frac{1}{2}\frac{d}{dt}\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) + \lambda\alpha\|u_m(t)\|_V^2 + a\int_\Omega |u_m(t)|^p$$
$$\leq c|\Omega| + \lambda|s(t)|\,\|\hat{\varphi}\|_{V'}\,\|u_m(t)\|_V + (\alpha+\mu)\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) .$$

On the other hand, by Young's inequality we have for all $\theta > 0$ the following

$$\lambda|s(t)|\,\|\hat{\varphi}\|_{V'}\,\|u_m(t)\|_V \leq \frac{1}{2\theta}\lambda^2 s(t)^2\,\|\hat{\varphi}\|_{V'}^2 + \frac{\theta}{2}\|u_m(t)\|_V^2 ,$$

then, by taking $\theta = \lambda\alpha$ we get the following inequality that will be useful a little later.

$$\frac{1}{2}\frac{d}{dt}\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) + \frac{\lambda\alpha}{2}\|u_m(t)\|_V^2 + a\int_\Omega |u_m(t)|^p$$
$$\leq c|\Omega| + \frac{\lambda}{2\alpha}s(t)^2\,\|\hat{\varphi}\|_{V'}^2 + (\alpha+\mu)\left(\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right) . \tag{42}$$

From (42) it follows immediately that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda\,\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right)$$

$$\leq c\,|\Omega| + \frac{\lambda}{2\alpha}s(t)^2\,\|\hat{\varphi}\|_{V'}^2 + (\alpha+\mu)\left(\lambda\,\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2\right).$$

Then, integrating with respect to t over the interval $[0, t_m)$ on both sides of the previous inequality we get

$$\lambda\,\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2 \leq 2c\,|\Omega|\,t_m + \frac{\lambda}{\alpha}\|s\|_\infty^2\,\|\hat{\varphi}\|_{V'}^2\,t_m + \lambda\,\|u_m(0)\|_H^2$$

$$+ \|w_m(0)\|_H^2 + 2(\alpha+\mu)\int_0^{t_m}\left(\lambda\,\|u_m(\tau)\|_H^2 + \|w_m(\tau)\|_H^2\right)d\tau.$$

Recall now that, there exist a constant $c > 0$, such that $\|u_m(0)\|_H \leq c$ y $\|w_m(0)\|_H \leq c$, also we have that Ω is bounded. Then, from the previous inequality and from Gronwall's inequality it follows that there is a constant $\mathcal{C}_1$ that depends only on $c, \sigma, f, g, u_0, w_0, \Omega, s, \hat{\varphi}$ and t_m, such that

$$\forall t \in [0, t_m), \qquad \lambda\,\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2 \leq \mathcal{C}_1.$$

As a consequence we have that (u_m, w_m) is bounded in any finite interval of time, this is $t_m = +\infty$. For $T > 0$ fixed we have shown (33).

In order to get (34) we begin by integrating (42) in the interval $[0, T]$

$$\frac{1}{2}\left(\lambda\,\|u_m(T)\|_H^2 + \|w_m(T)\|_H^2\right) + \frac{\lambda\alpha}{2}\|u_m\|_{L^2(0,T;V)}^2 + a\,\|u_m\|_{L^p(Q_T)}^p$$

$$\leq k_1 + (\alpha+\mu)\int_0^T\left(\lambda\,\|u_m(\tau)\|_H^2 + \|w_m(\tau)\|_H^2\right)d\tau,$$

with $k_1 = c|\Omega|T + \frac{\lambda}{2\alpha}\|s\|_\infty^2\|\hat{\varphi}\|_{V'}^2 + \frac{1}{2}(\lambda\|u_m(0)\|_H^2 + \|w_m(0)\|_H^2)$. Then, we use (33) on the integral on the right side of the previous inequality,

$$\frac{\lambda\alpha}{2}\|u_m\|^2_{L^2(0,T;V)} + a\|u_m\|^p_{L^p(Q_T)} \leq k_2,$$

with $k_2 = k_1 + (\alpha + \mu)C_1T$. Therefore, we have shown inequality (34) with

$$C_2 = \max\left\{\left(\frac{2}{\lambda\alpha}k_2\right)^{1/2}, \left(\frac{1}{a}k_2\right)^{1/p}\right\}.$$

Integrating (33) on $[0,T]$ we also get a bound for wm in $L^2(Q_T)$.

Now we will obtain the estimates for u'_m and w'_m. Consider the projection operator $P_m : V' \to V'$ defined by $u \in V' \mapsto P_m u = \sum_{i=0}^{m}\langle u, \psi_i\rangle\psi_i$. Equivalently, $P_m u$ is defined as the only element in V_m such that $\langle u, v\rangle = \langle P_m u, v\rangle$ for all $v \in V_m$. On the other hand, note that for all $v \in V$ and for all $t > 0$ we have

$$\frac{d}{dt}(u_m(t), v) = \langle u'_m(t), v\rangle, \quad \int_\Omega f(u_m(t), w_m(t))v = \langle f(u_m(t), w_m(t)), v\rangle,$$

because $u'_m(t) \in V_m \subset V'$, $f(u_m(t), w_m(t)) \in L^{p'}(Q_T)$ and $v \in V \subset L^p(Q_T)$. Thus, from (26) it follows that

$$\forall v \in V_m, \forall t > 0, \quad \langle u'_m(t), v\rangle = \langle -[Au_m(t) + f(u_m(t), w_m(t))] + s(t)\hat{\varphi}, v\rangle,$$

and then

$$\forall t > 0, \quad u'_m(t) = P_m\left(-[Au_m(t) + f(u_m(t), w_m(t))] + s(t)\hat{\varphi}\right), \tag{43}$$

where A is the weak operator defined in (19). For the continuity of A and the estimate (34) we have for all $T > 0$

$$\|Au_m\|_{L^2(0,T;V')} \leq M \left(\int_0^T \|u_m(t)\|_V^2 \, dt \right)^{1/2} \leq MC_2.$$

On the other hand, from the estimates (33), (34) and by lemma 6

$$\|f(u_m, w_m)\|_{L^{p'}(Q_T)} \leq A_1 \left(|\Omega|T\right)^{1/p'} + A_2 C_2^{p/p'} + A_3 \left(C_1 T\right)^{2/p'}.$$

The next thing will be to obtain a bound for the projection operator P_m. We begin by highlighting that, as $P_m(V') \subset V_m \subset V$, the restriction of P_m to V can be considered as an operator from V on V defined by $u \in V \mapsto P_m u = \sum_{i=0}^m (u, \psi_i)\psi_i$. If $u \in H$, $P_m u$ is the orthogonal projection of u in V_m, and then $\|P_m u\|_H \leq \|u\|_H$. The transpose operator P_m^T of $P_{m|V}$ identifies with $P_m : V' \to V'$, and therefore we have $\|P_m u\|_{\mathcal{L}(V',V')} = \|P_m u\|_{\mathcal{L}(V,V)}$. If $u \in V$ we can calculate

$$a(P_m u, P_m u) = \sum_{i=0}^m \lambda_i (u, \psi_i)(u, \psi_i) \leq \sum_{i=0}^\infty \lambda_i |(u, \psi_i)|^2 = a(u, u).$$

Therefore, for all $u \in V$ we have

$$\alpha \|P_m u\|_V^2 \leq a(P_m u, P_m u) + \alpha \|P_m u\|_H^2 \leq M \|u\|_V^2 + \alpha \|u\|_H^2 \leq (M + \alpha) \|u\|_V^2.$$

The previous inequality shows that the family of operators P_m is uniformly bounded in V',

$$\|P_m\|_{\mathcal{L}(V',V')} \leq \frac{M}{\alpha} + 1.$$

Then, the following inequalities are met

$$\|P_m(Au_m)\|_{L^2(0,T;V')} \leq \left(\frac{M}{\alpha}+1\right) MC_2.$$

$$\|P_m(f(u_m,w_m))\|_{L^{p'}(Q_T)} \leq \left(\frac{M}{\alpha}+1\right) \left(A_1(|\Omega|\,T)^{1/p'} + A_2 C_2^{p/p'} + A_3(C_1 T)^{1/p'}\right).$$

$$\|P_m(s\hat{\varphi})\|_{L^2(0,T;V')} \leq \left(\frac{M}{\alpha}+1\right) \|s\|_\infty \|\hat{\varphi}\|_{V'} T.$$

Inequality (35) is obtained from the previous inequalities and (43). We will proceed similarly to obtain the estimate for w'_m. From (27) it follows that

$$\forall v \in V_m \subset H, \ \forall > 0, \qquad \langle w'_m(t), v \rangle = -\langle g(u_m(t), w_m(t), v) \rangle,$$

and therefore

$$\forall t > 0, \qquad w'_m(t) = -P_m(g(u_m(t), w_m(t))),$$

where we take the operator P_m restricted to the orthogonal projection $P_{m|H}$, so $\|P_m\|_{\mathcal{L}(H,H)} \leq 1$. Then, for $T > 0$ fixed, from (33), (34) and by proposition 6, we have (36)

$$\|w'_m\|_{L^2(Q_T)} \leq \|g(u_m, w_m)\|_{L^2(Q_T)}$$
$$\leq B_1(|\Omega|\,T)^{1/2} + B_2(C_2)^{p/2} + B_3(C_1 T)^{1/2} := C_4.$$

Convergence of approximate solutions

In the previous section it was shown that the approximate solutions proposed in (25) exist and are defined for all $t > 0$. In this section we will use the a priori estimates (33) - (36) to show that, there exist subsequences of the approximate solutions (u_m, w_m) that converge, in a suitable form,

to a weak solution according to the definition 4. Furthermore, we prove that this weak solutions is also a strong solution.

Lemma 2. *There are subsequences, which for convenience are also denoted as* u_m, u'_m, w_m *and* w'_m *such that*

$$u_m \rightarrow u, \text{weakly in } L^p(Q_T) \cap L^2(0,T;V). \tag{44}$$

$$u'_m \rightarrow \tilde{u}, \text{ weakly in } L^{p'}(Q_T) + L^2(0,T;V'). \tag{45}$$

$$w_m \rightarrow w, \text{ weakly in } L^2(Q_T). \tag{46}$$

$$w'_m \rightarrow \tilde{w}, \text{ weakly in } L^2(Q_T). \tag{47}$$

and

$$u_m \rightarrow u \text{ strongly in } L^2(Q_T), \tag{48}$$

$$w_m \rightarrow w \text{ strongly in } L^2(Q_T). \tag{49}$$

Proof. Evidently $L^p(Q_T) \cap L^2(0,T;V)$ is a reflexive space since $L^p(Q_T)$ is reflexive, see (Brezis 2011, prop. 3.20, p. 60). By inequality (34), u_m is a bounded sequence in $L^p(Q_T) \cap L^2(0,T;V)$, then it has a subsequence that converge weakly, see (Brezis 2011, thm. 3.18, p. 69). So (44) has been proved. By a similar argument we obtain (45)-(47).

Note that, because $2 \geq p'$, we have $L^{p'}(Q_T) + L^2(0,T;V') \subset L^{p'}(0,T;V')$. By lemma 1 we know that u'_m is bounded in $L^{p'}(0,T;V')$ while u_m is bounded in $L^2(0,T;V)$ and then u_m is a bounded sequence in $W^{1,2,p'}(0,T;V,V')$, see theorem 2. Then, by the compact immersion of $W^{1,2,p'}(0,T;V,V')$ in $L^2(Q_T)$, there is a subsequence that converge in $L^2(Q_T)$.

Corollary 1. *The subquences* u_m, w_m *satisfy*

$$\frac{d}{dt}(u_m, v) \rightarrow \frac{d}{dt}(u, v), \quad \text{weakly in } \mathcal{D}'(0,T), \text{ for all } v \in V, \tag{50}$$

$$\frac{d}{dt}(w_m, h) \rightarrow \frac{d}{dt}(w, h), \quad \text{weakly in } \mathcal{D}'(0,T), \text{ for all } h \in H, \tag{51}$$

and also, it has that

$$\frac{d}{dt}(u, v) = \langle \tilde{u}, v \rangle, \quad v \in V,$$

$$\frac{d}{dt}(w, h) = \langle \tilde{w}, h \rangle, \quad h \in H,$$

in $\mathcal{D}'(0,T)$. That is, $u \in W^{1,2,p'}(0,T;\ V,V')$, and $w \in W^{1,2,2}(0,T;H,H)$.

Proof. Let us take $v \in V, \phi \in \mathcal{D}(0,T)$, and note that,

$$\int_0^T \frac{d}{dt}(u_m, v)\phi = \int_0^T (u'_m, v)\phi = -\int_0^T (u_m, v)\phi',$$

by taking limit in the above equality we obtain

$$\int_0^T \frac{d}{dt}(u_m, v)\phi \to -\int_0^T (u, v)\phi' = \int_0^T \frac{d}{dt}(u, v)\phi.$$

Thus, we have obtained (50). Also, by the weak converge of u'_m, we get

$$\int_0^T (u'_m, v)\phi \to \int_0^T \frac{d}{dt}(u, v)\phi,$$

and, due to the uniqueness the weak limit

$$\int_0^T \frac{d}{dt}(u, v)\phi = \int_0^T \langle \tilde{u}, v \rangle \phi,$$

that is

$$\frac{d}{dt}(u, v) = \langle \tilde{u}, v \rangle.$$

In a similar form are proved the affirmations for w.

Corollary 2. *For $\psi_i, i \geq 0$ and the bilinear form $a(\cdot, \cdot)$ defined in (15) we have*

$$\int_0^T a(u_m(t), \psi_i)\phi \rightarrow \int_0^T a(u(t), \psi_i)\phi_i \qquad \forall \phi \in \mathcal{D}(0, T).$$

Proof. Because $a(\cdot, \cdot)$ is a continuous bilinear form, the map

$$u_m \mapsto \int_0^T a(u_m(t), \psi_i)\phi,$$

is a continuous linear functional on $L^p(Q_T) \cap L^2(0, T; V)$, and then the result follows immediately from the fact that u_m converges to u weakly in $L^p(Q_T) \cap L^2(0, T; V)$.

Corollary 3. *For f and g defined in (7)-(8) and for all $\psi_i, i \geq 0$, we have*

$$\int_0^T \int_\Omega f(u_m(t), w_m(t))\psi_i\phi \rightarrow \int_0^T \int_\Omega f(u(t), w(t))\phi(t)\psi_i\phi, \quad \forall \phi \in \mathcal{D}(0, T),$$

$$\int_0^T \int_\Omega g(u_m(t), w_m(t))\psi_i\phi \rightarrow \int_0^T \int_\Omega g(u(t), w(t))\psi_i\phi, \quad \forall \phi \in \mathcal{D}(0, T).$$

Proof. Given that $u_m \rightarrow u$, and $w_m \rightarrow w$, in $L^2(Q_T)$, it obtains

$$u_m \rightarrow u, \quad \text{a.e. in } Q_T,$$

$$w_m \rightarrow w, \quad \text{a.e. in } Q_T,$$

and by the continuity of f,

$$f(u_m, w_m) \to f(u, w), \quad \text{a.e. in } Q_T,$$
$$g(u_m, w_m) \to g(u, w), \quad \text{a.e. in } Q_T.$$

Also,

$$\|f(u_m, w_m)\|_{L^{p'}(Q_T)}$$
$$\leq A_1 |\Omega|^{1/p'} T^{1/p'} + A_2 \left\| |u_m|^{p/p'} \right\|_{L^{p'}(Q_T)} + A_3 \left\| |w_m|^{2/p'} \right\|_{L^{p'}(Q_T)}$$
$$\leq A_1 |\Omega|^{1/p'} T^{1/p'} + A_2 C_2^{p/p'} + A_3 C_2^{2/p'},$$

And

$$\|g(u_m, w_m)\|_{L^2(Q_T)}$$
$$\leq B_1 |\Omega|^{1/2} T^{1/2} + B_2 \left\| |u_m|^{p/2} \right\|_{L^2(Q_T)} + B_3 \left\| |w_m| \right\|_{L^{p'}(Q_T)}$$
$$\leq B_1 |\Omega|^{1/2} T^{1/2} + B_2 C_2^{p/2} + B_3 C_2.$$

Using an argument of dominated convergence type, see (Lions 1969), we can affirm that $f(u_m, w_m)$, converges to $f(u, w)$, and $g(u_m, w_m)$, converges to $g(u, w)$, weakly in $L^{p'}(Q_T)$, and $L^2(Q_T)$, respectively, that is, for all $\zeta \in L^p(Q_T)$ and $\eta \in L^2(Q_T)$, it has

$$\int_0^T \int_\Omega f(u_m, w_m)\zeta \to \int_0^T \int_\Omega f(u, w)\zeta,$$
$$\int_0^T \int_\Omega g(u_m, w_m)\eta \to \int_0^T \int_\Omega g(u, w)\eta,$$

taking $\zeta = \phi v, \eta = \phi h$ with $\phi \in \mathcal{D}'(0, T)$, $v \in V$ and $h \in H$, it has the result.

Conclusion

By the three previous corollaries it is concluded that the functions u and w satisfy for all $i \geq 1$ the following

$$\frac{d}{dt}(u(t), \psi_i) + a(u(t), \psi_i) + \langle f(u(t), w(t)), \psi_i \rangle = \langle s(t)\varphi, \psi_i \rangle \tag{52}$$

$$\frac{d}{dt}(w(t), \psi_i) + \langle g(u(t), w(t)), \psi_i \rangle = 0 \tag{53}$$

where equality is considered in $\mathcal{D}'(0, T)$. Then, because functions $\psi_i, i \geq 0$ are dense in V, it follows that u and w satisfy the equations (29)-(30) in the definition of weak solution 4.

For other hand, by corollary 1, these weak solutions u, w belong to $W^{1,2,p'}(0, T; V, V')$ and $W^{1,2,2}(0, T; H, H)$, thus they are strong solutions , too.

In other words, we have proved that if the systems of Faedo-Galerkin (26)-(27) are considered with uniformly bounded initial conditions the corresponding solutions, u_m, w_m, have subsequences that converge, in a suitable form, to a weak solution of the considered problem.

Note that, in the case that the Cauchy problem be considered for the variational formulation, that is, initial conditions u_0, w_0 be given the systems of Faedo-Galerkin (26)-(27) have initial conditions u_{0m}, w_{0m} which are the projections of u_0, w_0 in the subspaces V_m, for each $m = 0, 1, ...,$ and are uniformly bounded. In fact,

$$\|u_{m0}\|_H \leq \|u_0\|_H, \quad \|w_{m0}\|_H \leq \|w_0\|_H,$$

thus, by applying the results previously exposed we obtain the existence of weak solution of the variational Cauchy problem.

Continuity

From the previous section we have that $u \in W^{1,2,p'}(0,T; V, V') \subset W^{1,2,2}(0,T; V'V')$, and $w \in W^{1,2,2}(0,T; H, H)$. Then, by theorem (3) it follows that the functions $u: t \in [0,T] \to u(t) \in V'$ and $w: t \in [0,T] \to w(t) \in H$ are continuous. Regarding u, it only shows that u it is weakly continuous in V.

By corollary 1 it follows that

$$\langle \partial_t u(t), u(t) \rangle = \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 ,$$

where equality is considered in $\mathcal{D}'(0,T)$. Then, from (52), we have

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 = -a(u(t), u(t)) - \langle f(u(t), w(t)), u(t) \rangle + \langle s(t)\varphi, u(t) \rangle \qquad (54)$$

so that the function $t \to \|u(t)\|_H^2$ is in $H^1(0,T)$, and then is continuous from $[0,T]$ to $\mathbb{R}$. Then, it follows that function $u: t \in [0,T] \mapsto u(t) \in H$ is continuous.

When we consider u_{m0} and w_{m0} as the orthogonal projections in H of u_0 and w_0 respectively, we obtain that $u(0) = u_0$ and $w(0) = w_0$.

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